A NEW PERSPECTIVE TO THE MAZUR-ULAM PROBLEM IN 2-FUZZY 2-NORMED LINEAR SPACES

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Abstract. In this paper, we introduce the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, we give a new generalization of the Mazur-Ulam theorem when X is a 2-fuzzy 2-normed linear space or \( S(X) \) is a fuzzy 2-normed linear space, that is, the Mazur-Ulam theorem holds, when the 2-isometry mapped to a 2-fuzzy 2-normed linear space is affine.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [25]. A satisfactory theory of 2-norms and n-norms on a linear space has been introduced and developed by Gähler in [9, 10]. Different authors introduced various definitions of fuzzy norms on a linear space. For reference, one may see [8, 11, 13, 14, 21, 23]. Following Cheng and Mordeson [3], Bag and Samanta [1] introduced a concept of fuzzy norm on a linear space.

Recently, Somasundaram and Beaula [20] introduced a concept of 2-fuzzy 2-normed linear space or fuzzy 2-normed linear space of the set of all fuzzy sets of a set. The authors gave the notion of \( \alpha \)-2-norm on a linear space corresponding to the 2-fuzzy 2-norm by using some ideas of [1] and also gave some fundamental properties of this space.

In 1932, Mazur and Ulam [15] proved the following theorem.

Mazur-Ulam Theorem. Every isometry of a real normal linear space onto a real normed linear space is a linear mapping up to translation.

Baker [2] showed an isometry from a real normed linear space into a strictly convex real normed linear space is affine. Also, Jian [12] investigated the generalizations of the Mazur-Ulam theorem in \( F^\alpha \)-spaces. Rassias and Wagner [19] described all volume preserving mappings from a real finite dimensional vector space into itself and Väisälä [22] gave a short and simple proof of the Mazur-Ulam theorem. Chu [6] proved that the Mazur-Ulam theorem holds when X is a linear 2-normed space. Chu et al. [7] generalized the Mazur-Ulam theorem when X is a linear n-normed space, that is, the Mazur-Ulam theorem holds, when the n-isometry mapped to a linear n-normed space is affine. In addition, Moslehian and Sadeghi [16] investigated the Mazur-Ulam theorem in non-archimedean spaces. Chu et al. [7] also

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obtained extensions of Rassias and Šemrl’s theorem [18]. Cho et al. [5] investigated the Mazur–Ulam theorem in probabilistic 2-normed spaces. The Mazur–Ulam theorem has been extensively studied by many authors (see [17, 19, 24]). In the present paper, we introduce the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, we give a new generalization of the Mazur–Ulam theorem when \( X \) is a 2-fuzzy 2-normed linear space or \( \mathcal{S}(X) \) is a fuzzy 2-normed linear space, that is, the Mazur–Ulam theorem holds, when the 2-isometry mapped to a 2-fuzzy 2-normed linear space is affine.

2. On 2-Fuzzy 2-Normed Linear Spaces

In this section at first we give a concept of linear 2-normed space and later a concept of 2-fuzzy 2-normed linear space and it’s fundamental properties by using some ideas of [20]. For more details we refer the readers to [1, 4, 20].

**Definition 2.1.** [4] Let \( X \) be a real vector space of dimension greater than 1 and let \( \|\cdot,\cdot\| \) be a real valued function on \( X \times X \) satisfying the following four properties:

1. \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent,
2. \( x, y \| = \| y, x \| \),
3. \( \| x, \alpha y \| = \| x \| \) for any \( \alpha \in \mathbb{R} \),
4. \( \| x, y + z \| \leq \| x, y \| + \| x, z \| \).

\( \| \cdot, \cdot \| \) is called a 2-norm on \( X \) and the pair \((X, \| \cdot, \cdot \|)\) is called a linear 2-normed space.

