MINIMAL SOLUTION OF FUZZY LINEAR SYSTEMS

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Abstract. In this paper, we use parametric form of fuzzy number and we convert a fuzzy linear system to two linear system in crisp case. Conditions for the existence of a minimal solution to $m \times n$ fuzzy linear equation systems are derived and a numerical procedure for calculating the minimal solution is designed. Numerical examples are presented to illustrate the proposed method.

1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [31], Dubois and Prade [14]. We refer the reader to [20] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy metric spaces [29], fuzzy differential equations [5] and fuzzy linear systems [3, 4, 11].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems [7, 8, 9, 10, 24, 25, 26, 28]. Several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. Friedman *et al.* [16] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where A and B are real $n \times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers and the constant y is vector consisting of n fuzzy numbers, in [17]. In [1, 3, 4, 11] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Abbasbandy *et al.* [6] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $Ax + f = Bx + c$, where A and B are real $m \times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers and the constant f and c are vectors consisting of m fuzzy numbers.

Recently, Ezzati [15] proposed a new method for solving a $n \times n$ fuzzy linear system whose coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector by using the embedding method given by Cong-Xin and Min [13] and replace the original $n \times n$ fuzzy linear system by two $n \times n$ crisp linear systems. Since perturbation analysis is very important in numerical methods, Wang *et al.* [30] presented the perturbation analysis for a class of fuzzy

Received: June 2013; Revised: June 2014; Accepted: December 2014

Key words and phrases: Fuzzy linear system, Pseudo-inverse, Minimal solution.

linear systems which could be solved by an embedding method. Now, according to the presented method in this paper, we can investigate perturbation analysis in two $m \times n$ crisp linear systems.

The paper is organized as follows. In section 2, we introduce the notation, the definitions and preliminary results that will be used throughout the paper. In this section, we review the method proposed by Friedman *et al.* [16], Abbasbandy *et al.* [6] and Ezzati [15]. In section 3, the method for solving fuzzy linear system is proposed. The proposed method is illustrated by solving some examples in section 4 and conclusions are drawn in section 5.

2. **Preliminaries**

The minimal solution of an arbitrary linear system is formally defined such that:

a) If the system is consistent and has a unique solution, then this solution is also the minimal solution.
b) If the system is consistent and has a set solution, then the minimal solution is a member of this set that has the least Euclidean norm.
c) If the system is inconsistent and has a unique least squares solution, then this solution is also the minimal solution.
d) If the system is inconsistent and has a set of least squares solutions, then the minimal solution is a member of this set that has the least Euclidean norm.

Definition 2.1. [21] A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0, 1]$ such that
i. u is upper semi-continuous;
ii. $u(x) = 0$ outside some interval $[a, d]$;
iii. There are real numbers b and c, $a \leq b \leq c \leq d$, for which
1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,
3. $u(x) = 1, b \leq x \leq c$.

The set of all the fuzzy numbers (as given in Definition 2.1) is denoted by E^1.

An alternative definition which yields the same E^1 is given by Kaleva [19] and Ming *et al.* [23].

Parametric form of an arbitrary fuzzy number is given in [23] as follows. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r),\ 0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

The set of all these fuzzy numbers is denoted by E which is a complete metric space with Hausdorff distance. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leq r \leq 1$.

For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r)),\ y = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition of fuzzy numbers and the multiplication of real number by fuzzy number by using the extension principle as [23]

a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,
b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,
c) $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Remark 2.2. [2] Let $u = (\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$
$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \geq 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$. A fuzzy number $u \in E$ is said symmetric if $u^c(r)$ is a constant function of r for all $0 \leq r \leq 1$.

Remark 2.3. Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then

$$w^c(r) = ku^c(r) + sv^c(r),$$
$$w^d(r) = |k|u^d(r) + |s|v^d(r).$$

Definition 2.4. The $m \times n$ linear system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & y_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & y_2, \\ \vdots & & & & & & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & y_m, \end{cases} \tag{1}$$

where the matrix of coefficients $A = (a_{ij}), 1 \leq i \leq m$ and $1 \leq j \leq n$, is a given real $m \times n$ matrix, $y_i \in E$, $1 \leq i \leq m$, are given fuzzy numbers with the unknowns $x_j \in E$, $1 \leq j \leq n$ is called a fuzzy linear system (FLS). In this paper, we assume the matrix A is full rank, i.e., $rank(A) = m$ (for $m \leq n$) or $rank(A) = n$ (for $n < m$).

