ON UPPER AND LOWER ALMOST WEAKLY CONTINUOUS FUZZY MULTIFUNCTIONS

S. E. ABBAS, M. A. HEBESHI AND I. M. TAHA

Abstract. The aim of this paper is to introduce the concepts of fuzzy upper and fuzzy lower almost continuous, weakly continuous and almost weakly continuous multifunctions. Several characterizations and properties of these multifunctions along with their mutual relationships are established in L-fuzzy topological spaces

1. Introduction

Kubiak [14] and Šostak [24] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang [6] and Goguen [8]). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological type structures on fuzzy powersets was developed in [9-11,14,15,24-26].

Berge [5] introduced the concept multimapping $F : X \to Y$ where X and Y are topological spaces and Popa [21,22] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [6], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (e.g. see [2,3,17,18,20]). Also, Tsiporkova et al., [27,28] introduced the Continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [6]. Later, Abbas et al., [1] introduced the concepts of fuzzy upper and fuzzy lower semi-continuous multifunctions on L-fuzzy topological spaces in Šostak sense.

Throughout this paper, nonempty sets will be denoted by X, Y etc.. Let a complete lattice $L = (L,\leq,\vee,\wedge,')$ be a complete distributive complete lattice with an order-reversing involution on it, and with the smallest element $\perp$ and the largest element $\top$ ($\perp \neq \top$). The family of all L-fuzzy sets in X is denoted by L^X and $L_\circ = L - \{0\}$. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The complement of an L-fuzzy set λ is denoted by λ^c. This symbol $\multimap$ for a multifunction. All other notations are standard notations of L-fuzzy set theory.

Definition 1.1. [1] Let $F : X \multimap Y$, then F is called a fuzzy multifunction (FM, for short) iff $F(x) \in L^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$.

Received: February 2013; Revised: May 2014; Accepted: December 2014

Key words and phrases: L-fuzzy topology, Fuzzy multifunction, Fuzzy upper and lower almost continuous, Weakly continuous, Almost weakly continuous, Composition, Union.

The domain of F, denoted by $dom(F)$ and the range of F, denoted by $rng(F)$, for any $x \in X$ and $y \in Y$, are defined by:

$$dom(F)(x) = \bigvee_{y \in Y} G_F(x, y) \qquad and \qquad rng(F)(y) = \bigvee_{x \in X} G_F(x, y).$$

Definition 1.2. [1] Let $F : X \multimap Y$ be a FM. Then F is called:

(1) Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x, y_0) = \top$.
(2) A crisp iff $G_F(x, y) = \top$ for each $x \in X$ and $y \in Y$.

Definition 1.3. [1] Let $F : X \multimap Y$ be a FM. Then,

(1) The image of $\lambda \in L^X$ is an L-fuzzy set $F(\lambda) \in L^Y$ defined by:

$$F(\lambda)(y) = \bigvee_{x \in X} [G_F(x, y) \wedge \lambda(x)].$$

(2) The lower inverse of $\mu \in L^Y$ is an L-fuzzy set $F^l(\mu) \in L^X$ defined by:

$$F^l(\mu)(x) = \bigvee_{y \in Y} [G_F(x, y) \wedge \mu(y)].$$

(3) The upper inverse of $\mu \in L^Y$ is an L-fuzzy set $F^u(\mu) \in L^X$ defined by:

$$F^u(\mu)(x) = \bigwedge_{y \in Y} [G^c_F(x, y) \vee \mu(y)].$$

Theorem 1.4. [1] Let $F : X \multimap Y$ be a FM. Then,

(1) $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.
(2) $F^l(\mu_1) \leq F^l(\mu_2)$ and $F^u(\mu_1) \leq F^u(\mu_2)$ if $\mu_1 \leq \mu_2$.
(3) $F^l(\mu^c) = (F^u(\mu))^c$.
(4) $F^u(\mu^c) = (F^l(\mu))^c$.
(5) $F(F^u(\mu)) \leq \mu$ if F is a crisp.
(6) $F^u(F(\lambda)) \geq \lambda$ if F is a crisp.