**Definition 2.2.** [1] Let \( X \) be a linear space over \( S \) (field of real or complex numbers). A fuzzy subset \( N \) of \( X \times \mathbb{R} \) (the set of real numbers) is called a fuzzy norm on \( X \) if and only if:

1. For all \( t \in \mathbb{R} \) with \( t \leq 0 \), \( N(x, t) = 0 \).
2. For all \( t \in \mathbb{R} \) with \( t > 0 \), \( N(x, t) = 1 \) if and only if \( x = 0 \).
3. For all \( t \in \mathbb{R} \) with \( t > 0 \), \( \lambda N(x, t) = N(x, \frac{t}{|\lambda|}) \) if \( \lambda \neq 0 \), \( \lambda \in S \).
4. For all \( s, t \in \mathbb{R} \), \( x, y \in X \), \( N(x + y, s + t) \geq \min \{ N(x, s), N(y, t) \} \).
5. \( N(x, \cdot) \) is a non-decreasing function of \( t \in \mathbb{R} \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Then \((X, N)\) is called a fuzzy normed linear space or in short f-NLS.

**Theorem 2.3.** [1] Let \((X, N)\) be a f-NLS. Assume the condition that

1. \( N(x, t) > 0 \) for all \( t > 0 \) implies \( x = 0 \).
2. Define \( \| x \|_\alpha = \inf \{ t : N(x, t) \geq \alpha \} \), \( \alpha \in (0, 1) \). Then \( \{ \| \cdot \|_\alpha : \alpha \in (0, 1) \} \) is an ascending family of norms on \( X \). We call these norms as \( \alpha \)-norms on \( X \) corresponding to the fuzzy norm on \( X \).

**Definition 2.4.** Let \( X \) be any non-empty set and \( \mathcal{S}(X) \) be the set of all fuzzy sets on \( X \). For \( U, V \in \mathcal{S}(X) \) and \( \lambda \in S \) the field of real numbers, define

\[ U + V = \{ (x + y, \lambda \nu) : (x, \nu) \in U, (y, \mu) \in V \} \]

and \( \lambda U = \{ (\lambda x, \nu) : (x, \nu) \in U \} \).
Definition 2.5. A fuzzy linear space $\tilde{X} = X \times (0, 1]$ over the number field $S$ where the addition and scalar multiplication operation on $X$ are defined by $(x, \nu) + (y, \mu) = (x + y, \nu \land \mu)$, $\lambda (x, \nu) = (\lambda x, \nu)$ is a fuzzy normed space if to every $(x, \nu) \in \tilde{X}$ there is associated a non-negative real number, $|| (x, \nu) ||$, called the fuzzy norm of $(x, \nu)$, in such away that

(i) $|| (x, \nu) || = 0$ iff $x = 0$ the zero element of $X$, $\nu \in (0, 1]$, 
(ii) $|| \lambda (x, \nu) || = |\lambda| || (x, \nu) ||$ for all $(x, \nu) \in \tilde{X}$ and all $\lambda \in S$,
(iii) $|| (x, \nu) + (y, \mu) || \leq || (x, \nu) || + || (y, \mu) ||$ for all $(x, \nu), (y, \mu) \in \tilde{X}$,
(iv) $|| (x, \lor (x, v_i)) || = \lor_i || (x, v_i) ||$ for all $v_i \in (0, 1]$.

Definition 2.6. [20] Let $X$ be a non-empty set and $\mathcal{S}(X)$ be the set of all fuzzy sets in $X$. If $f \in \mathcal{S}(X)$ then $f = \{(x, \mu) : x \in X$ and $\mu \in (0, 1]\}$. Clearly $f$ is a bounded function, since $|f(x)| \leq 1$. Let $S$ be the space of real numbers, then $\mathcal{S}(X)$ is a linear space over the field $S$ where the addition and scalar multiplication are defined by

$$f + g = \{(x, \mu) + (y, \eta)\} = \{(x + y, \mu \land \eta) : (x, \mu) \in f$ and $(y, \eta) \in g\}$$

and

$$\lambda f = \{(\lambda x, \mu) : (x, \mu) \in f\}$$

where $\lambda \in S$.

