APPROXIMATION THEOREMS FOR FUZZY SET MULTIFUNCTIONS IN VIETORIS TOPOLOGY. PHYSICAL IMPLICATIONS OF REGULARITY

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Abstract. In this paper, we consider continuity properties (especially, regularity, also viewed as an approximation property) for $\mathcal{P}_0(X)$-valued set multifunctions (X being a linear, topological space), in order to obtain Egoroff and Lusin type theorems for set multifunctions in the Vietoris hypertopology. Some mathematical applications are established and several physical implications of the mathematical model of regularity are presented, which allows a classification of the physical models.

1. Introduction

Non-additive measures can be used for modeling problems in non-deterministic environment. In the last years, this area has been widely developed and a wide variety of topics have been investigated. Also, non-additive integrals have very interesting properties from a mathematical point of view, which have been studied and applied to various fields (as information sciences, decision-making problems, monotone expectation, aggregation approach etc.).

The idea of modeling the behavior of phenomena at multiple scales has become an useful tool in pure and applied mathematics. Fractal-based techniques lie at the heart of this area, since fractals are multiscale objects, which often describe such phenomena better than traditional mathematical models. In Kunze *et al.* [23] and Wicks [43], certain hyperspace theories concerning the Hausdorff metric and the Vietoris topology, as a foundation for self-similarity and fractality are developed. In fact, from a mathematical point of view, fractality has important commonalities, both with various hypertopologies (e.g., Hausdorff, Vietoris [2], [3], [6], [23], [42]) and with the regularity of Borel measures.

For many years, topological methods were used in many fields to study the chaotic nature in dynamical systems (Sharma and Nagar [40], Wang *et al.* [42], Goméz-Rueda *et al.* [17], Li [27], Liu *et al.* [28], Ma *et al.* [30], Fu and Xing [14] etc.), which seems to be collective phenomenon emerging out of many segregated components. Most of these systems are collective (set-valued) dynamics of many units of individual systems. It therefore arises the need of a topological treatment of such collective dynamics. Recent studies of dynamical systems, in engineering and

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physical sciences, have revealed that the underlying dynamics is set-valued (collective), and not of a normal, individual kind, as it was usually studied before. We mention interesting approaches of topology in psychology, in the studies of Lewin *et al.* [24] and Brown [8]. We also note different considerations concerning generalized fractals in hyperspaces endowed with the Hausdorff, or, more generally, with the Vietoris hipertopology (Andres and Fišer [2], Andres and Rypka [3], Banakh and Novosad [6], Kunze *et al.* [23]).

Because of its numerous applications in problems of optimization, convex analysis, economy, the study of hypertopologies (for instance, Hausdorff, Vietoris, Wijsman, Fell, Attouch-Wets) has become of a great interest. Important results concerning the Vietoris topology can be found in Beer [7], Apreutesei [4], Hu and Papageorgiou [20] etc. Interesting results involving for instance the Hausdorff distance were obtained by Lorenzo and Maio [9] in melodic similarity, Lu *et al.* [29] - an approach to word image matching etc.

Due to their various applications in economics, games theory, artificial intelligence and many other important fields, many considerations from non-additive measures theory (especially, continuity properties) were extended to the set-valued case (see Guo and Zhang [18] in Kuratowski topology, Gavriluţ [15], [16] and Precupanu and Gavriluţ [37], [38] in Hausdorff topology).

Regularity property is an important continuity property in different topologies, but, at the same time, it can be interpreted as an approximation property. Precisely, in this way, we can approximate no matter how well different "unknown" sets by other sets which we have more information. Usually, from a mathematical perspective, this approximation is done from the left by open, or more restrictive, by compact sets and/or from the right by open sets. In this paper, we shall point out important mathematical and physical applications of regularity.

The classical Lusin's theorem concerning the existence of continuous restrictions of measurable functions is very important and useful for discussing different kinds of approximation of measurable functions defined on special topological spaces and for numerous applications in the study of convergence of sequences of Sugeno and Choquet integrable functions (Li and Yasuda [26]). In the last years, Lusin's result was generalized for non-additive set functions by many authors: Kawabe [22], Li and Yasuda [26], Song and Li [41], Jiang and Suzuki [21], Pap [33] etc. Their proofs are based on one hand, on different versions of Egoroff's theorem and, on the other hand, on various types of regularity of the non-additive measures. An interesting application of the Lusin theorem is due to Li *et al.* [25], in the study of the approximation properties of neural networks, as the learning ability of a neural network is closely related to its approximating capabilities. Recently, Lusin's theorem was considered in the set-valued case by Averna [5] for measurable multifunctions.

In [15] we have introduced and studied different notions of continuity, including regularity, in the Vietoris hypertopology, for $\mathcal{P}_0(X)$-valued set multifunctions, X being a Hausdorff, linear topological space.

The important theoretical and practical problem of obtaining Lusin type theorems in the set-valued case in different hypertopologies has been very recently investigated. Precisely, Precupanu and Gavriluţ [38] generalized these results for regular null-additive monotone set multifunctions in the Hausdorff hypertopology.

In this paper, based on some results established in [15], we continue our investigations concerning regularity in the set-valued setting of Vietoris hypertopology. Namely, we are now able to establish set-valued Egoroff and Lusin type theorems in Vietoris hypertopology and also to point out several mathematical applications of these results. A Lebesgue type theorem for real valued measurable functions with respect to $\mathcal{P}_0(X)-$valued fuzzy set multifunctions in the Vietoris hypertopology is also obtained. In the last two sections, some physical implications of the mathematical model of regularity are analyzed: regularization by sets of functions of $\varepsilon-$approximation type scale, which involves scale relativity type theories, regularization with known sets, which involves the trans-physics type theories and also simultaneous regularization with sets of functions of $\varepsilon-$approximation type scale and with known sets, which involves fractal string type theories.

2. **Terminology, Basic Notions and Results**

For the beginning, we briefly recall some notions and notations involved in the Vietoris topology.

Let (X, τ) be a Hausdorff, linear topological space with the origin 0, T an abstract, nonvoid set and $\mathcal{C}$ a ring of subsets of T. By $\mathcal{P}(T)$ we denote the family of all subsets of T, by $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X and by $\mathcal{V}(0)$ ($\mathcal{U}(0)$, respectively) we denote the system (base - fundamental system, respectively) of neighborhoods of 0.

We recall the following notions and notations (Hu and Papageorgiou [20] - Ch. 1, Precupanu *et al.* [35] - Ch. 1, Precupanu *et al.* [36] - Ch. 8):

$M^- = \{C \in \mathcal{P}_0(X); M \cap C \neq \emptyset\}$, $M^+ = \{C \in \mathcal{P}_0(X); C \subseteq M\}$,

$S_{UV} = \{D^+; D \in \tau\}$ and $S_{LV} = \{D^-; D \in \tau\}$.