Finally, we conclude this subsection by reviewing the methods for solving fuzzy linear system proposed by Friedman *et al.* [16], Ezzati [15] and Abbasbandy *et al.* [6].

3. **Solving Fuzzy Linear System**

3.1. **Solution Method for Square System by Friedman et al.**

Friedman *et al.* [16] wrote the linear system (1) for $m = n$ as follows:

$$S\hat{X} = \hat{Y}, \tag{2}$$

where s_{ij} are determined as follows:

$$a_{ij} \geq 0 \Longrightarrow s_{ij} = a_{ij},\ s_{i+n,j+n} = a_{ij},$$
$$a_{ij} < 0 \Longrightarrow s_{i,j+n} = -a_{ij},\ s_{i+n,j} = -a_{ij},$$

and any s_{ij} which is not determined by (2) is zero and

$$\hat{X} = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix},\ \hat{Y} = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

The structure of S implies that $s_{ij} \geq 0, 1 \leq i, j \leq 2n$ and that

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{3}$$

where B contains the positive entries of A, and C contains the absolute values of the negative entries of A, i.e., $A = B - C$.

Theorem 3.1. [16] *The matrix S is nonsingular if and only if the matrices $A = B - C$ and $B + C$ are both nonsingular.*

Theorem 3.2. [16] *If S^{-1} exists it must have the same structure as S, i.e.*

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{4}$$

where

$$D = \frac{1}{2}[(B + C)^{-1} + (B - C)^{-1}],\ E = \frac{1}{2}[(B + C)^{-1} - (B - C)^{-1}].$$

We know that if S is nonsingular then

$$\hat{X} = S^{-1}\hat{Y}. \tag{5}$$

3.2. **Solution Method for Square System by Ezzati.** Ezzati [15] considered fuzzy linear system (1) for $m = n$ and solved by using the embedding approach. Unfortunately he has not indicated conditions for the existence of a unique fuzzy solution to $n \times n$ linear system. Ezzati [15] wrote the linear system (1) as follows:

$$A(\underline{x} + \overline{x}) = \underline{y} + \overline{y}, \tag{6}$$

where $h = \underline{x} + \overline{x} = (\underline{x}_1 + \overline{x}_1, \underline{x}_2 + \overline{x}_2, \ldots, \underline{x}_n + \overline{x}_n)^T$ and $\underline{y} + \overline{y} = (\underline{y}_1 + \overline{y}_1, \underline{y}_2 + \overline{y}_2, \ldots, \underline{y}_n + \overline{y}_n)^T$.

Theorem 3.3. [15] *Suppose the inverse of matrix A in Eq. (1) exists and $x = (x_1, x_2, \ldots, x_n)^T$ is a fuzzy solution of this equation. Then $\underline{x}(r) + \overline{x}(r)$ is the solution of the following system*

$$A(\underline{x}(r) + \overline{x}(r)) = \underline{y}(r) + \overline{y}(r). \tag{7}$$

We know that if A is nonsingular then

$$h(r) = A^{-1}(\underline{y}(r) + \overline{y}(r)). \tag{8}$$

Let matrices B and C have defined as equation (3). Using matrix notation for equation (1), we get

$$\begin{cases} B\underline{x}(r) - C\overline{x}(r) = \underline{y}(r), \\ B\overline{x}(r) - C\underline{x}(r) = \overline{y}(r). \end{cases}$$

By substituting of $\overline{x}(r) = h(r) - \underline{x}(r)$ and $\underline{x}(r) = h(r) - \overline{x}(r)$ in the first and second equation of above system, respectively, we have

$$(B + C)\underline{x}(r) = \underline{y}(r) + Ch(r) \tag{9}$$

and

$$(B + C)\overline{x}(r) = \overline{y}(r) + Ch(r). \tag{10}$$

If $B + C$ is nonsingular then

$$\underline{x}(r) = (B + C)^{-1}(\underline{y}(r) + Ch(r))$$

and

$$\overline{x}(r) = h(r) - \underline{x}(r).$$

Therefore, the solution of the fuzzy system (1) can be calculated by formulas (8), (9) and (10).