The linear space $\mathcal{S}(X)$ is said to be normed linear space if, for every $f \in \mathcal{S}(X)$, there exists an associated non-negative real number $||f||$ (called the norm of $f$ ) that satisfies

(i) $||f|| = 0$ if and only if $f = 0$. For

$$||f|| = 0$$

$\iff \{(x, \mu) : (x, \mu) \in f\} = 0$

$\iff x = 0, \mu \in (0, 1] \iff f = 0.$

(ii) $||\lambda f|| = |\lambda| ||f||$, $\lambda \in S$. For

$$||\lambda f|| = \{|\lambda| (x, \mu) : (x, \mu) \in f, \lambda \in S\}$$

$$= \{|\lambda| (x, \mu) : (x, \mu) \in f\} = |\lambda| ||f||.$$

(iii) $||f + g|| \leq ||f|| + ||g||$ for all $f, g \in \mathcal{S}(X)$. For

$$||f + g|| = \{|(x, \mu) + (y, \eta) : x, y \in X, \mu, \eta \in (0, 1]\}$$

$$= \{|(x + y, (\mu \land \eta)) : x, y \in X, \mu, \eta \in (0, 1]\}$$

$$= \{|(x, \mu \land \eta) + (y, \mu \land \eta) : (x, \mu) \in f, (y, \eta) \in g\}$$

$$= ||f|| + ||g||.$$
(2-N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(f_1, f_2, t) = 0$.
(2-N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, t) = 1$ if and only if $f_1$ and $f_2$ are linearly dependent.
(2-N3) $N(f_1, f_2, t)$ is invariant under any permutation of $f_1$, $f_2$.
(2-N4) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, \lambda f_2, t) = N(f_1, f_2, \frac{t}{\lambda})$, if $\lambda \neq 0$, $\lambda \in S$.
(2-N5) for all $s, t \in \mathbb{R}$,
$$N(f_1, f_2 + f_3, s + t) \geq \min \{N(f_1, f_2, s), N(f_1, f_3, t)\}.$$ 
(2-N6) $N(f_1, f_2, \cdot ) : (0, \infty) \to [0, 1]$ is continuous.
(2-N7) $\lim_{t \to \infty} N(f_1, f_2, t) = 1$.

Then $(\mathfrak{N}(X), N)$ is a fuzzy 2-normed linear space or $(X, N)$ is a 2-fuzzy 2-normed linear space.

**Remark 2.9.** In a 2-fuzzy 2-normed linear space $(X, N)$, $N(f_1, f_2, \cdot)$ is a non-decreasing function of $t$ for all $f_1, f_2 \in \mathfrak{N}(X)$.

**Theorem 2.10.** [20] Let $(\mathfrak{N}(X), N)$ be a fuzzy 2-normed linear space. Assume that
(2-N8) $N(f_1, f_2, t) > 0$ for all $t > 0$ implies that $f_1$ and $f_2$ are linearly dependent.

Define $\|f_1, f_2\|_\alpha = \inf \{t : N(f_1, f_2, t) \geq \alpha, \alpha \in (0, 1)\}$.
Then $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2-norms on $\mathfrak{N}(X)$. These 2-norms are called $\alpha$-2-norms on $\mathfrak{N}(X)$ corresponding to the 2-fuzzy 2-norm on $X$.

3. On the Mazur-Ulam Problem

In this section, we give a new generalization of the Mazur-Ulam theorem when $X$ is a 2-fuzzy 2-normed linear space or $\mathfrak{N}(X)$ is a fuzzy 2-normed linear space. Hereafter we use the notion of fuzzy 2-normed linear space on $\mathfrak{N}(X)$ instead of 2-fuzzy 2-normed linear space on $X$.