Vietoris topology, denoted by $\widehat{\tau}_V$, is the topology on $\mathcal{P}_0(X)$ which has as a subbase the class $S_{UV} \cup S_{LV}$.

Also, $\widehat{\tau}_V = \widehat{\tau}_V^+ \cup \widehat{\tau}_V^-$, where $\widehat{\tau}_V^+$ is *the upper Vietoris topology* and $\widehat{\tau}_V^-$ is *the lower Vietoris topology.*

$\widehat{\tau}_V^+$ has as a subbase the class S_{UV} and $\widehat{\tau}_V^-$ has as a subbase the class S_{LV}.

If $U, V \in \tau$, then the family of subsets $\mathcal{B}_{U,V} = \{M \in \mathcal{P}_0(X); M \subseteq U, M \cap V \neq \emptyset\}$ is a base for the topology $\widehat{\tau}_V$.

The family of subsets $\mathcal{B}_U = \{M \in \mathcal{P}_0(X); M \subseteq U\}$ ($\mathcal{B}_V = \{M \in \mathcal{P}_0(X); M \cap V \neq \emptyset\}$, respectively) is a base for $\widehat{\tau}_V^+$ ($\widehat{\tau}_V^-$, respectively).

Vietoris topology is in fact an example of a hit and miss topology. Like some physical concepts, Vietoris topology, although it is composed of two independent parts of its, topologies $\widehat{\tau}_V^+$ and $\widehat{\tau}_V^-$, it becomes consistent when viewed as $\widehat{\tau}_V^+ \cup \widehat{\tau}_V^-$.

For instance, in physical terms, the non-differentiability of the physical object motion curve involves the simultaneous definition of two differentials (left and right) at any point of the curve. Since we can not favor one of the two differentials, the only solution is to simultaneously consider them through a complex differential. Its application, multiplied by dt, where t is an affine parameter, to the field of space coordinates implies complex speed fields.

Definition 2.1. [15] A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, with $\mu(\emptyset) = \{0\}$ is said to be:

i) *monotone* (or, *fuzzy*) with respect to the inclusion of sets if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \subseteq B$.

ii) a *multisubmeasure* if it is monotone and *subadditive*, i.e., $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every (disjoint) sets $A, B \in \mathcal{C}$.

Unless stated otherwise, all over the paper we assume that (X, τ) is a Hausdorff, linear topological space and $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a monotone set multifunction. By $\mathbb{N}$ we denote the set of all naturals and by $i = \overline{1, n}$, we usually mean $i \in \{1, 2, ...n\}$. By $\mathbb{N}^*$ we mean $\mathbb{N}\backslash\{0\}$.

We now recall the following notions and results concerning several types of continuity in the Vietoris topology, that will be used throughout the paper.

Definition 2.2. [15] μ is said to be:

i) $\widehat{\tau}_V^+-$*decreasing convergent* if for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow A \in \mathcal{C}$, $\mu(A_n) \overset{\widehat{\tau}_V^+}{\to} \mu(A)$, i.e., for every $U \in \tau$, with $\mu(A) \subset U$, there exists $n_0(U) \in \mathbb{N}$ so that $\mu(A_n) \subset U$, for every $n \geq n_0$.

ii) $\widehat{\tau}_V^--$*increasing convergent* if for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$, $\mu(A_n) \overset{\widehat{\tau}_V^-}{\to} \mu(A)$, i.e., for every $V \in \tau$, with $\mu(A) \cap V \neq \emptyset$, there exists $n_0(V) \in \mathbb{N}$ so that $\mu(A_n) \cap V \neq \emptyset$, for every $n \geq n_0$.

iii) $\widehat{\tau}_V^+-$*order continuous* if for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow \emptyset$, $\mu(A_n) \overset{\widehat{\tau}_V^+}{\to} \{0\}$, i.e., for every $U_0 \in \mathcal{V}(0)$, there exists $n_0(U_0) \in \mathbb{N}$ so that $\mu(A_n) \subset U_0$, for every $n \geq n_0$.

Proposition 2.3. [15] *i) If μ is $\widehat{\tau}_V^+-$decreasing convergent, then μ is $\widehat{\tau}_V^+-$order continuous.*

ii) If X is a metrisable linear topological space and μ is a $\widehat{\tau}_V^+-$order continuous multisubmeasure, then μ is $\widehat{\tau}_V^--$increasing convergent and $\widehat{\tau}_V^+-$decreasing convergent.

iii) If $\mathcal{C}$ is finite, then any set multifunction with $\mu(\emptyset) = \{0\}$ is $\widehat{\tau}_V^+-$decreasing convergent, $\widehat{\tau}_V^--$increasing convergent and $\widehat{\tau}_V^+-$order continuous.

Lemma 2.4. [15] *If μ is $\widehat{\tau}_V^--$increasing convergent, $U_0 \in \mathcal{V}(0)$, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$ are arbitrary, fixed and if $\mu(A_n) \subset U_0$, for every $n \in \mathbb{N}$, then $\mu(A) \subset U_0$.*

We now establish the following lemma, that will be used in Section 3:

Lemma 2.5. *Suppose $\mathcal{C}$ is a σ-ring, μ is $\widehat{\tau}_V^--$increasing convergent and $\widehat{\tau}_V^+-$decreasing convergent, $U_0 \in \mathcal{V}(0)$ is arbitrary and, for every $k \in \mathbb{N}^*, (B_n^{(k)})_n \subset \mathcal{C}$. Then:*

i) If for every $k \in \mathbb{N}^*$, $B_n^{(k)} \underset{n\to\infty}{\searrow} \emptyset$, there exists $(n_k)_k$ so that $\mu\left(\bigcup_{k=1}^{\infty} B_{n_k}^{(k)}\right) \subset U_0$.

ii) If, moreover, μ is a multisubmeasure and for every $k \in \mathbb{N}^*$, $B_n^{(k)} \underset{n\to\infty}{\searrow} C_k$, with $\mu(C_k) = \{0\}$, there exists $(n_k)_k$ so that $\mu\left(\overset{\infty}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) \subset U_0$.

Proof. i) Since $U_0 \in \mathcal{V}(0)$, there is $D_0 \in \tau$ so that $0 \in D_0 \subset U_0$ (evidently, $D_0 \in \mathcal{V}(0)$).

Because μ is $\widehat{\tau}_V^+-$decreasing convergent, by Proposition 2.3 i), it is also $\widehat{\tau}_V^+-$order continuous, so, since $B_n^{(1)} \underset{n\to\infty}{\searrow} \emptyset$, there exists $n_1 \in \mathbb{N}$ so that $\mu\left(B_{n_1}^{(1)}\right) \subset D_0(\subset U_0)$. Then, since $B_{n_1}^{(1)} \cup B_n^{(2)} \underset{n\to\infty}{\searrow} B_{n_1}^{(1)}$ and μ is $\widehat{\tau}_V^+-$decreasing convergent, there exists $n_2 \in \mathbb{N}$ so that $n_2 > n_1$ and $\mu\left(B_{n_1}^{(1)} \cup B_{n_2}^{(2)}\right) \subset D_0(\subset U_0)$.