Recently, Otadi and Mosleh [27] adopt Ezzati's method to the non-square system case. The formulas obtained by Otadi and Mosleh are identical to Ezzati's formulas. The only difference is that the adopted method uses pseudo-inverses for non-square matrices where the original method uses inverses for square matrices.

3.3. **Solution Method for Non-square System by Abbasbandy et al.** Abbasbandy *et al.* [6] considered fuzzy linear system (1) as follows:

$$S\hat{X} = \hat{Y}. \tag{11}$$

Corollary 3.4. *Let W be a $p \times q$ real, full rank matrix. There exists a $p \times p$ orthogonal matrix U, a $q \times q$ orthogonal matrix V, and a $p \times q$ diagonal matrix Σ with $\langle \Sigma \rangle_{ij} = 0$ for $i \neq j$ and $\langle \Sigma \rangle_{ii} = \sigma_i > 0$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s > 0$, where $s = \min\{p, q\}$, such that the singular value decomposition*

$$W = U\Sigma V^T,$$

is valid. If Σ^+ is that $q \times p$ matrix whose only nonzero entries are $\langle \Sigma^+ \rangle_{ii} = 1/\sigma_i$ for $1 \leq i \leq s$, then $W^+ = V\Sigma^+U^T$ is the unique pseudo-inverse of W.

We refer the reader to [12] for more information on finding pseudo-inverse of an arbitrary matrix. When we work with full rank matrices, there are not any problem and all calculations are stable and well-posed.

Theorem 3.5. [6]. *The pseudo-inverse of non-negative full rank matrix*

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

is

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{12}$$

where

$$D = \frac{1}{2}[(B + C)^+ + (B - C)^+], \ E = \frac{1}{2}[(B + C)^+ - (B - C)^+].$$

We know that the minimal solution is

$$\hat{X} = S^+\hat{Y}. \tag{13}$$

4. A New Solution Method

In this section, we propose a new method for solving the $m \times n$ fuzzy linear system (1).

Consider fuzzy linear system equation (1). By referring to Remark 2 we have

$$\begin{cases} Ax^c(r) = y^c(r), \\ Fx^d(r) = y^d(r) \end{cases} \tag{14}$$

where $x^c(r) = (x_1^c(r), x_2^c(r), \ldots, x_n^c(r))^T$, $x^d(r) = (x_1^d(r), x_2^d(r), \ldots, x_n^d(r))^T$, $y^c(r) = (y_1^c(r), y_2^c(r), \ldots, y_m^c(r))^T$, $y^d(r) = (y_1^d(r), y_2^d(r), \ldots, y_m^d(r))^T$ and F contains the absolute values of entries of A. Also, we assume that A and F are full rank matrices. The minimal solution of (14) is obtained by [22]

$$\begin{cases} x^c(r) = A^+ y^c(r), \\ x^d(r) = F^+ y^d(r). \end{cases} \tag{15}$$

Therefore, we can solve fuzzy linear system equation (1) by solving equation (14) and we have

$$\begin{aligned} \underline{x}(r) &= x^c(r) - x^d(r), \\ \overline{x}(r) &= x^c(r) + x^d(r), \end{aligned} \tag{16}$$

where $X = ((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2), \ldots, (\underline{x}_n, \overline{x}_n))^T$ is minimal solution vector. If X represents a fuzzy vector, we name it as minimal fuzzy solution vector.