**Lemma 3.1.** For all $f, h \in \mathfrak{N}(X)$, $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}$. Then
$$\|f, h\|_\alpha = \|f, h + \lambda f\|_\alpha.$$ 

**Proof.** The proof of Lemma is clear from [4, Theorem 2.1.6]. □

As an immediate consequence of Lemma 3.1, we have the following.

**Remark 2.3.** For all $f, g, h \in \mathfrak{N}(X)$, $\alpha \in (0, 1)$,
$$\|f - h, f - g\|_\alpha = \|f - h, g - h\|_\alpha.$$ 

**Lemma 3.3.** For $g, h \in \mathfrak{N}(X)$, if $g$ and $h$ are linearly dependent with the same direction, that is, $h = \lambda g$ for some $\lambda > 0$, then
$$\|f, g + h\|_\alpha = \|f, g\|_\alpha + \|f, h\|_\alpha$$
for all $f \in \mathfrak{N}(X), \alpha \in (0, 1)$.

**Proof.** For all $f \in \mathfrak{N}(X)$, $\|f, g + h\|_\alpha = \|f, g + \lambda g\|_\alpha = \|f, (1 + \lambda)g\|_\alpha = (1 + \lambda)\|f, g\|_\alpha + \lambda\|f, g\|_\alpha = \|f, g\|_\alpha + \|f, h\|_\alpha$. □
Definition 3.4. Let $\mathcal{Z}(X)$ and $\mathcal{Z}(Y)$ be fuzzy 2-normed linear spaces and $\Psi : \mathcal{Z}(X) \to \mathcal{Z}(Y)$ a mapping. We call $\Psi$ a 2-isometry if

$$\|f - h, g - h\|_\alpha = \|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_\beta$$

for all $f, g, h \in \mathcal{Z}(X)$ and $\alpha, \beta \in (0, 1)$.

For a map $\Psi$, consider the following condition which is called the Area One Preserving Property (AOPP).

(AOPP) Let $f$, $g$, $h \in \mathcal{Z}(X)$ with $\|f - h, g - h\|_\alpha = 1$.

Then $\|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_\beta = 1$.

Definition 3.5. The elements $f$, $g$ and $h$ are said to be collinear if and only if $g - h = r(f - h)$ for some real number $r$.

Now we define the concept of 2-Lipschitz mapping.

Definition 3.6. We call $\Psi$ a 2-Lipschitz mapping if there is a $\kappa \geq 0$ such that

$$\|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_\beta \leq \kappa\|f - h, g - h\|_\alpha$$

for all $f, g, h \in \mathcal{Z}(X)$ and $\alpha, \beta \in (0, 1)$. The constant $\kappa$ is called the 2-Lipschitz constant.

Lemma 3.7. Assume that if $f$, $g$ and $h$ are collinear, then $\Psi(f), \Psi(g)$ and $\Psi(h)$ are collinear, and that $\Psi$ satisfies (AOPP). Then $\Psi$ preserves the area $k$ for each $k \in \mathbb{N}$.

Proof. Suppose that there exist $f$, $g \in \mathcal{Z}(X)$ with $f \neq g$ such that $\Psi(f) = \Psi(g)$. Since $\dim\mathcal{Z}(X) \geq 2$, there is $h' \in \mathcal{Z}(X)$ such that $g - f$ and $h' - f$ are linearly independent. Since $\|h' - f, g - f\|_\alpha \neq 0$, we can set

$$h = f + \frac{1}{\|h' - f, g - f\|_\alpha}(h' - f).$$

Then we have

$$\|h - f, g - f\|_\alpha = \left\|\frac{1}{\|h' - f, g - f\|_\alpha}(h' - f), g - f\right\|_\alpha = 1.$$

Since $\Psi$ preserves the unit distance, $\|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_\beta = 1$. But it follows from $\Psi(f) = \Psi(g)$ that

$$\|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_\beta = 0,$$

which is a contradiction. Thus $\Psi$ is injective.