Recurrently, there exists $(n_k)_k$ so that $\mu\left(\overset{s}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) \subset D_0(\subset U_0)$, for every $s \in \mathbb{N}^*$. Because μ is $\widehat{\tau}_V^-$−increasing convergent, by Lemma 2.4, we get that $\mu\left(\overset{\infty}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) \subset U_0$. ii) For $U_0 \in \mathcal{V}(0)$, there exists $W_0 \in \mathcal{V}(0)$ so that $W_0 + W_0 \subset U_0$.

By i) applied for W_0, there exists $(n_k)_k$ so that $\mu\left(\overset{\infty}{\underset{k=1}{\cup}} (B_{n_k}^{(k)} \backslash C_k)\right) \subset W_0$.

Therefore,

$$\mu\left(\left(\overset{\infty}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) \setminus \left(\overset{\infty}{\underset{k=1}{\cup}} C_k\right)\right) \subset \mu\left(\overset{\infty}{\underset{k=1}{\cup}} \left(B_{n_k}^{(k)} \backslash C_k\right)\right) \subset W_0.$$

On the other hand, because μ is a multisubmeasure, then, for every $s \in \mathbb{N}^*$, $\mu\left(\overset{s}{\underset{k=1}{\cup}} C_k\right) = \{0\}$, so, since μ is $\widehat{\tau}_V^-$−increasing convergent, by Lemma 2.4, $\mu\left(\overset{\infty}{\underset{k=1}{\cup}} C_k\right) \subset W_0$. Consequently,

$$\begin{aligned} \mu\left(\overset{\infty}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) &\subset \mu\left(\left(\overset{\infty}{\underset{k=1}{\cup}} B_{n_k}^{(k)}\right) \setminus \left(\overset{\infty}{\underset{k=1}{\cup}} C_k\right)\right) + \mu\left(\overset{\infty}{\underset{k=1}{\cup}} C_k\right) \subset \\ &\subset W_0 + W_0 \subset U_0. \end{aligned}$$

□

3. Convergences for Real Valued Measurable Functions with Respect to $\mathcal{P}_0(X)$-valued Monotone Set Multifunctions in Vietoris Topology

In the following, suppose $\mathcal{A}$ is a σ-algebra of subsets of T. Let be arbitrary $A \in \mathcal{A}$. By $A \cap \mathcal{A}$ we denote $\{B \in \mathcal{A}, B \subset A\}$.

We consider the class $\mathcal{M}$ of all $\mathcal{A}$-measurable real-valued functions on $(T, \mathcal{A}, \mu)$ (i.e., functions $f : T \to \mathbb{R}$ so that there exists a sequence $(f_n)_n$ of simple functions which converges to f on T). Let $f \in \mathcal{M}$ and $\{f_n\} \subset \mathcal{M}$.

Since $\mu(\emptyset) = \{0\}$ and μ is monotone, one can easily verify that it is natural to give only the following notions in the upper Vietoris topology $\widehat{\tau}_V^+$:

Definition 3.1. We say that $\{f_n\}$ *converges to* f *on* A:

i) *μ-almost everywhere* (and denote it by $f_n \overset{a.e.}{\underset{A}{\longrightarrow}} f$) if there exists a subset $B \in A \cap \mathcal{A}$ such that $\mu(B) = \{0\}$ and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$.

ii) in *μ-measure* (and denote it by $f_n \xrightarrow[A]{\mu} f$) if for every $\varepsilon > 0$ and every $U \in \mathcal{V}(0)$, there exists $n_0(U,\varepsilon) \in \mathbb{N}$ so that for every $n \geq n_0, \mu(A_n(\varepsilon)) \subset U$, where $A_n(\varepsilon) = \{t \in A; |f_n(t) - f(t)| \geq \varepsilon\}$.

iii) *μ-almost uniformly* (and denote it by $f_n \xrightarrow[A]{a.u.} f$) if there exists a decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$, such that:

a) for every $U \in \mathcal{V}(0)$, there exists $k_0 = k_0(U) \in \mathbb{N}$ so that $\mu(A_{k_0}) \subset U$ and

b) for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A \backslash A_k$ $\left(f_n \xrightarrow[A\backslash A_k]{u} f\right)$.

Proposition 3.2. *If* $f_n \xrightarrow[A]{a.u.} f$ *and if in Definition 3.1 iii),* $A_k \searrow B$*, then* $\mu(B) = \{0\}$.

Proof. By the hypothesis, for every $U \in \mathcal{U}(0)$, there exists $k_0(U) \in \mathbb{N}$ so that for every $k \geq k_0, \mu(A_k) \subset U$. Since $A_k \searrow B$ and μ is monotone, then $\mu(B) \subset \mu(A_{k_0}) \subset U$, whence $\{0\} \subset \mu(B) \subset \bigcap_{U\in\mathcal{U}(0)} U = \{0\}$, so $\mu(B) = \{0\}$. □

Theorem 3.3. *i) If* $f_n \xrightarrow[A]{a.u.} f$*, then* $f_n \xrightarrow[A]{a.e.} f$.

ii) (Egoroff type) Conversely, if μ *is a* $\widehat{\tau}_V^{-}$*−increasing convergent and* $\widehat{\tau}_V^{+}$*−decreasing convergent multisubmeasure and if* $f_n \xrightarrow[A]{a.e.} f$*, then* $f_n \xrightarrow[A]{a.u.} f$.

Proof. i) If $f_n \xrightarrow[A]{a.u.} f$, by Proposition 3.2, there exists a decreasing sequence $\{C_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that $C_k \searrow C$, $\mu(C) = \{0\}$ and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A\backslash C_k$.

For every $x \in A\backslash C$, there is $k_0 \in \mathbb{N}^*$ so that $x \in A\backslash C_{k_0}$ and, therefore, $f_n(x)$ converges to $f(x)$, whence $f_n \xrightarrow[A]{a.e.} f$.

ii) Let $f \in \mathcal{M}, \{f_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ be such that $f_n \xrightarrow[A]{a.e.} f$. Let also $U \in \mathcal{V}(0)$ be arbitrary.

For every $m, n \in \mathbb{N}$, we define $A_n^{(m)} = \bigcup_{i=n}^{\infty} \{t \in A; |f_i(t) - f(t)| \geq \frac{1}{m}\}$.

We see that $A_n^{(m)} \supset A_{n+1}^{(m)}$, for every $m, n \in \mathbb{N}$ and $\mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} A_n^{(m)}\right) = \{0\}$, so $\mu\left(\bigcap_{n=1}^{\infty} A_n^{(m)}\right) = \{0\}$, for every $m \in \mathbb{N}$.

