COUPLED COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR SINGLE-VALUED AND FUZZY MAPPINGS

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Abstract. In this paper, we study the existence of coupled coincidence and coupled common fixed points for single-valued and fuzzy mappings under a contractive condition in metric space. Presented theorems extend and improve the main results of Abbas and Ćirić *et al.* [M. Abbas, L. Ćirić, *et al.*, Coupled coincidence and common fixed point theorems for hybrid pair of mappings, Fixed Point Theory Appl. (4) (2012) doi:10.1186/1687-1812-2012-4].

1. Introduction

In 2006, Bhaskar and Lakshmikantham [10] initiated the study of coupled fixed point in partially ordered metric spaces. Bhaskar and Lakshmikantham introduced the notion of a mixed monotone mapping and proved some coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. Lakshmikantham and Ćirić [30] obtained coupled coincidence and coupled common fixed theorems for nonlinear contractive mappings in partially ordered complete metric spaces using mixed g-monotone property. Afterwards, many authors established various coupled fixed point theorems in partially ordered metric spaces [12, 17, 20, 21, 33, 38, 40], in partially ordered cone metric spaces [4, 28, 37], as well as in partially ordered G-metric spaces [3, 6, 7, 13, 11, 34].

In 1968 and 1969, Markin [35] and Nadler [36] extended the Banach contraction principle from the single-valued mappings to the multi-valued mappings. Since then, some further results on fixed points for multi-valued mappings have appeared, see e.g. [39, 9, 14, 23, 29]. In 1989, Kaneko and Sessa [27] obtained fixed point theorems for compatible multi-valued and single-valued mappings. Recently, the study of fixed point theory for hybrid mappings is interesting and well developed. We refer the reader to [25, 41, 18, 16, 32, 42] and the references therein. Very recently, Abbas and Ćirić *et al.* [1] and Hussain and Alotaibi [24] proved some coupled coincidence and coupled common fixed point theorems for hybrid pair of mappings. A result of Abbas and Ćirić *et al.* [1] is stated as follows.

Theorem 1.1. [1] *Let (X, d) be a metric space and $CB(X)$ be the class of all nonempty bounded and closed subset of X. $T : X \times X \to CB(X)$ and $g : X \to X$*

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are mappings satisfying

$$\begin{aligned} H(T(x,y),T(u,v)) \quad \leq \quad & a_1d(gx,gu)+a_2d(T(x,y),gx)+a_3d(gy,gv) \\ & a_4d(T(u,v),gu)+a_5d(T(x,y),gu)+a_6d(T(u,v),gx), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i = a_i(x, y, u, v)$, $i = 1, 2, \cdots, 6$, are nonnegative real numbers such that

$$a_1+a_2+a_3+a_4+a_5+a_6 \leq h < 1,$$

where h is a fixed number. If $T(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then T and g have coupled coincidence point.

In 1981, Heilpern [22] introduced the concept of fuzzy contraction mappings and developed Banach contraction principle for fuzzy mappings in complete metric linear spaces. Since then, a lot of fixed point theorems for fuzzy mappings have been established (see [2, 5, 8, 15, 26, 31] and references).

Inspired by the above works, in this paper, we introduce the concepts of coupled coincidence and coupled common fixed points of single-valued and fuzzy mappings, and establish some coupled coincidence and coupled common fixed point theorems for hybrid mappings.

For the sake of convenience, we first recall some concepts.

Let (X, d) be a metric space, a fuzzy set in X is a function with domain X and values in $I = [0, 1]$. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by I^X. For $x \in X$, we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X. For $\alpha \in (0, 1]$, the fuzzy point x_α of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $x_\alpha(z) = 0$ if $z \neq x$ [19].

The α-level set of A, denoted by $[A]_\alpha$, is defined as

$$[A]_\alpha = \{x : A(x) \geq \alpha\}, \text{ if } \alpha \in (0,1],$$

$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

where $\overline{B}$ is the closure of the non-fuzzy set B.