Let $f$, $g$ and $h$ be elements of $\mathcal{Z}(X)$ and $k \in \mathbb{N}$ and $\|h - f, g - f\|_\alpha = k$. We put

$$f_i = f + \frac{i}{k}(g - f), \quad i = 0, 1, \ldots, k.$$
Thus
\[ \| h - f_i f_{i+1} - f_i \|_{\alpha} = \left\| h - f, f + \frac{i+1}{k} (g - f) - \left( f + \frac{i}{k} (g - f) \right) \right\|_{\alpha} \]
\[ = \left\| h - f, f + \frac{1}{k} (g - f) \right\|_{\alpha} = \frac{1}{k} \| h - f, g - f \|_{\alpha} = \frac{k}{k} = 1 \]

for all \( i = 0, 1, \ldots, k \). Since \( \Psi \) satisfies (AOPP),
\[ \| \Psi (h) - \Psi (f), \Psi (f_{i+1}) - \Psi (f_i) \|_{\beta} = 1 \]

for all \( i = 0, 1, \ldots, k \). Since \( f_0, f_1 \) and \( f_2 \), are collinear, \( \Psi (f_0), \Psi (f_1) \) and \( \Psi (f_2) \) are also collinear. Thus there is a real number \( r_0 \) such that \( \Psi (f_2) = \Psi (f_1 + \Psi (f_0)) \).

Since
\[ \| \Psi (h) - \Psi (f), \Psi (f_1) - \Psi (f_0) \|_{\beta} = \| \Psi (h) - \Psi (f), \Psi (f_1) - \Psi (f) \|_{\beta} \]
\[ \| (\Psi (h) - \Psi (f)) r_0 (\Psi (f_1) - \Psi (f_0)) \|_{\beta} = r_0 \| \Psi (h) - \Psi (f), \Psi (f_1) - \Psi (f_0) \|_{\beta} , \]

we have \( r_0 = 1 \) or \(-1\). If \( r_0 = -1 \), \( \Psi (f_2) - \Psi (f_1) = -\Psi (f_1) + \Psi (f_0) \), that is, \( \Psi (f_2) = \Psi (f_0) \). Since \( \Psi \) is injective, \( f_2 = f_0 \), which is a contradiction. Thus \( r_0 = 1 \). Then we have \( \Psi (f_2) - \Psi (f_1) = \Psi (f_1) - \Psi (f_0) \). Similarly, one can obtain that \( \Psi (f_{i+1}) - \Psi (f_i) = \Psi (f_i) - \Psi (f_{i-1}) \) for all \( i = 0, 1, \ldots, k - 1 \). Thus
\[ \Psi (g) - \Psi (f) = \Psi (f_k) - \Psi (f_0) \]
\[ = \Psi (f_k) - \Psi (f_{k-1}) + \Psi (f_{k-1}) - \Psi (f_0) + \ldots + \Psi (f_1) - \Psi (f_0) \]
\[ = k (\Psi (f_1) - \Psi (f_0)) . \]

Hence we obtain
\[ \| \Psi (h) - \Psi (f), \Psi (g) - \Psi (f) \|_{\beta} = \| \Psi (h) - \Psi (f), k (\Psi (f_1) - \Psi (f_0)) \|_{\beta} \]
\[ = k \| \Psi (h) - \Psi (f), \Psi (f_1) - \Psi (f_0) \|_{\beta} = k. \]

This completes the proof.

**Theorem 3.8.** Let \( \Psi \) be a 2-Lipschitz mapping with the 2-Lipschitz constant \( \kappa \leq 1 \). Assume that if \( f, g \) and \( h \) are collinear, then \( \Psi (f), \Psi (g) \) and \( \Psi (h) \) are collinear, and that \( \Psi \) satisfies (AOPP). Then \( \Psi \) is a 2-isometry.

**Proof.** From Lemma 3.7, \( \Psi \) preserves distances \( k \) for all \( k \in \mathbb{N} \). For \( f, g, h \in \mathcal{H} (X) \), there are two cases depending on whether \( \| h - f, g - f \|_{\alpha} = 0 \) or not.