By Lemma 2.5 ii), there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of naturals such that $\mu\left(\bigcup_{i=1}^{\infty} A_{n_i}^i\right) \subset U$.

For every $k \in \mathbb{N}$, we denote $B_k = \bigcup_{i=k}^{\infty} A_{n_i}^i$. We observe that for every $k \in \mathbb{N}$, $B_k \supset B_{k+1}, U \supset \mu(B_k)$ and $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f on every set $A\backslash B_k$. Consequently, $f_n \xrightarrow[A]{a.u.} f$. □

Corollary 3.4. *If* $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ *is a* $\widehat{\tau}_V^{-}$*−increasing convergent and* $\widehat{\tau}_V^{+}$*−decreasing convergent multisubmeasure, then* $f_n \xrightarrow[A]{a.u.} f$ *iff* $f_n \xrightarrow[A]{a.e.} f$.

Theorem 3.5 (Lebesgue type). *Suppose X is a metrisable, linear topological space and $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ is a multisubmeasure. Then $f_n \xrightarrow[A]{a.e.} f \Rightarrow f_n \xrightarrow[A]{\mu} f$ if and only if μ is $\widehat{\tau}_V^{+}-$decreasing convergent.*

Proof. The set C of points $t \in A$ at which $\{f_n\}$ is pointwise convergent to f can be written as $C = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{i=n}^{\infty} (A \backslash B_i(\frac{1}{m}))$, where $B_i(\frac{1}{m}) = \{t \in A; |f_i(t) - f(t)| \geq \frac{1}{m}\}$, for every $m, i \in \mathbb{N}^*$.

If for every $m, n \in \mathbb{N}^*$, we denote $A_n^{(m)} = \bigcup\limits_{i=n}^{\infty} B_i(\frac{1}{m})$ and $A^{(m)} = \bigcap\limits_{n=1}^{\infty} A_n^{(m)}$, then $A \backslash A^{(m)} = \bigcup\limits_{n=1}^{\infty} (A \backslash A_n^{(m)}) = \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{i=n}^{\infty} (A \backslash B_i(\frac{1}{m}))$ and $C = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} (A \backslash A_n^{(m)}) = \bigcap\limits_{m=1}^{\infty} (A \backslash A^{(m)})$.

We observe that for every fixed $m \in \mathbb{N}^*$, $A_n^{(m)} \underset{n\to\infty}{\searrow} A^{(m)}$ and so, $A \backslash A_n^{(m)} \underset{n\to\infty}{\nearrow} A \backslash A^{(m)}$.

If there exists a set $B \in A \cap \mathcal{A}$ and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$, then for every $m \in \mathbb{N}^*$,

$$A \backslash B \subset C \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \left(A \backslash B_i \left(\frac{1}{m} \right) \right) = A \backslash A^{(m)} \subset A.$$

We also observe that $B_n(\frac{1}{m}) \subset A_n^{(m)}$, for every $m, n \in \mathbb{N}^*$.

Necessity. We prove that μ is $\widehat{\tau}_V^{+}-$decreasing convergent. By Proposition 2.3, this is equivalent to prove that μ is $\widehat{\tau}_V^{+}-$order continuous. So, we consider arbitrary $(A_n)_{n\in\mathbb{N}} \subset A \cap \mathcal{A}$, with $A_n \searrow \emptyset$ and arbitrary $U \in \mathcal{V}(0)$.

For every $n \in \mathbb{N}$, we define $f_n(t) = \begin{cases} 0, \text{ if } t \in A_n \\ 1, \text{ if } t \in A \backslash A_n. \end{cases}$

Evidently, $\{f_n\} \subset \mathcal{M}$ and $\{f_n\}$ is pointwise convergent to 1 on A, so $f_n \xrightarrow[A]{a.e.} 1$.

By the hypothesis, $f_n \xrightarrow[A]{\mu} f$, whence, for $\varepsilon = \frac{1}{2}$, there exists $n_0(U) \in \mathbb{N}^*$ so that, for every $n \geq n_0$, $\mu(A_n) = \mu(\{t \in A; |f_n(t) - 1| \geq \frac{1}{2}\}) \subset U$, that is, μ is $\widehat{\tau}_V^{+}-$order continuous, and so, equivalently, it is $\widehat{\tau}_V^{+}-$decreasing convergent.

Sufficiency. Suppose μ is $\widehat{\tau}_V^{+}-$decreasing convergent and $f_n \xrightarrow[A]{a.e.} f$.

Consequently, there exists $B \in A \cap \mathcal{A}$ such $\mu(B) = \{0\}$ and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$. Therefore, for every $m \in \mathbb{N}^*$, $A^{(m)} \subset B$ and so, $\mu(A^{(m)}) = \{0\}$.

Let be arbitrary $U_0 \in \mathcal{V}(0)$. There is $D_0 \in \tau$ so that $0 \in D_0 \subset U_0$.

Since for every fixed $m \in \mathbb{N}^*$, $A_n^{(m)} \underset{n\to\infty}{\searrow} A^{(m)}$ and μ is $\widehat{\tau}_V^{+}-$decreasing convergent, then for every $m \in \mathbb{N}^*$, there exists $n_m(D_0) \in \mathbb{N}^*$ so that, for every $n \geq n_m$, $\mu(A_n^{(m)}) \subset D_0 \subset U_0$.

If $\varepsilon > 0$ is arbitrary, there exists $m_0(\varepsilon) \in \mathbb{N}^*$ so that $\frac{1}{m_0} < \varepsilon$, whence $B_n(\varepsilon) \subset B_n(\frac{1}{m_0})$. On the other hand, for every $m, n \in \mathbb{N}^*, B_n(\frac{1}{m}) \subset A_n^{(m)}$. Consequently, there exists $n_0(m_0, D_0) = n_0(U_0, \varepsilon) \in \mathbb{N}^*$ so that, for every $n \geq n_0$,

$$\mu\left(B_n(\varepsilon)\right) \subset \mu\left(B_n\left(\frac{1}{m_0}\right)\right) \subset \mu\left(A_n^{(m_0)}\right) \subset D_0 \subset U_0,$$

whence $f_n \xrightarrow[A]{\mu} f$. □

4. Set-valued Egoroff and Lusin Type Theorems and Applications in the Vietoris Topology

In what follows, without any other special assumptions, we suppose that (X, τ) is a Hausdorff, linear topological space with the origin 0, T is a locally compact Hausdorff space and $\mathcal{C}$ is a ring of subsets of T.