Denote by $W(X)$ the totality of fuzzy sets which satisfy that for each $\alpha \in I$, $[A]_\alpha$ is non-empty convex and compact in X and $\sup_{x \in X} A(x) = 1$. Let $A, B \in W(X)$, then A is said to be more accurate than B, denoted by $A \subset B$, if and only if $A(x) \leq B(x)$, for each $x \in X$.

For $A, B \in W(X), \alpha \in [0, 1]$, define

$$P_\alpha(A,B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x,y),$$

$$D_\alpha(A,B) = H([A]_\alpha,[B]_\alpha),$$

$$P(A,B) = \sup_\alpha P_\alpha(A,B),$$

$$D(A,B) = \sup_\alpha D_\alpha(A,B),$$

where H is the Hausdorff metric induced by the metric d.

Let X be an arbitrary set and Y a metric space. A mapping F is called a fuzzy mapping if F is a mapping from $X \times X$ into I^Y, that is, $F(x,y) \in I^Y$ for each $x, y \in X$.

Definition 1.2. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. An element $(x,y) \in X \times X$ is said to be (i) fuzzy coupled fixed point of F if there exists $\alpha \in (0,1]$ such that $x \in [F(x,y)]_\alpha$ and $y \in [F(y,x)]_\alpha$ (ii) coupled fixed point of F if $\{x\} \subset F(x,y)$ and $\{y\} \subset F(y,x)$ (iii) fuzzy coupled coincidence point of F and g if there exists $\alpha \in (0,1]$ such that $gx \in [F(x,y)]_\alpha$ and $gy \in [F(y,x)]_\alpha$ (iv) fuzzy coupled common fixed point of F and g if there exists $\alpha \in (0,1]$ such that $x = gx \in [F(x,y)]_\alpha$ and $y = gy \in [F(y,x)]_\alpha$. We denote by $CC_\alpha(F,g) = \{(x,y) \in X \times X | gx \in [F(x,y)]_\alpha, gy \in [F(y,x)]_\alpha\}$ the set of fuzzy coupled coincidence points of F and g. Note that if $(x,y) \in CC_\alpha(F,g)$, then (y,x) is also in $CC_\alpha(F,g)$.

Definition 1.3. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. An element $(x,y) \in X \times X$ is said to be (i) coupled coincidence point of F and g if $\{gx\} \subset F(x,y)$ and $\{gy\} \subset F(y,x)$ (ii) coupled common fixed point of F and g if $x = gx \in [F(x,y)]_1$ and $y = gy \in [F(y,x)]_1$. We denote by $CC(F,g) = \{(x,y) \in X \times X | \{gx\} \subset F(x,y), \{gy\} \subset F(y,x)\}$ the set of coupled coincidence points of F and g.

Definition 1.4. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. The hybrid pair $\{F,g\}$ is said to be α-ω-compatible if there exists $\alpha \in (0,1]$ such that $g[F(x,y)]_\alpha \subseteq [F(gx,gy)]_\alpha$ whenever $(x,y) \in CC_\alpha(F,g)$.

Definition 1.5. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. The hybrid pair $\{F,g\}$ is said to be ω-compatible if $g[F(x,y)]_1 \subseteq [F(gx,gy)]_1$ whenever $(x,y) \in CC(F,g)$.

Definition 1.6. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. The mapping g is said to be α-F-weakly commuting at $(x,y) \in X \times X$ if there exists $\alpha \in (0,1]$ such that $g^2x \in [F(gx,gy)]_\alpha$ and $g^2y \in [F(gy,gx)]_\alpha$.

Definition 1.7. Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. The mapping g is said to be F-weakly commuting at $(x,y) \in X \times X$ if $\{g^2x\} \subset F(gx,gy)$ and $\{g^2y\} \subset F(gy,gx)$.

2. Main Results

The following lemma is important in proving our main results.