Theorem 4.1. *Vector X defined by equations (15) and (16) is a minimal fuzzy solution if F^+, A^+ and $F^+ - A^+$ are nonnegative matrices.*

Proof. Let $F^+ \geq 0$, then

$$\underline{x}(r) = A^+ y^c(r) - F^+ y^d(r), \tag{17}$$

$$\overline{x}(r) = A^+ y^c(r) + F^+ y^d(r), \tag{18}$$

and by subtracting equation (17) from equation (18) we get

$$\overline{x}(r) - \underline{x}(r) = 2F^+ y^d(r). \tag{19}$$

Thus, if y is arbitrary input vector which represents a fuzzy vector, i.e $\overline{y}(r) - \underline{y}(r) \geq 0$, then $y^d(r) \geq 0$, therefore the necessary condition $\overline{x}(r) - \underline{x}(r) \geq 0$ is satisfied. By using equations (17) and (18), we have

$$\underline{x}(r) = (F^+ + A^+)\frac{\underline{y}(r)}{2} - (F^+ - A^+)\frac{\overline{y}(r)}{2}, \tag{20}$$

$$\overline{x}(r) = (F^+ + A^+)\frac{\overline{y}(r)}{2} - (F^+ - A^+)\frac{\underline{y}(r)}{2}. \tag{21}$$

Since $\overline{y}(r)$ is monotonically decreasing and $\underline{y}(r)$ is monotonically increasing, the previous condition due to equations (20) and (21) is also necessary for $\overline{x}(r)$ and $\underline{x}(r)$ to be monotonically decreasing and increasing, respectively. The bounded left continuity of $\overline{x}(r)$ and $\underline{x}(r)$ is obvious since they are linear combinations of $\overline{y}(r)$ and $\underline{y}(r)$. $\square$

Theorem 4.2. *Assume that m_A, m_O and m_M are the numbers of multiplication operations that are required to calculate $\hat{X} = S^+\hat{Y}$ (the method proposed by Abbasbandy et al.* [6]*), $X = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ from equations (14) and (16) and X (the method proposed by Otadi and Mosleh), respectively. Then $m_O \leq m_M \leq m_A$, $m_A - m_O = 2mn$ and $m_M - m_O = mn$.*

Proof. According to equation (12), we have

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

where

$$D = \frac{1}{2}[(B+C)^+ + (B-C)^+], \; E = \frac{1}{2}[(B+C)^+ - (B-C)^+].$$

Therefore, for determining S^+, we need to compute $(B+C)^+$ and $(B-C)^+$. Now, assume that M is $m\times n$ matrix and denote by $h_m(M)$ the number of multiplication operations that are required to calculate M^+. It is clear that

$$h_m(S) = h_m(B+C) + h_m(B-C) = 2h_m(A)$$

and hence

$$m_A = 2h_m(A) + 4mn.$$

For computing $X = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$, from equations (14) and (16), the number of multiplication operations are $h_m(A)+mn$ and $h_m(D)+mn$. Therefore

$$m_O = 2h_m(A) + 2mn$$

and hence $m_A - m_O = 2mn$. Also, for computing $\underline{x}+\overline{x} = (\underline{x}_1+\overline{x}_1, \underline{x}_2+\overline{x}_2, \ldots, \underline{x}_n+\overline{x}_n)^T$ and $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)^T$ from the method proposed by Otadi and Mosleh, the number of multiplication operations are $h_m(A)+mn$ and $h_m(B+C)+2mn$, respectively. Clearly $h_m(B+C) = h_m(A)$, so

$$m_M = 2h(A) + 3mn$$

and hence $m_M - m_O = mn$. This proves Theorem. □

Now, we assume in fuzzy linear system (1) $n = m$ and the coefficients matrix A is nonsingular, then $A^+ = A^{-1}$. Therefore, we have the following theorem.