For the consistency of the following notions of regularity, one may suppose that $\mathcal{C}$ is, for instance, $\mathcal{B}_0$ ($\mathcal{B}'_0$, respectively) - the Baire δ-ring (σ-ring, respectively) generated by compact sets, which are G_δ (i.e., countable intersections of open sets) or $\mathcal{C}$ is $\mathcal{B}$ ($\mathcal{B}'$, respectively) - the Borel δ-ring (σ-ring, respectively) generated by the compact sets of T.

By $\mathcal{K}$ we denote the family of all compact subsets of T and by $\mathcal{D}$, the family of all open subsets of T.

We now recall the following notions and results, that will be used throughout this section:

Remark 4.1. i) Obviously, $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}'$ and $\mathcal{B}_0 \subset \mathcal{B}'_0$.

Also, if T is metrisable or if it has a countable base, then any compact set $K \subset T$ is G_δ, so, in this case, $\mathcal{B}_0 = \mathcal{B}$ (Dinculeanu [10], Ch. III) and, consequently, $\mathcal{B}'_0 = \mathcal{B}'$.

ii) [15] For every $A \in \mathcal{C}$, there always exists $D \in \mathcal{D} \cap \mathcal{C}$ so that $A \subset D$.

iii) [15] If $\mathcal{C}$ is $\mathcal{B}$ or $\mathcal{B}'$, then for every $A \in \mathcal{C}$, there always exist $K \in \mathcal{K} \cap \mathcal{C}$ and $D \in \mathcal{D} \cap \mathcal{C}$ so that $K \subset A \subset D$.

iv) If T is a compact, G_δ, Hausdorff topological space or a metrisable, compact space, then $\mathcal{B}'_0$ is a $\sigma-$algebra.

Definition 4.2. [15] I) A set $A \in \mathcal{C}$ is said to be (with respect to μ):

i) $R_l'^{+}$-*regular* if for every $V_0 \in \mathcal{V}(0)$, there is $K = K_{V_0,A} \in \mathcal{K} \cap \mathcal{C}$ so that $K \subset A$ and $\mu(A \backslash K) \subset V_0$.

ii) R'^{+}-*regular* if for every $V_0 \in \mathcal{V}(0)$, there are $K = K_{V_0,A} \in \mathcal{K} \cap \mathcal{C}$ and $D = D_{V_0,A} \in \mathcal{D} \cap \mathcal{C}$ such that $K \subset A \subset D$ and $\mu(D \backslash K) \subset V_0$.

II) μ is said to be $R_l'^{+}$-*regular* (R'^{+}-*regular*, respectively) if the same is every set $A \in \mathcal{C}$.

Example 4.3. Any set $K \in \mathcal{K} \cap \mathcal{C}$ is $R_l'^{+}$-regular.

In fact, the notions of regularity (given in the above definition in its abstract form) becomes consistent if it is considered for the case when $\mathcal{C}$ is $\mathcal{B}_0, \mathcal{B}, \mathcal{B}'$ or $\mathcal{B}'_0$. The local compacity of T is thus strongly needed.

Theorem 4.4 ([15] Alexandroff type theorem)**.** *Suppose X is a metrisable, linear topological space. If μ is a $R_l'^{+}$-regular multisubmeasure, then μ is $\widehat{\tau}_V^{+}-$order continuous.*

We now prove a converse of Theorem 4.3 when $\mathcal{C}$ is the σ-ring $\mathcal{B}'_0$:

Theorem 4.5. *If T is a metrisable, locally compact space, X is a metrisable, linear topological space and if $\mu : \mathcal{B}'_0 \to \mathcal{P}_0(X)$ is a $\widehat{\tau}_V^{+}-$ order continuous multisubmeasure, then μ is R'^{+}-regular on $\mathcal{B}'_0$.*

Proof. Denote $\mathcal{A} = \{A \in \mathcal{B}_0'; A \text{ is } R'^{+}\text{-regular with respect to } \mu\}$.

We shall prove that $\mathcal{A} = \mathcal{B}_0'$. By the proof of Theorem 4.13 from [15], $\mathcal{A}$ is nonvoid, and also, by Lemma 4.12 [15], $\mathcal{A}$ is a δ-ring which contains all G_δ, compact sets of T.

We now prove that $\mathcal{A}$ is a σ-ring. For this, it is sufficient to establish that for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A_n \nearrow A$, we have $A \in \mathcal{A}$.

Since X is a metrisable linear topological space, by Precupanu [39] there exists a fundamental system of neighborhoods of the origin, $\mathcal{U}(0) = \{V_n\}_{n\in\mathbb{N}}$ so that, for every arbitrary fixed $n \in \mathbb{N}$,

$$V_{n+1} + V_{n+2} + ... + V_{n+p} \subset V_n, \text{ for every } p \in \mathbb{N}^*.$$

Consider arbitrary $V \in \mathcal{V}(0)$. There exists $\widetilde{V} \in \mathcal{U}(0)$ so that $\widetilde{V} \subset V$. Obviously, there is $i_0 \in \mathbb{N}$ so that $\widetilde{V} = V_{i_0}$ and for every $p \in \mathbb{N}^*$,

$$V_{i_0+1} + V_{i_0+2} + ... + V_{i_0+p} \subset \widetilde{V} \subset V.$$

Since for every $r \in \mathbb{N}$, A_r is R'^{+}-regular, there exist $K_r \in \mathcal{K}\cap\mathcal{B}_0'$ and $D_r \in \mathcal{D}\cap\mathcal{B}_0'$ so that $K_r \subset A_r \subset D_r$ and $\mu(D_r\backslash K_r) \subset V_{i_0+r+1}$.

Denote $C_r = \overset{r}{\underset{i=1}{\cup}} K_i, E_r = \overset{r}{\underset{i=1}{\cup}} D_i$ and $D = \overset{\infty}{\underset{i=1}{\cup}} E_r$. Evidently, for every $r \in \mathbb{N}^*, C_r \in \mathcal{K} \cap \mathcal{B}_0'$ and $E_r, D \in \mathcal{D} \cap \mathcal{B}_0'$.

Since $E_r \nearrow D$ and μ is $\widehat{\tau}_V^{+}-$order continuous, there exists $r_0 \in \mathbb{N}^*$ so that $\mu(D\backslash E_{r_0}) \subset V_{i_0+1}$.

We observe that $C_{r_0} \subset E_{r_0} \subset D, C_{r_0} \subset A \subset D$ and $E_{r_0}\backslash C_{r_0} = (\overset{r_0}{\underset{i=1}{\cup}} D_i)\backslash(\overset{r_0}{\underset{i=1}{\cup}} K_i) \subset \overset{r_0}{\underset{i=1}{\cup}}(D_i\backslash K_i)$, so

$$\mu(E_{r_0}\backslash C_{r_0}) \subset \mu\,(D_1\backslash K_1) + ... + \mu\,(D_{r_0}\backslash K_{r_0}) \subset V_{i_0+2} + ... + V_{i_0+r_0+1}.$$

Then

$$\begin{aligned} \mu(D\backslash C_{r_0}) &= \mu((D\backslash E_{r_0}) \cup (E_{r_0}\backslash C_{r_0})) \subset \\ &\subset V_{i_0+1} + V_{i_0+2} + ... + V_{i_0+r_0+1} \subset \widetilde{V} \subset V, \end{aligned}$$

so, A is R'^{+}-regular.