In the first case \( \| h - f, g - f \|_{\alpha} = 0 \), \( h - f \) and \( g - f \) are linearly dependent. So \( f, g \) and \( h \) are collinear. Thus \( \Psi (f), \Psi (g) \) and \( \Psi (h) \) are collinear, that is, \( \Psi (h) - \Psi (f) \) and \( \Psi (g) - \Psi (f) \) are linearly dependent. Hence \( \| \Psi (h) - \Psi (f), \Psi (g) - \Psi (f) \|_{\beta} = 0 \).

In the case \( \| h - f, g - f \|_{\alpha} > 0 \), there exists an \( r_0 \in \mathbb{N} \) such that \( r_0 > \| h - f, g - f \|_{\alpha} \). Assume that
\[ \| \Psi (h) - \Psi (f), \Psi (g) - \Psi (f) \|_{\beta} < \| h - f, g - f \|_{\alpha} . \]
We can set
\[ w = f + \frac{n_0}{\|h - f, g - f\|_\alpha} (g - f). \]

Then we get
\[
\|h - f, w - f\|_\alpha = \left\| h - f, f + \frac{n_0}{\|h - f, g - f\|_\alpha} (g - f) - f \right\|_\alpha
\]
\[ = \frac{n_0}{\|h - f, g - f\|_\alpha} \|h - f, g - h\|_\alpha = n_0. \]

Thus,
\[ \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta = n_0. \]

By the definition of \( w \),
\[ w - g = \left( \frac{n_0}{\|h - f, g - f\|_\alpha} - 1 \right) (g - f). \]

Since
\[ \frac{n_0}{\|h - f, g - f\|_\alpha} > 1, \]
\( h - f_1 \) and \( f_1 - f_0 \) have the same direction. From Lemma 3.3,
\[ \|h - f, w - f\|_\alpha = \|h - f, w - g\|_\alpha + \|h - f, g - f\|_\alpha. \]

Thus we have
\[
\|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta \\
\leq \|h - f, w - g\|_\alpha \\
= n_0 - \|h - f, g - f\|_\alpha.
\]

By the assumption,
\[ n_0 = \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta \\
\leq \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta + \|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_\beta \\
< n_0 - \|h - f, g - f\|_\alpha + \|h - f, g - f\|_\alpha = n_0,
\]
which is a contradiction. Hence \( \Psi \) is a 2-isometry. This completes the proof. \( \square \)

**Lemma 3.9.** Let \( f, g \) be elements of \( \mathcal{S}(X) \). Then \( v = \frac{f + g}{2} \) is the unique element of \( \mathcal{S}(X) \) satisfying
\[ \|f - h, f - v\|_\alpha = \|g - v, g - h\|_\alpha = \frac{1}{2} \|f - h, g - h\|_\alpha \]
for some \( h \in \mathcal{S}(X) \) with \( \|f - h, g - h\|_\alpha \neq 0 \) and \( v, f, g \) are collinear.

**Proof.** Let \( \|f - h, g - h\|_\alpha \neq 0 \) and \( v = \frac{f + g}{2} \). Then \( v, f, g \) are 2-collinear. From Lemma 3.1, \( v \) satisfies
\[ \|f - h, f - v\|_\alpha = \|g - v, g - h\|_\alpha = \frac{1}{2} \|f - h, g - h\|_\alpha \]
for all \( h \in \mathcal{S}(X) \) with \( \|f - h, g - h\|_\alpha \neq 0 \).