Theorem 4.3. *Assume that m_F, m_E and m_O are the number of multiplication operations that are required to calculate $\hat{X} = S^{-1}\hat{Y}$ (the method proposed by Friedman et al.* [16]*), X from equations (7)-(9) (the method proposed by Ezzati* [15]*) and X from equations (14) and (16), respectively. Then $m_O \le m_E \le m_F$ and $m_F - m_E = m_E - m_O = n^2$.*

Proof. According to equation (4), we have

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

where

$$D = \frac{1}{2}[(B+C)^{-1} + (B-C)^{-1}],\ E = \frac{1}{2}[(B+C)^{-1} - (B-C)^{-1}].$$

Therefore, for determining S^{-1}, we need to compute $(B+C)^{-1}$ and $(B-C)^{-1}$. Now, assume that M is a $n\times n$ matrix and denote by $h_n(M)$ the number of multiplication operations that are required to calculate M^{-1}. It is clear that

$$h(S) = h(B+C) + h(B-C) = 2h_n(A)$$

and hence

$$m_F = 2h_n(A) + 4n^2.$$

For computing $\underline{x}+\overline{x} = (\underline{x}_1+\overline{x}_1, \underline{x}_2+\overline{x}_2, \ldots, \underline{x}_n+\overline{x}_n)^T$ from equation (7) and $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)^T$ from equation (9), the number of multiplication operations are $h_n(A)+n^2$ and $h_n(B+C)+2n^2$, respectively. Clearly $h_n(B+C) = h_n(A)$, so

$$m_E = 2h_n(A) + 3n^2$$

and hence $m_E - m_F = n^2$. For computing $X = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$, from equations (14) and (16), the number of multiplication operations are $h_n(A)+n^2$ and $h_n(D)+n^2$. Therefore

$$m_O = 2h_n(A) + 2n^2$$

and hence $m_F - m_E = m_E - m_O = n^2$. This proves Theorem. $\square$

5. **Numerical Examples**

Example 5.1. Consider the 2×3 fuzzy linear system

$$\begin{cases} x_1 - x_3 = (-1+2r, 4-3r), \\ x_1 + 2x_2 = (1+4r, 10-5r). \end{cases}$$

By simple calculation

$$A^+ = \begin{bmatrix} 0.5415 & 0.5122 & 0.6667 \\ 0.8313 & -0.4449 & -0.3333 \\ -0.1258 & -0.7347 & 0.6667 \end{bmatrix} \times \begin{bmatrix} 0.4343 & 0 \\ 0 & 0.7676 \\ 0 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.2898 & 0.9571 \\ 0.9571 & -0.2898 \end{bmatrix}$$

and

$$F^+ = \begin{bmatrix} 0.5415 & 0.5122 & -0.6667 \\ 0.8313 & -0.4449 & 0.3333 \\ 0.1258 & 0.7347 & 0.6667 \end{bmatrix} \times \begin{bmatrix} 0.4343 & 0 \\ 0 & 0.7676 \\ 0 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.2898 & 0.9571 \\ 0.9571 & -0.2898 \end{bmatrix}.$$

By using equation (15), we have:

$$\begin{bmatrix} x_1^c(r) \\ x_2^c(r) \\ x_3^c(r) \end{bmatrix} = \begin{bmatrix} 1.2778 - 0.2778r \\ 2.1111 - 0.1111r \\ -0.2222 + 0.2222r \end{bmatrix}$$

and

$$\begin{bmatrix} x_1^d(r) \\ x_2^d(r) \\ x_3^d(r) \end{bmatrix} = \begin{bmatrix} 1.6111 - 1.6111r \\ 1.4444 - 1.4444r \\ 0.8889 - 0.8889r \end{bmatrix},$$

and hence

$$x_1(r) = (-0.3333 + 1.3333r, 2.8889 - 1.8889r),$$
$$x_2(r) = (0.6667 + 1.3333r, 3.5555 - 1.5555r),$$
$$x_3(r) = (-1.1111 + 1.1111r, 0.6667 - 0.6667r).$$

According to this fact that $\underline{x}_i \leq \overline{x}_i,\ i = 1, 2, 3$ are monotonic decreasing functions then the fuzzy solution $x_i(r) = (\underline{x}_i(r), \overline{x}_i(r)),\ i = 1, 2, 3$, is a minimal fuzzy solution.