Consequently, $\mathcal{A} \subset \mathcal{B}_0'$ is a σ-ring which contains all G_δ, compact sets of T, so $\mathcal{A} = \mathcal{B}_0'$. This means μ is R'^{+}-regular on $\mathcal{B}_0'$. □

We are now able to establish our set-valued variants of Egoroff and Lusin type theorems for set multifunctions in the Vietoris topology:

Theorem 4.6 (Egoroff type theorem)**.** *Suppose $\mathcal{C}$ is a σ-algebra of subsets of a locally compact Hausdorff space T and $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a $\widehat{\tau}_V^{+}-$decreasing convergent, $\widehat{\tau}_V^{-}-$increasing convergent and $R_l'^{+}$-regular multisubmeasure. If $(f_n) \subset \mathcal{M}, f \in \mathcal{M}$ are so that (f_n) converges to f μ-almost everywhere on $A \in \mathcal{C}$, then for every $U \in \mathcal{V}(0)$, there exists $K = K_U \in \mathcal{K} \cap \mathcal{C}$ so that $\mu(A\backslash K) \subset U$ and (f_n) uniformly converges to f on K.*

Proof. Generally, if $\mu(M) = \{0\}$ and $\mu(N) \subset U$ (where $M, N \in \mathcal{C}$ are disjoint), then $\mu(M \cup N) \subset U$, so, we can reduce our considerations to the case when (f_n) converges to f everywhere on T.

Consider arbitrary $U \in \mathcal{V}(0)$. There exists $W \in \mathcal{V}(0)$ so that $W + W \subset U$.

If $\widetilde{A}$ denotes the set of points $t \in T$ so that $f_n(t) \to f(t)$, we observe that $\widetilde{A} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n^{(k)}$, where for every $k \in \mathbb{N}^*$, $A_n^{(k)} = \bigcap_{j=n}^{\infty} \{t \in T; |f_j(t) - f(t)| < \frac{1}{k}\}$.

Since (f_n) converges to f everywhere on T, then for every $k \in \mathbb{N}^*$, $A_n^{(k)} \underset{n\to\infty}{\nearrow} T$, whence $T \backslash A_n^{(k)} \underset{n\to\infty}{\searrow} \emptyset$.

By Lemma 2.5 i), for $W \in \mathcal{V}(0)$, there exists $(n_k)_k$ so that $\mu\left(\bigcup_{k=1}^{\infty}\left(T \backslash A_{n_k}^{(k)}\right)\right) \subset W$.

If we denote $B = \bigcap_{k=1}^{\infty} A_{n_k}^{(k)}$, then $\mu(T\backslash B) = \mu\left(\bigcup_{k=1}^{\infty}\left(T \backslash A_{n_k}^{(k)}\right)\right) \subset W$. We also observe that (f_n) uniformly converges to f on B.

On the other hand, since μ is $R_l'^{+}$-regular, for B there exists $K \in \mathcal{K} \cap \mathcal{C}$ so that $K \subset B$ and $\mu(B\backslash K) \subset W$.

Obviously, (f_n) uniformly converges to f on K and, because μ is a multisubmeasure, we have

$$\mu(T\backslash K) \subset \mu(T\backslash B) + \mu(B\backslash K) \subset W + W \subset U.$$

□

By Theorem 4.3, Proposition 2.3 ii), Remark 4.1 iv) and Theorem 4.5, we get:

Corollary 4.7 (Egoroff type theorem)**.** *Suppose T is a metrisable, compact space, X is a metrisable, linear topological space and $\mu : \mathcal{B}_0' \to \mathcal{P}_0(X)$ is a $R_l'^{+}$-regular multisubmeasure. If $(f_n) \subset \mathcal{M}, f \in \mathcal{M}$ are so that (f_n) converges to f μ-almost everywhere on T, then for every $U \in \mathcal{V}(0)$, there exists $K = K_U \in \mathcal{K} \cap \mathcal{B}_0'$ so that $\mu(T\backslash K) \subset U$ and (f_n) uniformly converges to f on K.*

We now state our two main results:

Theorem 4.8 (Lusin type theorem)**.** *Suppose $\mathcal{C}$ is a σ-algebra of subsets of a normal, locally compact Hausdorff space T, X is a metrisable, linear topological space and $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a $\widehat{\tau}_V^{+}-$decreasing convergent, $\widehat{\tau}_V^{-}-$increasing convergent and $R_l'^{+}$-regular multisubmeasure. If $f \in \mathcal{M}$, then for every $U \in \mathcal{V}(0)$, there exists $K = K_U \in \mathcal{K} \cap \mathcal{C}$ so that $\mu(T\backslash K) \subset U$ and f is continuous on K.*

Proof. Let be arbitrary $U \in \mathcal{V}(0)$. We prove the theorem stepwise:

Step 1: *Suppose f is a simple function*, i.e., for every $t \in T, f(t) = \sum_{i=1}^{p} a_i \aleph_{A_i}$, where $(A_i)_{i=\overline{1,p}}$ is a partition of T.

For $U \in \mathcal{V}(0)$, there exists $V_p \in \mathcal{V}(0)$ so that $\underbrace{V_p + V_p + ... + V_p}_{p} \subset U$.

Since μ is $R_l'^{+}$-regular, then for every $i = \overline{1,p}$, there exists $K_i \in \mathcal{K} \cap \mathcal{C}$ so that $K_i \subset A_i$ and $\mu(A_i \backslash K_i) \subset V_p$. If we denote $K = \bigcup_{i=1}^{p} K_i$, then $K \in \mathcal{K} \cap \mathcal{C}$ and, since μ is a multisubmeasure,

$$\mu(T\backslash K) \subset \mu\left(\bigcup_{i=1}^{p}(A_i \backslash K_i)\right) \subset \underbrace{V_p + V_p + ... + V_p}_{p} \subset U.$$

Moreover, since T is normal, then f is continuous on K.

Step 2: *Suppose $f \in \mathcal{M}$ is arbitrary.* Then f is the limit of an increasing sequence (f_m) of simple functions.