Lemma 2.1. [36] *Let A and B be nonempty closed and bounded subsets of a metric space (X,d). If $a \in A$, then $d(a,B) \leq H(A,B)$.*

Theorem 2.2. *Let (X,d) be a metric space and $\alpha \in (0,1]$. $F : X \times X \to W(X)$ and $g : X \to X$ are mappings satisfying*

$$\begin{aligned}&\psi(D_\alpha(F(x,y),F(u,v)))\\ &\leq \psi(M(x,y,u,v,\alpha)) - \varphi(\psi(M(x,y,u,v,\alpha))) + \theta(N(x,y,u,v,\alpha)),\end{aligned} \tag{1}$$

for all $x, y, u, v \in X$, where
(i) $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and also $\limsup\limits_{s\to 0^+} \frac{s}{\psi(s)} < \infty$;
(ii) $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$ and also for any sequence $\{t_n\}$ with $\lim\limits_{n\to\infty} t_n = 0$, there exists $k \in (0,1)$ and $n_0 \in N$ such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$;
(iii) $\theta : [0,\infty) \to [0,\infty)$ is a continuous function such that $\theta(t) = 0$ if and only if $t = 0$;
(iv)

$$\begin{aligned} M(x,y,u,v,\alpha) &= \max\{d(gx,gu), d(gy,gv), d(gx,[F(x,y)]_\alpha), d(gu,[F(u,v)]_\alpha), \\ &\quad \frac{1}{2}[d(gx,[F(u,v)]_\alpha) + d(gu,[F(x,y)]_\alpha)] \end{aligned}$$

and

$$N(x,y,u,v,\alpha) = \min\{d(gx,[F(u,v)]_\alpha), d(gu,[F(x,y)]_\alpha)\}.$$

Suppose also that
(v) $[F(x,y)]_\alpha \subseteq g(X)$ for all $x, y \in X$;
(vi) $g(X)$ is complete subset of X.

Then F and g have at least one fuzzy coupled coincidence point in X. Moreover F and g have at least one fuzzy coupled common fixed point if one of the following conditions holds.
(a) F and g are α-ω-compatible, $\lim\limits_{n\to\infty} g^n x = u$ and $\lim\limits_{n\to\infty} g^n y = v$ for some $(x,y) \in CC_\alpha(F,g)$, $u, v \in X$ and g is continuous at u and v.
(b) g is α-F-weakly commuting for some $(x,y) \in X \times X$, $g^2 x = gx$ and $g^2 y = gy$.
(c) g is continuous at x, y for some $(x,y) \in CC_\alpha(F,g)$ and for some $u, v \in X$, $\lim\limits_{n\to\infty} g^n u = x$ and $\lim\limits_{n\to\infty} g^n v = y$.
(d) $g(CC_\alpha(F,g))$ is singleton subset of $CC_\alpha(F,g)$.

Proof. **Step 1:** Let $x_0, y_0 \in X$ be arbitrary. Then $[F(x_0,y_0)]_\alpha$ and $[F(y_0,x_0)]_\alpha$ are well defined. From *(v)*, there exist $x_1, y_1 \in X$ such that $gx_1 \in [F(x_0,y_0)]_\alpha$ and $gy_1 \in [F(y_0,x_0)]_\alpha$. Again from *(v)* and Lemma 2.1, in view of the compactness of $[F(x_0,y_0)]_\alpha$ and $[F(y_0,x_0)]_\alpha$, there exist $x_2, y_2 \in X$ such that $gx_2 \in [F(x_1,y_1)]_\alpha, gy_2 \in [F(y_1,x_1)]_\alpha$ and

$$d(gx_1, gx_2) = d(gx_1, [F(x_1,y_1)]_\alpha) \leq D_\alpha(F(x_0,y_0), F(x_1,y_1)), \tag{2}$$

$$d(gy_1, gy_2) = d(gy_1, [F(y_1,x_1)]_\alpha) \leq D_\alpha(F(y_0,x_0), F(y_1,x_1)). \tag{3}$$

Continuing this process, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} \in [F(x_n,y_n)]_\alpha, gy_{n+1} \in [F(y_n,x_n)]_\alpha, n = 0,1,2,\ldots,$$

and

$$d(gx_n, gx_{n+1}) = d(gx_n, [F(x_n,y_n)]_\alpha) \leq D_\alpha(F(x_{n-1},y_{n-1}), F(x_n,y_n)), \tag{4}$$

$$d(gy_n, gy_{n+1}) = d(gy_n, [F(y_n,x_n)]_\alpha) \leq D_\alpha(F(y_{n-1},x_{n-1}), F(y_n,x_n)), n = 1,2,\ldots. \tag{5}$$