Example 5.2. Consider the 3×2 fuzzy linear system

$$\begin{cases} x_1 + x_2 = (r, 2-r), \\ x_1 - 2x_2 = (2+r, 3), \\ 2x_1 + x_2 = (-2, -1-r). \end{cases}$$

By a simple calculation

$$A^{+}=\begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}\times\begin{bmatrix} 0.378 & 0 & 0 \\ 0 & 0.4472 & 0 \end{bmatrix}\times$$

$$\begin{bmatrix} -0.5345 & 0.2673 & -0.8018 \\ 0 & -0.9487 & -0.3162 \\ -0.8452 & -0.169 & 0.5071 \end{bmatrix}$$

and

$$F^{+}=\begin{bmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}\times\begin{bmatrix} 0.378 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\times$$

$$\begin{bmatrix} -0.4264 & -0.6396 & -0.6396 \\ 0 & -0.7071 & 0.7071 \\ -0.9045 & 0.3015 & 0.3015 \end{bmatrix}.$$

By using equation (15), we have:

$$\begin{bmatrix} x_1^c(r) \\ x_2^c(r) \end{bmatrix}=\begin{bmatrix} 0.2429-0.0429r \\ -0.9571-0.2429r \end{bmatrix}$$

and

$$\begin{bmatrix} x_1^d(r) \\ x_2^d(r) \end{bmatrix}=\begin{bmatrix} 0.2273-0.2273r \\ 0.2273-0.2273r \end{bmatrix},$$

and hence

$$\begin{aligned} x_1(r) &= (0.0156+0.1844r, 0.4702-0.2702r), \\ x_2(r) &= (-1.1844-0.0156r, -0.7298-0.4702r). \end{aligned}$$

The fact that x_2 is not fuzzy number, therefore the fuzzy linear system is not fuzzy minimal solution.

Example 5.3. [18] Consider the 3×2 fuzzy linear system

$$\begin{cases} -x_1+2x_2=(2r-1, 3-2r), \\ 3x_1+4x_2=(2r+15, 20-3r), \\ 2x_1-x_2=(r+2, 6-3r). \end{cases}$$

By a simple calculation

$$A^{+}=\begin{bmatrix} -0.5473 & -0.8369 \\ -0.8369 & 0.5473 \end{bmatrix}\times\begin{bmatrix} 0.1952 & 0 & 0 \\ 0 & 0.3377 & 0 \end{bmatrix}\times$$

$$\begin{bmatrix} -0.2199 & -0.9742 & -0.0503 \\ 0.6523 & -0.1085 & -0.7501 \\ 0.7253 & -0.1978 & 0.6594 \end{bmatrix}$$

and

$$F^{+}=\left[\begin{array}{cc}-0.627 & 0.779\\ -0.779 & -0.627\end{array}\right]\times\left[\begin{array}{ccc}0.1718 & 0 & 0\\ 0 & 0.9442 & 0\end{array}\right]\times$$

$$\left[\begin{array}{ccc}-0.3754 & -0.8585 & -0.3493\\ -0.4485 & -0.1615 & 0.8791\\ -0.8111 & 0.4867 & -0.3244\end{array}\right].$$

By using equation (15), we have:

$$\left[\begin{array}{c}x_1^c(r)\\ x_2^c(r)\end{array}\right]=\left[\begin{array}{c}3.0674-0.2848r\\ 2.0696+0.0609r\end{array}\right]$$

and

$$\left[\begin{array}{c}x_1^d(r)\\ x_2^d(r)\end{array}\right]=\left[\begin{array}{c}0.7237-0.7237r\\ 0.2105-0.2105r\end{array}\right],$$

and hence

$$x_1(r)=(2.3437+0.4389r, 3.7911-1.0085r),$$
$$x_2(r)=(1.8591+0.2714r, 2.2801-0.1496r).$$

6. Conclusions

In this paper, we propose a general model for solving fuzzy linear system. The original system with crisp coefficient matrix A is replaced by two $m \times n$ crisp linear systems. Also, conditions for the existence of a minimal fuzzy solution to the fuzzy linear system, is presented. The proposed method possesses several properties which makes it better, or at least more suitable, than the existing methods.

Acknowledgements. We would like to present our sincere thanks to the Prof. M. Mashinchi, Prof. R. A. Borzooei, Prof. S. Abbasbandy and referees for their valuable suggestions.

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