For $U \in \mathcal{V}(0)$, there exists $W \in \mathcal{V}(0)$ so that $W + W \subset U$. For W there exists $W' \in \mathcal{U}(0)$ so that $W' \subset W$. On the other hand, since X is a metrisable, linear topological space, by Precupanu [39] there exists a countable fundamental system of neighborhoods of the origin, $\mathcal{U}(0) = \{V_n\}_{n\in\mathbb{N}}$ so that, for every arbitrary fixed $n \in \mathbb{N}$,

$$V_{n+1} + V_{n+2} + ... + V_{n+p} \subset V_n, \text{ for every } p \in \mathbb{N}^*. \quad (*)$$

Evidently, there exists $n_0 \in \mathbb{N}$ so that $W' = V_{n_0}$, so, $V_{n_0+1}+V_{n_0+2}+...+V_{n_0+p} \subset V_{n_0} = W' \subset W$, for every $p \in \mathbb{N}^*$.

By Step 1, there exists a sequence $(K_m) \subset \mathcal{K} \cap \mathcal{C}$ so that, for every $m \in \mathbb{N}^*, \mu(T\backslash K_m) \subset V_{n_0+m}$ and f_m is continuous on K_m.

Consequently, since μ is a multisubmeasure, for every $s \in \mathbb{N}^*$,

$$\begin{aligned} \mu\left(\overset{s}{\underset{m=1}{\cup}}(T\backslash K_m)\right) &\subset \mu(T\backslash K_1) + \mu(T\backslash K_2) + ... + \mu(T\backslash K_s) \subset \\ &\subset V_{n_0+1} + V_{n_0+2} + ... + V_{n_0+s} \subset V_{n_0} = W. \end{aligned}$$

By Lemma 2.4, we get that $\mu\left(\overset{\infty}{\underset{m=1}{\cup}}(T\backslash K_m)\right) \subset W$, so, if we denote $\widetilde{K} = \overset{\infty}{\underset{m=1}{\cap}} K_m$, then

$$\mu(T\backslash\widetilde{K}) = \mu\left(\overset{\infty}{\underset{m=1}{\cup}}(T\backslash K_m)\right) \subset W.$$

On the other hand, by Theorem 4.5, there exists $K_0 \in \mathcal{K}\cap\mathcal{C}$ so that $\mu(T\backslash K_0) \subset W$ and $f_m \underset{K_0}{\overset{u}{\rightarrow}} f$. Denote $K = \widetilde{K} \cap K_0 \in \mathcal{K}\cap\mathcal{C}$. Then $f_m \underset{K}{\overset{u}{\rightarrow}} f$ and

$$\mu(T\backslash K) \subset \mu(T\backslash K_0) + \mu(T\backslash\widetilde{K}) \subset W + W \subset U,$$

so, the proof is finished. □

Remark 4.9. In some particular cases (for instance, when $\mathcal{C}$ in $\mathcal{B}_0'$) and μ is a finitely additive (or at least subadditive) set (multi)functions regularity is equivalent to order continuity which in turn is equivalent to decreasing convergence. Although, in the above theorem, the assumption of regularity is compulsory since $\mathcal{C}$ is an arbitrary $\sigma-$algebra and so, the existence of the compact sets $K_i \in \mathcal{K}\cap\mathcal{C}, i = \overline{1,p}$, cannot be otherwise assured.

By Remark 4.1 iv) and Theorem 4.7, we get:

Corollary 4.10 (Lusin type theorem)**.** *Suppose T is a metrisable, compact space, X is a metrisable, linear topological space and $\mu : \mathcal{B}_0' \to \mathcal{P}_0(X)$ is a $R_l'^+$-regular multisubmeasure. If $f \in \mathcal{M}$, then for every $U \in \mathcal{V}(0)$, there exists $K = K_U \in \mathcal{K}\cap\mathcal{B}_0'$ so that $\mu(T\backslash K) \subset U$ and f is continuous on K.*

We now present several applications of the set-valued Lusin theorems in the Vietoris topology. In what follows, we suppose that the conditions of Corollary 4.9 are fulfilled.

Corollary 4.11. *If $f \in \mathcal{M}$, there exists a sequence $(f_n)_n$ of continuous functions on T so that $f_n \underset{T}{\overset{\mu}{\rightarrow}} f$. Moreover, if $|f| \leq M$, then $|f_n| \leq M$, for every $n \in \mathbb{N}$.*

Proof. Evidently, $\mathcal{U}(0) = \{V_n\}_{n\geq 1}$, where $\bigcap_{n=1}^{\infty} V_n = \{0\}$ and $V_i \supset V_{i+1}$, for every $i \in \mathbb{N}^*$.

For every $U \in \mathcal{V}(0)$, there exists $W \in \mathcal{U}(0)$, with $W = V_{i_0} \subset U$.

By Corollary 4.9, for every $n \geq 1$, there is a compact set $K_n \subset T$ so that $\mu(T\backslash K_n) \subset V_n$ and f is continuous on K_n. By Tietze's extension theorem, for every $n \geq 1$, there is $f_n : T \to \mathbb{R}$ so that f_n is continuous on T, $f_n(t) = f(t)$, for every $t \in K_n$ and if $|f| \leq M$, then $|f_n| \leq M$.

Consequently, for every $\varepsilon > 0$ and every $U \in \mathcal{V}(0)$, there exists $i_0 = i_0(U)(= i_0(U, \varepsilon)) \in \mathbb{N}$ so that for every $n \geq i_0$,

$$\mu(\{t \in T; |f_n(t) - f(t)| \geq \varepsilon\}) \subset \mu(T\backslash K_n) \subset V_n \subset V_{i_0} = W \subset U,$$

whence $f_n \underset{T}{\overset{\mu}{\to}} f$. □

Corollary 4.12. *If $f \in \mathcal{M}$, there exists a sequence $(P_n)_n$ of polynomials on $[a, b]$ so that $P_n \underset{[a,b]}{\overset{\mu}{\to}} f$. Moreover, if $|f| \leq M$, then $|P_n| \leq M + 1$, for every $n \in \mathbb{N}$.*

Proof. By Corollary 4.10 on the reduced space $([a, b], [a, b] \cap \mathcal{B}_0', \mu)$, there exists a sequence $(f_n)_n$ of continuous functions on $[a, b]$, so that $f_n \underset{[a,b]}{\overset{\mu}{\to}} f$.

Consequently, there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ so that, for every $k \geq 1, \mu(\{t \in T; |f_{n_k}(t) - f(t)| \geq \frac{1}{2k}\}) \subset V_k$.

Applying Weierstrass' theorem for every $f_{n_k}, k \geq 1$, on $[a, b]$, there exists a polynomial P_k on $[a, b]$ so that $|f_{n_k}(t) - P_k(t)| < \frac{1}{2k}$, whence $\mu(\{t \in T; |f_{n_k}(t) - P_k(t)| \geq \frac{1}{2k}\}) = \mu(\emptyset) = \{0\}$.