By the monotone property of ψ and inequality (1), we have for $n = 1, 2, \ldots$,

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &\le \psi(D_\alpha(F(x_{n_1}, y_{n-1}), F(x_n, y_n))) \\ &\le \psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)) - \varphi(\psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha))) \\ &\quad +\theta(N(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)), \end{aligned} \tag{6}$$

where

$$\begin{aligned} & N(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) \\ = \;& \min\{d(gx_{n-1}, [F(x_n, y_n)]_\alpha), d(gx_n, [F(x_{n-1}, y_{n-1})]_\alpha)\} = 0 \end{aligned}$$

and

$$\begin{aligned} & M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) \\ = \;& \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, [F(x_{n-1}, y_{n-1})]), \\ & d(gx_n, [F(x_n, y_n)]_\alpha), \frac{1}{2}[d(gx_{n-1}, [F(x_n, y_n)]_\alpha) + d(gx_n, [F(x_{n-1}, y_{n-1})]_\alpha)]\} \\ = \;& \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gx_n, gx_{n+1})\}. \end{aligned}$$

Suppose that for some $n \ge 1$,

$$0 \le \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\} < d(gx_n, gx_{n+1}),$$

then

$$\psi(d(gx_n, gx_{n+1})) \le \psi(d(gx_n, gx_{n+1})) - \varphi(\psi(d(gx_n, gx_{n+1}))) + \theta(0),$$

it follows from *(i)*, *(ii)* and *(iii)* that $d(gx_n, gx_{n+1}) = 0$, it is a contradiction. Hence

$$d(gx_n, gx_{n+1}) \le M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) = \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\} \tag{7}$$

and

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &\le \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &\quad -\varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\})). \end{aligned} \tag{8}$$

Similarly it can be shown that

$$d(gy_n, gy_{n+1}) \le M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) = \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\} \tag{9}$$

and

$$\begin{aligned} \psi(d(gy_n, gy_{n+1})) &\le \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &\quad -\varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\})). \end{aligned} \tag{10}$$

By (7) and (9), we have

$$\begin{aligned} & 0 \le \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \\ & \le M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) = \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \end{aligned} \tag{11}$$

It follows that $\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}$ is a nonincreasing sequence. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = \lim_{n\to\infty} M(x_n, y_n, x_{n+1}, y_{n+1}, \alpha) = r.$$

By (8), (10) and the monotone property of ψ, we have

$$\begin{aligned} & \psi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) \\ & = \max\{\psi(d(gx_n, gx_{n+1})), \psi(d(gy_n, gy_{n+1}))\} \\ & \le \psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)) - \varphi(\psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha))). \end{aligned} \tag{12}$$

Letting $n \to +\infty$ in (12), and using the properties of ψ and φ, we get

$$\psi(r) \leq \psi(r) - \varphi(\psi(r)),$$

this implies that $r = 0$. Therefore

$$\lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = \lim_{n\to\infty} M(x_n, y_n, x_{n+1}, y_{n+1}, \alpha) = 0. \quad (13)$$

Step 2: Now, we show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in gX.

Since $\lim_{n\to\infty} M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha) = 0$, by the properties of ψ and φ, there exists $k \in (0, 1)$ and $n_0 \in N$ such that for each $n \geq n_0$,

$$\varphi(\psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha))) \geq k\psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)). \quad (14)$$

Hence, for each $n \geq n_0$, we have

$$\begin{aligned} & \psi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) \\ \leq\ & \psi((M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)) - \varphi(\psi((M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha))) \\ \leq\ & (1-k)\psi(M(x_{n-1}, y_{n-1}, x_n, y_n, \alpha)) \leq (1-k)^n\psi(M(x_0, y_0, x_1, y_1, \alpha)). \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{n=1}^{\infty} \psi(d(gx_n, gx_{n+1})) \\ \leq\ & \sum_{n=1}^{n_0-1} \psi(d(gx_n, gx_{n+1})) + \sum_{n=n_0}^{\infty} (1-k)^n \psi(\max\{d(gx_0, gx_1), d(gy_0, gy_1)\}) < \infty. \end{aligned}$$