Now we prove the uniqueness.
Let $u$ be an element of $\mathfrak{Z}(X)$ satisfying the above properties. That is,
\[ \| f - h, f - u \|_\alpha = \| g - u, g - h \|_\alpha = \frac{1}{2} \| f - h, g - h \|_\alpha \]
for some $h \in \mathfrak{Z}(X)$ with $\| f - h, g - h \|_\alpha \neq 0$ and $u, f, g$ are collinear. Since $u, f, g$ are collinear, there exists a real number $t$ such that $u = tf + (1 - t)g$. From Lemma 3.1, we get

\[
\begin{align*}
\frac{1}{2} \| f - h, g - h \|_\alpha &= \| f - h, f - u \|_\alpha \\
&= \| f - h, f - (tf + (1 - t)g) \|_\alpha \\
&= \| 1 - t \| f - h, f - g \|_\alpha \\
&= \| 1 - t \| f - h, g - h \|_\alpha \\
\end{align*}
\]
and

\[
\begin{align*}
\frac{1}{2} \| f - h, g - h \|_\alpha &= \| g - u, g - h \|_\alpha \\
&= \| g - (tf + (1 - t)g), g - h \|_\alpha \\
&= \| -tf + tg, g - h \|_\alpha \\
&= |t| \| f - g, g - h \|_\alpha \\
&= |t| \| f - h, g - h \|_\alpha.
\end{align*}
\]

Since $\| f - h, g - h \|_\alpha \neq 0$, thus we have $\frac{1}{2} = |1 - t| = |t|$. Therefore, we get $t = \frac{1}{2}$ and hence $v = u$. This completes the proof. \(\Box\)

**Theorem 3.10.** Assume that $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear when $f$, $g$ and $h$ are collinear. If $\Psi$ is a 2-isometry, then $\Psi$ is affine.

**Proof.** Let $\Psi$ be a 2-isometry and $\Phi(f) = \Psi(f) - \Psi(0)$. Then $\Phi$ is a 2-isometry and $\Phi(0) = 0$. Thus we may assume that $\Psi(0) = 0$. Hence it suffices to show that $\Psi$ is linear.

Let $f, g \in \mathfrak{Z}(X)$ with $f \neq g$. Since $\dim \mathfrak{Z}(X) > 1$, there exist an element $h \in \mathfrak{Z}(X)$ such that

\[ \| f - h, g - h \|_\alpha \neq 0. \]

Since $\Psi$ is a 2-isometry, we have
\[
\begin{align*}
\left\| \Psi(f) - \Psi(h), \Psi(f) - \Psi\left(\frac{f + g}{2}\right) \right\|_\beta \\
&= \left\| f - h, f - \frac{f + g}{2} \right\|_\alpha \\
&= \left\| f - h, \frac{f - g}{2} \right\|_\alpha \\
&= \frac{1}{2} \left\| f - h, f - g \right\|_\alpha \\
&= \frac{1}{2} \left\| f - h, g - h \right\|_\alpha = \frac{1}{2} \left\| \Psi(f) - \Psi(h), \Psi(g) - \Psi(h) \right\|_\beta
\end{align*}
\]
from Remark 3.2. Similarly, we can obtain
\[
\left\| \Psi(g) - \Psi \left( \frac{f + g}{2} \right), \Psi(g) - \Psi(h) \right\|_\beta = \frac{1}{2} \left\| \Psi(f) - \Psi(h), \Psi(g) - \Psi(h) \right\|_\beta.
\]
Since \( \frac{f + g}{2} \), \( f \) and \( g \) are collinear, \( \Psi \left( \frac{f + g}{2} \right) \), \( \Psi(f) \) and \( \Psi(g) \) are also collinear. By Lemma 3.9 we have
\[
\Psi \left( \frac{f + g}{2} \right) = \frac{\Psi(f) + \Psi(g)}{2}
\]
for all \( f, g \in \mathcal{Y}(X) \), \( \alpha, \beta \in (0, 1) \). Since \( \Psi(0) = 0 \), we can easily show that \( \Psi \) is additive. It follows that \( \Psi \) is \( \mathbb{Q} \)-linear.