Therefore, for every $k \geq 1$,

$$\begin{aligned} \mu(\{t &\in T; |f(t) - P_k(t)| \geq \tfrac{1}{k}\}) \subset \mu(\{t \in T; |f_{n_k}(t) - f(t)| \geq \tfrac{1}{2k}\}) + \\ +\mu(\{t &\in T; |f_{n_k}(t) - P_k(t)| \geq \tfrac{1}{2k}\}) = \\ &= \mu(\{t \in T; |f_{n_k}(t) - f(t)| \geq \tfrac{1}{2k}\}) \subset V_k. \end{aligned}$$

We shall prove that $P_n \underset{[a,b]}{\overset{\mu}{\to}} f$. Indeed, for every $U \in \mathcal{V}(0)$, there exists $W \in \mathcal{U}(0)$ so that $W \subset U$. Evidently, $W = V_{i_0'}$, where $i_0' = i_0'(U)$ and for every $i \geq i_0'$,

$$\mu\left(\{t \in T; |f(t) - P_i(t)| \geq \frac{1}{i}\}\right) \subset V_i \subset V_{i_0'} = W \subset U.$$

On the other hand, for every $\varepsilon > 0$, there exists $i_0'' = i_0''(\varepsilon)$ so that $\frac{1}{i} < \varepsilon$, for every $i \geq i_0''$.

Let $i_0 = i_0(U, \varepsilon) = \max\{i_0', i\}$. Then, for every $i \geq i_0$,

$$\mu(\{t \in T; |f(t) - P_i(t)| \geq \varepsilon\}) \subset \mu(\{t \in T; |f(t) - P_i(t)| \geq \frac{1}{i}\}) \subset U,$$

which says that $P_n \underset{[a,b]}{\overset{\mu}{\to}} f$.

Moreover, if $|f| \leq M$, then according to Corollary 4.10, $|f_{n_k}| \leq M$, for every $k \geq 1$.

Since for every $t \in [a,b]$ and every $k \geq 1$, we have $|P_k(t) - f_{n_k}(t)| < \frac{1}{2k}$, then $|P_k(t)| \leq M + 1$. □

5. Regularization by Sets of Functions of ε-approximation Type Scale

Let us consider a fractal function $f(x)$, with $x \in [a,b]$, for instance, one of the trajectory's equation. We now consider the sequence of the values of the variable x:

$$x_a = x_0, x_1 = x_0 + \varepsilon, ..., x_k = x_0 + k\varepsilon, ..., x_n = x_0 + n\varepsilon = x_b. \tag{1}$$

We shall denote by $f(x,\varepsilon)$, the fractured (broken) line connecting the points

$$f(x_0), ..., f(x_k), ..., f(x_n).$$

The broken line will be considered as an approximation which is different from the one used before. We shall say that $f(x,\varepsilon)$ is an $\varepsilon-$*approximation scale*.

Now let us consider the $\overline{\varepsilon}-$approximation scale $f(x,\overline{\varepsilon})$ of the same function. Since $f(x)$ is similar almost everywhere, if ε and $\overline{\varepsilon}$ are small enough, then the two approximations $f(x,\varepsilon)$ and $f(x,\overline{\varepsilon})$ must lead to the same results when we study a fractal phenomenon by approximation. If we compare the two cases, then to an infinitesimal increase $d\varepsilon$ of ε, it corresponds an increase $d\overline{\varepsilon}$ of $\overline{\varepsilon}$, if the scale is dilated. But in this case we have $\frac{d\varepsilon}{\varepsilon} = \frac{d\overline{\varepsilon}}{\overline{\varepsilon}}$, that is,

$$\frac{d\varepsilon}{\varepsilon} = d\rho \tag{2}$$

is the ratio of the scale $\varepsilon + d\varepsilon$ and $d\varepsilon$ must be preserved. We can then consider the infinitesimal transformation of the scale as

$$\varepsilon' = \varepsilon + d\varepsilon = \varepsilon + \varepsilon d\rho. \tag{3}$$

By such transformation, in the case of function $f(x,\varepsilon)$, we have:

$$f(x,\varepsilon') = f(x, \varepsilon + \varepsilon d\rho), \tag{4}$$

respectively, if we stop after the first approximation,

$$f(x,\varepsilon') = f(x,\varepsilon) + \frac{\partial f}{\partial \varepsilon}(\varepsilon' - \varepsilon), \tag{5}$$

that is,

$$f(x,\varepsilon') = f(x,\varepsilon) + \frac{\partial f}{\partial \varepsilon}\varepsilon d\rho. \tag{6}$$

Let us also observe that, for arbitrary, but fixed ε_0,

$$\frac{\partial \ln \frac{\varepsilon}{\varepsilon_0}}{\partial \varepsilon} = \frac{\partial(\ln \varepsilon - \ln \varepsilon_0)}{\partial \varepsilon} = \frac{1}{\varepsilon}, \tag{7}$$

so, (5) becomes

$$f(x,\varepsilon') = f(x,\varepsilon) + \frac{\partial f(x,\varepsilon)}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} d\rho. \tag{8}$$

Finally, we have

$$f(x,\varepsilon') = \left(1 + \frac{\partial}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} d\rho\right) f(x,\varepsilon). \tag{9}$$

The operator

$$\widetilde{D} = \frac{\partial}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} \tag{10}$$

is called the *dilation operator*.

The relation (10) shows that the intrinsic variable of the resolution is not ε, but $\ln \frac{\varepsilon}{\varepsilon_0}$.

6. Physical Implications

Simultaneous invariance with respect to both space-time coordinates and the resolution scale induces general Scale Relativity type Theory [11], [12]. These theories are more general than Einstein's general relativity theory, being invariant with respect to the generalized Poincaré group (standard Poincaré group and dilatation group) [19], [11], [12], [34].

Basically, we discuss various physical theories built on manifolds of space-time fractal type. They all turn out to be reducible to one of the following classes:

i) Scale Relativity Theory [31], [32] and its possible extensions [1]. It is considered that microparticles' motion takes place on non-differentiable continuous curves. In such context, regularization works using sets of functions of $\varepsilon-$approximation type scale.

ii) Transition in which to each point of the motion trajectory, a transfinite set is assigned (in particular, a Cantor type set - see the El Naschie [13] $\varepsilon^{(\infty)}$ model of space-time), in order to mimic the continuous (the trans-physics). In such context, the regularization of "vague" sets by known sets works.

iii) Fractal string theories simultaneously containing relativity and trans-physics [19], [34].

The reduction of the complex dimensions to their real part is equivalent to Scale Relativity type Theory, while the reduction to the imaginary part of their complex dimensions generates the trans-physics. In such context, the simultaneous regularization by sets of functions of $\varepsilon-$approximation type scale and also by "known" sets works. The "reduction" of the complex dimensions to their real part requires the regularization by sets of functions of $\varepsilon-$approximation type scale, while the "reduction" to their imaginary part requires regularization with "known" sets.

7. Conclusions

In this paper, some mathematical applications of regularity in Vietoris topology are established and several physical implications of the mathematical model of regularity are presented, which allows a classification of the physical models.

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