From

$$\limsup_{n\to\infty} \frac{d(gx_n, gx_{n+1})}{\psi(d(gx_n, gx_{n+1}))} \leq \limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty$$

we obtain

$$\sum_{n=1}^{\infty} d(gx_n, gx_{n+1}) < \infty.$$

It follows that $\{gx_n\}$ is a Cauchy sequence. Similarly, we can show that $\{gy_n\}$ is also a Cauchy sequence. Since $g(X)$ is complete, there exist $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$.

Step 3: We can get from (1) that

$$\begin{aligned} & \psi(|d(gx, [F(x,y)]_\alpha) - d(gx, gx_{n+1})|) \\ & \leq \psi(d(gx_{n+1}, [F(x,y)]_\alpha)) \leq \psi(D_\alpha(F(x_n, y_n), F(x,y))) \\ & \leq \psi(M(x_n, y_n, x, y, \alpha)) - \varphi(\psi(M(x_n, y_n, x, y, \alpha))) + \theta(N(x_n, y_n, x, y, \alpha)), \end{aligned} \quad (15)$$

where

$$N(x_n, y_n, x, y, \alpha) = \min\{d(gx_n, [F(x,y)]_\alpha), d(gx, [F(x_n, y_n)]_\alpha)\}$$

and

$$\begin{aligned} & M(x_n, y_n, x, y, \alpha) \\ =\ & \max\{d(gx_n, gx), d(gy_n, gy), d(gx_n, [F(x_n, y_n)]_\alpha), d(gx, [F(x,y)]_\alpha), \\ & \frac{1}{2}[d(gx_n, [F(x,y)]_\alpha) + d(gx, [F(x_n, y_n)]_\alpha)]\}. \end{aligned}$$

Using $gx_n \to gx$ and $gy_n \to gy$, we have

$$\lim_{n\to\infty} N(x_n, y_n, x, y, \alpha) = 0$$

and

$$\lim_{n\to\infty} M(x_n, y_n, x, y, \alpha) = d(gx, [F(x, y)]_\alpha).$$

Letting $n \to \infty$ in the inequality (15), one can deduce from the properties of ψ, φ and θ that

$$\psi(d(gx, [F(x, y)]_\alpha)) \leq \psi(d(gx, [F(x, y)]_\alpha)) - \varphi(\psi(d(gx, [F(x, y)]_\alpha))),$$

this implies that $d(gx, [F(x, y)]_\alpha) = 0$, that is, $gx \in [F(x, y)]_\alpha$. Similarly, we can get $gy \in [F(y, x)]_\alpha$. Hence (x, y) is a fuzzy coupled coincidence point of the mappings F and g.

Suppose that *(a)* holds. Then for some $(x, y) \in CC_\alpha(F, g)$, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$, where $u, v \in X$. Since g is continuous at u and v, we obtain