Let \( r \in \mathbb{R}^+ \) with \( r \neq 1 \) and \( f \in \mathcal{Y}(X) \). Since \( 0, f \) and \( rf \) are collinear, \( \Psi(0), \Psi(f) \) and \( \Psi(rf) \) are also collinear. Since \( \Psi(0) = 0 \), there exists a real number \( k \) such that \( \Psi(rf) = k \Psi(f) \). Since \( \dim \mathcal{Y}(X) > 1 \), there exist an element \( g \) of \( \mathcal{Y}(X) \) such that \( \| r, g \|_\alpha \neq 0 \). Then we get
\[
r \| r, g \|_\alpha = \| rf - 0, g - 0 \|_\alpha
= \| \Psi(rf) - \Psi(0), \Psi(g) - \Psi(0) \|_\beta
= \| \Psi(rf), \Psi(g) \|_\beta = k \| \Psi(f), \Psi(g) \|_\beta
= \| k \| \| \Psi(f) - \Psi(0), \Psi(g) - \Psi(0) \|_\beta
= \| k \| \| f - 0, g - 0 \|_\alpha = \| k \| \| f, g \|_\alpha.
\]
Since \( \| f, g \|_\alpha \neq 0 \), \( |k| = r \). Then \( \Psi(rf) = r \Psi(f) \) or \( \Psi(rf) = -r \Psi(f) \). Firstly, assume that \( k = -r \), that is, \( \Psi(rf) = -r \Psi(f) \). Then there exist positive rational numbers \( q_1, q_2 \) such that \( 0 < q_1 < r < q_2 \). Since \( \dim \mathcal{Y}(X) > 1 \), there exist an element \( h \in \mathcal{Y}(X) \) such that \( \| rf - q_2f, h - q_2f \|_\alpha \neq 0 \). Then we have
\[
(q_2 + r) \| \Psi(f), \Psi(h) - \Psi(q_2f) \|_\beta
= \| r \Psi(f) + q_2 \Psi(f), \Psi(h) - \Psi(q_2f) \|_\beta
= \| \Psi(rf) - q_2f, h - q_2f \|_\beta
= \| rf - q_2f, h - q_2f \|_\alpha
= \| r - q_2 \| f, h - q_2f \|_\alpha
= \| q_2 - r \| f, h - q_2f \|_\alpha
\leq (q_2 - r) \| f, h - q_2f \|_\alpha
= \| q_1f - q_2f, h - q_2f \|_\alpha
= \| \Psi(q_1f) - \Psi(q_2f), \Psi(h) - \Psi(q_2f) \|_\beta
= \| q_1 \Psi(f) - q_2 \Psi(f), \Psi(h) - \Psi(q_2f) \|_\beta
= \| q_1 - q_2 \| \Psi(f), \Psi(h) - \Psi(q_2f) \|_\beta
= \| q_1 - q_2 \| \Psi(f), \Psi(h) - \Psi(q_2f) \|_\beta.
\]
Since \( \| rf - q_2f, h - q_2f \|_\alpha \neq 0 \),
\[
\| \Psi(rf) - \Psi(q_2f), \Psi(h) - \Psi(q_2f) \|_\beta \neq 0.
\]
Thus we have \( r + q_2 \leq q_2 - q_1 \), which is a contradiction. Hence \( k = r \), that is, \( \Psi(rf) = r\Psi(f) \) for all positive real number \( r \). Thus for every real number \( r \), \( \Psi(rf) = r\Psi(f) \). This completes the proof. \( \square \)

We get the following corollary from Theorem 3.8 and Theorem 3.10.

**Corollary 3.11.** Let \( \Psi \) be a 2-Lipschitz mapping with the 2-Lipschitz constant \( \kappa \leq 1 \). Suppose that \( \Psi(f), \Psi(g) \) and \( \Psi(h) \) are collinear when \( f, g \) and \( h \) are collinear. If \( \Psi \) satisfies (AOPP), then \( \Psi \) is an affine 2-isometry.

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**References**


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