$$u = \lim_{n\to\infty} g^{n+1} x = \lim_{n\to\infty} g(g^n x) = gu$$

and

$$v = \lim_{n\to\infty} g^{n+1} y = \lim_{n\to\infty} g(g^n y) = gv.$$

As F and g are α-ω-compatible, we have

$$g[F(x, y)]_\alpha \subseteq [F(gx, gy)]_\alpha \tag{16}$$

and

$$g[F(y, x)]_\alpha \subseteq [F(gy, gx)]_\alpha. \tag{17}$$

From $gx \in [F(x, y)]_\alpha$ and $gy \in [F(y, x)]_\alpha$, we can get that

$$g(gx) \in [F(gx, gy)]_\alpha \tag{18}$$

and

$$g(gy) \in [F(gy, gx)]_\alpha. \tag{19}$$

Thus,

$$(gx, gy) \in CC_\alpha(F, g).$$

Again by the α-ω-compatibility of F and g, we obtain that

$$g[F(gx, gy)]_\alpha \subseteq [F(g(gx), g(gy))]_\alpha. \tag{20}$$

(18) and (20) imply that

$$g(g^2 x) \subseteq [F(g^2 x, g^2 y)]_\alpha.$$

Similarly, we have

$$g(g^2 y) \subseteq [F(g^2 y, g^2 x)]_\alpha.$$

So,

$$(g^2 x, g^2 y) \in CC_\alpha(F, g).$$

Continuing this process, we can get that $g^n x \in [F(g^{n-1} x, g^{n-1} y)]_\alpha$, $g^n y \in [F(g^{n-1} y, g^{n-1} x)]_\alpha$ and $(g^n x, g^n y) \in CC_\alpha(F, g)$ for all $n \geq 1$.

Then

$$\begin{aligned} &\psi(|d(gu, [F(u, v)]_\alpha) - d(gu, g^n x)|) \\ &\leq \psi(d(g^n x, [F(u, v)]_\alpha)) \leq \psi(D_\alpha(F(g^{n-1} x, g^{n-1} y), F(u, v))) \\ &\leq \psi(M(g^{n-1} x, g^{n-1} y, u, v, \alpha)) - \varphi(\psi(M(g^{n-1} x, g^{n-1} y, u, v, \alpha))) \\ &\quad +\theta(N(g^{n-1} x, g^{n-1} y, u, v, \alpha)), \end{aligned} \tag{21}$$

where

$$N(g^{n-1}x, g^{n-1}y, u, v, \alpha) = \min\{d(g^n x, [F(u,v)]_\alpha), d(gu, [F(g^{n-1}x, g^{n-1}y)]_\alpha)\}$$

and

$$\begin{aligned} & M(g^{n-1}x, g^{n-1}y, u, v, \alpha) \\ = \; & \max\{d(g^n x, gu), d(g^n y, gv), d(g^n x, [F(g^{n-1}x, g^{n-1}y)]_\alpha), d(gu, [F(u,v)]_\alpha), \\ & \frac{1}{2}[d(g^n x, [F(u,v)]_\alpha) + d(gu, [F(g^{n-1}x, g^{n-1}y)]_\alpha)]\}. \end{aligned}$$

Letting $n \to \infty$ in (21), we have

$$\psi(d(gu, [F(u,v)]_\alpha)) \le \psi(d(gu, [F(u,v)]_\alpha)) - \varphi(\psi(d(gu, [F(u,v)]_\alpha))),$$

this implies that $(d(gu, [F(u,v)]_\alpha) = 0$, that is $gu \in [F(u,v)]_\alpha$. Similarly, $gv \in [F(u,v)]_\alpha$. Hence (u,v) is a fuzzy coupled common fixed point of F and g.

Suppose that *(b)* holds. Then for some $(x,y) \in X \times X$, $gx = g(gx) \in [F(gx, gy)]_\alpha$ and $gy = g(gy) \in [F(gy, gx)]_\alpha$, hence (gx, gy) is a fuzzy coupled common fixed point of F and g.

Suppose that *(c)* holds. It follows from the continuity of g at x, y that $x = gx \in [F(x,y)]_\alpha$ and $y = gy \in [F(y,x)]_\alpha$. Hence (x,y) is a fuzzy coupled common fixed point of F and g.

Finally, suppose that *(d)* holds. Let $g(CC_\alpha(F,g)) = \{(x,y)\} \subseteq CC_\alpha(F,g)$, then $(x,y) = (gx, gy) \in CC_\alpha(F,g)$. Hence (x,y) is a fuzzy coupled common fixed point of F and g. $\square$

Remark 2.3. In Theorem 2.2, if let

$$F(x,y) = \begin{cases} 1, & (x,y) \in S(x,y); \\ 0, & otherwise. \end{cases}$$

where $S(x,y) \in 2^X$, then we can get the specific form of inequatlity (1) as follows:

$$\psi(H(S(x,y), S(u,v)))$$
$$\le \psi(M(x,y,u,v,\alpha)) - \varphi(\psi(M(x,y,u,v,\alpha))) + \theta(N(x,y,u,v,\alpha)),$$

for all $x, y, u, v \in X$, where

$$\begin{aligned} M(x,y,u,v,\alpha) \;\; = \;\; & \max\{d(gx, gu), d(gy, gv), d(gx, S(x,y)), d(gu, S(u,v)), \\ & \frac{1}{2}[d(gx, S(u,v)) + d(gu, S(x,y))]\} \end{aligned}$$

and

$$N(x,y,u,v,\alpha) = \min\{d(gx, S(u,v)), d(gu, S(x,y))\}.$$

Moreover, in this case, $S(X \times X) \subseteq g(X)$. By Theorem 2.2, we conclude that S and y have coupled coincidence point. Thus, the above analysis implies that Theorem 2.2 is an extension of Theorem 1.1.

If in Theorem 2.2 $g = I_X$ (I_X being the identity map on X), we obtain the following corollary.

Corollary 2.4. *Let (X,d) be a complete metric space, $F : X \times X \to W(X)$ be a mappings satisfying*

$$\begin{aligned} &\psi(D_\alpha(F(x,y),F(u,v))) \\ \leq\ &\psi(M^*(x,y,u,v,\alpha)) - \varphi(\psi(M^*(x,y,u,v,\alpha))) + \theta(N^*(x,y,u,v,\alpha)), \end{aligned}$$

for all $x,y,u,v \in X$ and $\alpha \in (0,1]$, where
(i) $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and also $\limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty$;
(ii) $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$ and also for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0,1)$ and $n_0 \in N$ such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$;
(iii) $\theta : [0,\infty) \to [0,\infty)$ is a continuous function such that $\theta(t) = 0$ if and only if $t = 0$;
(iv)

$$\begin{aligned} M^*(x,y,u,v,\alpha) &= \max\{d(x,u), d(y,v), d(x,[F(x,y)]_\alpha), d(u,[F(u,v)]_\alpha), \\ &\quad \frac{1}{2}[d(x,[F(u,v)]_\alpha) + d(u,[F(x,y)]_\alpha)]\} \end{aligned}$$

and

$$N^*(x,y,u,v,\alpha) = \min\{d(x,[F(u,v)]_\alpha), d(u,[F(x,y)]_\alpha)\}.$$

Then F has at least one fuzzy coupled fixed point in X.

Theorem 2.5. *Let (X,d) be a metric space, $F : X \times X \to W(X)$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} &\psi(D(F(x,y),F(u,v))) \\ \leq\ &\psi(m(x,y,u,v)) - \varphi(\psi(m(x,y,u,v))) + \theta(n(x,y,u,v)), \end{aligned}$$

for all $x,y,u,v \in X$, where
(i) $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and also $\limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty$;
(ii) $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous and nonincreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and also for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0,1)$ and $n_0 \in N$ such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$;
(iii) $\theta : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\theta(t) = 0$ if and only if $t = 0$;
(iv)

$$\begin{aligned} m(x,y,u,v) &= \max\{d(gx,gu), d(gy,gv), P(gx,F(x,y)), P(gu,F(u,v)), \\ &\quad \frac{1}{2}[P(gx,F(u,v)) + P(gu,F(x,y))]\} \end{aligned}$$

and

$$n(x,y,u,v) = \min\{P(gx,F(u,v)), P(gu,F(x,y))\}.$$

Suppose also that
(v) $[F(x,y)]_1 \subseteq g(X)$ for all $x,y \in X$;
(vi) $g(X)$ is complete subset of X.

Then F and g have at least one coupled coincidence point in X. Moreover F and g have at least one coupled common fixed point if one of the following conditions holds.
(a) F and g are ω-compatible, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x,y) \in CC(F,g)$, $u,v \in X$ and g is continuous at u and v.
(b) g is F-weakly commuting for some $(x,y) \in X \times X$, $g^2x = gx$ and $g^2y = gy$.
(c) g is continuous at x,y for some $(x,y) \in CC(F,g)$ and for some $u,v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$.
(d) $g(CC(F,g))$ is singleton subset of $CC(F,g)$.

Proof. Choose $x,y \in X$, since for all $x,y \in X$, $[F(x,y)]_1 \subseteq [F(x,y)]_\alpha \in W(X)$, for each $\alpha \in (0,1]$, therefore $d(gx,[F(x,y)]_\alpha) \leq d(gx,[F(x,y)]_1)$, for each $\alpha \in (0,1]$, and it implies that

$$P(gx,F(x,y)) \leq d(gx,[F(x,y)]_1).$$

Similarly, for $u,v \in X$

$$\begin{aligned}P(gu,F(u,v)) &\leq d(gu,[F(u,v)]_1),\\ P(gx,F(u,v)) &\leq d(gx,[F(u,v)]_1),\\ P(gu,F(x,y)) &\leq d(gu,[F(x,y)]_1).\end{aligned}$$

Now, for all $x,y,u,v \in X$,

$$\begin{aligned} m(x,y,u,v) &= \max\{d(gx,gu), d(gy,gv), P(gx,F(x,y)), P(gu,F(u,v)),\\ &\quad \frac{1}{2}[P(gx,F(u,v)) + P(gu,F(x,y))]\}\\ &\leq \max\{d(gx,gu), d(gy,gv), d(gx,[F(x,y)]_1), d(gu,[F(u,v)]_1),\\ &\quad \frac{1}{2}[d(gx,[F(u,v)]_1) + d(gu,[F(x,y)]_1)]\}\\ &= M(x,y,u,v,1), \end{aligned}$$

$$\begin{aligned} &n(x,y,u,v) = \min\{P(gx,F(u,v)), P(gu,F(x,y))\}\\ \leq\ &\min\{d(gx,[F(u,v)]_1), d(gu,[F(x,y)]_1)\} = N(x,y,u,v,1) \end{aligned}$$

and

$$D_1(F(x,y),F(u,v)) \leq D(F(x,y),F(u,v)).$$

Hence

$$\begin{aligned} &\psi(D_1(F(x,y),F(u,v))) \leq \psi(D(F(x,y),F(u,v)))\\ \leq\ &\psi(m(x,y,u,v)) - \varphi(\psi(m(x,y,u,v))) + \theta(n(x,y,u,v))\\ \leq\ &\psi(M(x,y,u,v,1)) - \varphi(\psi(M(x,y,u,v,1))) + \theta(N(x,y,u,v,1)). \end{aligned}$$

By Theorem 2.2, there exists $x,y \in CC_1(F,g)$, that is $gx \in [F(x,y)]_1$ and $gy \in [F(y,x)]_1$. So F and g have at least one coupled coincidence point in X. Similar to the proof of Theorem 2.2, we can prove that F and g have at least one coupled common fixed point if one of conditions *(a)*, *(b)*, *(c)* and *(d)* holds. □

If in Theorem 2.5 $g = I_X$ (I_X being the identity map on X), we obtain the following corollary.

Corollary 2.6. *Let (X, d) be a complete metric space, $F : X \times X \to W(X)$ be a mappings satisfying*

$$\begin{aligned} & \psi(D(F(x,y), F(u,v))) \\ \leq \quad & \psi(m^*(x,y,u,v)) - \varphi(\psi(m^*(x,y,u,v))) + \theta(n^*(x,y,u,v)), \end{aligned}$$

for all $x, y, u, v \in X$, where
(i) $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and also $\limsup\limits_{s\to 0^+} \frac{s}{\psi(s)} < \infty$;
(ii) $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous and nonincreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and also for any sequence $\{t_n\}$ with $\lim\limits_{n\to\infty} t_n = 0$, there exists $k \in (0,1)$ and $n_0 \in N$ such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$;
(iii) $\theta : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\theta(t) = 0$ if and only if $t = 0$;
(iv)

$$\begin{aligned} m^*(x,y,u,v) \quad = \quad & \max\{d(x,u), d(y,v), P(x, F(x,y)), P(u, F(u,v)), \\ & \frac{1}{2}[P(x, F(u,v)) + P(u, F(x,y))]\} \end{aligned}$$

and

$$n^*(x,y,u,v) = \min\{P(x, F(u,v)), P(u, F(x,y))\}.$$

Then F has at least one coupled fixed point in X.

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