Definition 1.5. [1] Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FMs. Then the composition $H \circ F$ is defined by: $((H \circ F)(x))(z) = \bigvee_{y \in Y} [G_F(x, y) \wedge G_H(y, z)]$.

Theorem 1.6. [1] Let $F : X \multimap Y$ and $H : Y \multimap Z$ be FMs. Then we have the following:

(1) $(H \circ F) = F(H)$.
(2) $(H \circ F)^u = F^u(H^u)$.
(3) $(H \circ F)^l = F^l(H^l)$.

Theorem 1.7. [1] Let $F_i : X \multimap Y$ be a FM. Then,

(1) $(\bigcup_{i \in \Gamma} F_i)(\lambda) = \bigvee_{i \in \Gamma} F_i(\lambda)$.
(2) $(\bigcup_{i \in \Gamma} F_i)^l(\mu) = \bigvee_{i \in \Gamma} F^l_i(\mu)$.
(3) $(\bigcup_{i \in \Gamma} F_i)^u(\mu) = \bigwedge_{i \in \Gamma} F^u_i(\mu)$.

Definition 1.8. [11, 14, 16, 24] An L-fuzzy topological space (L-fts, for short) is a pair (X, τ), where X is a nonempty set and $\tau : L^X \to L$ is a mapping satisfying the following properties:

(O1) $\tau(\underline{\top}) = \tau(\underline{\perp}) = \top$,
(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$, for any $\lambda_1, \lambda_2 \in L^X$,
(O3) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$, for any $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

Then τ is called an L-fuzzy topology on X. For every $\lambda \in L^X$, $\tau(\lambda)$ is called the degree of openness of the L-fuzzy set λ.

A mapping $f : (X, \tau) \to (Y, \eta)$ is said to be continuous with respect to L-fuzzy topologies τ and η iff $\tau(f^{-1}(\mu)) \geq \eta(\mu)$ for each $\mu \in L^Y$.

Theorem 1.9. [7, 12, 13, 16] Let (X, τ) be an L-fts. Then for each $\lambda \in L^X$, $r \in L_\circ$ we define L-fuzzy operators C_τ and $I_\tau : L^X \times L_\circ \to L^X$ as follows:

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in L^X \ : \lambda \leq \mu, \tau(\mu^c) \geq r\}.$$
$$I_\tau(\lambda, r) = \bigvee\{\mu \in L^X \ : \mu \leq \lambda, \tau(\mu) \geq r\}.$$

For λ, $\mu \in L^X$ and $r, s \in L_\circ$ the operator C_τ satisfies the following statements:

(C1) $C_\tau(\underline{\perp}, r) = \underline{\perp}$.
(C2) $\lambda \leq C_\tau(\lambda, r)$.
(C3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(C4) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.
(C5) $C_\tau(\lambda, r) = \lambda$ iff $\tau(\lambda^c) \geq r$.
(C6) $C_\tau(\lambda^c, r) = (I_\tau(\lambda, r))^c$ and $I_\tau(\lambda^c, r) = (C_\tau(\lambda, r))^c$.

Definition 1.10. [4, 12, 23] Let (X, τ) be an L-fts. Then for each $\lambda, \mu \in L^X$ and $r \in L_\circ$. Then λ is called:

(1) r-fuzzy semi-open (r-fso, for short) iff $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.
(2) r-fuzzy preopen (r-fpo, for short) iff $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$.
(3) r-fuzzy regular open (r-fro, for short) iff $\lambda = I_\tau(C_\tau(\lambda, r), r)$.
(4) r-fuzzy semi-closed (r-fsc, for short) iff $I_\tau(C_\tau(\lambda, r), r) \leq \lambda$.
(5) r-fuzzy preclosed (r-fpc, for short) iff $C_\tau(I_\tau(\lambda, r), r) \leq \lambda$.
(6) r-fuzzy regular closed (r-frc, for short) iff $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Theorem 1.11. [12] Let (X, τ) be an L-fts. Then for each $\lambda \in L^X$, $r \in L_\circ$ we define L-fuzzy operators PC_τ and $PI_\tau : L^X \times L_\circ \to L^X$ as follows:

$$PC_\tau(\lambda, r) = \bigwedge\{\mu \in L^X \ : \lambda \leq \mu,\ \mu \ is \ r - fpc\}.$$
$$PI_\tau(\lambda, r) = \bigvee\{\mu \in L^X \ : \mu \leq \lambda,\ \mu \ is \ r - fpo\}.$$

For λ, $\mu \in L^X$ and $r, s \in L_\circ$ the operators PC_τ and PI_τ satisfies the following: $PC_\tau(\lambda^c, r) = (PI_\tau(\lambda, r))^c$ and $PI_\tau(\lambda^c, r) = (PC_\tau(\lambda, r))^c$.

Lemma 1.12. [19] Let (X, τ) be an L-fts. Then $PC_\tau(\lambda) = \lambda \vee C_\tau(I_\tau(\lambda, r), r)$ for each $\lambda \in L^X$, $r \in L_\circ$.

Definition 1.13. [1] Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η) and let $r \in L_\circ$. Then F is called:

(1) FUS-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.

(2) FLS-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exist $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(3) Continuous if it is both FUS-continuous and FLS-continuous.

Proposition 1.14. [1] If F is normalized, then F is FUS-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.

Theorem 1.15. [1] Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η) and let $\mu \in L^Y$. Then we have the following:

(1) If F is normalized, then F is FUS-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.

(2) F is FLS-continuous iff $\tau(F^l(\mu)) \geq \eta(\mu)$.

2. Almost Continuous Fuzzy Multifunctions

Definition 2.1. Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η) and let $r \in L_\circ$. Then F is called:

(1) Fuzzy upper almost continuous (FUA-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(I_\eta(C_\eta(\mu, r), r)$.

(2) Fuzzy lower almost continuous (FLA-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exist $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(I_\eta(C_\eta(\mu, r), r)$.

(3) FUA-continuous (resp. FLA-continuous) iff it is FUA-continuous (resp. FLA-continuous) at every $x_t \in dom(F)$.

Proposition 2.2. If F is normalized, then F is FUA-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exist $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(I_\eta(C_\eta(\mu, r), r)$.

The following implications hold:

1. FUS-continuous $\Rightarrow$ FUA-continuous.
2. FLS-continuous $\Rightarrow$ FLA-continuous.

In general the converses are not true.

Theorem 2.3. Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η) and let $\mu \in L^Y$, then the following are equivalent:

(1) F is FLA-continuous.

(2) $F^l(\mu) \leq I_\tau(F^l(I_\eta(C_\eta(\mu, r), r)), r)$, if $\eta(\mu) \geq r$.

(3) $\tau(F^l(\mu)) \geq r$, if μ is r-fro set.

(4) $\tau(F^l(I_\eta(C_\eta(\mu, r), r))) \geq r$, if $\eta(\mu) \geq r$.

Proof. (1) $\Rightarrow$ (2). Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(I_\eta(C_\eta(\mu, r), r)$. Thus $x_t \in \lambda \leq F^l(I_\eta(C_\eta(\mu, r), r)$ and hence $F^l(\mu) \leq I_\tau(F^l(I_\eta(C_\eta(\mu, r), r)), r)$.

(2) $\Rightarrow$ (3) Let μ be r-fro set and hence by (2), $F^l(\mu) \leq I_\tau(F^l(I_\eta(C_\eta(\mu, r), r)), r)$ $= I_\tau(F^l(\mu), r)$. Then $\tau(F^l(\mu)) \geq r$.

(3) ⇒ (4) Since $\eta(\mu) \geq r$ then $I_\eta(C_\eta(\mu, r), r)$ is r-fro, and so

$$\tau(F^l(I_\eta(C_\eta(\mu, r), r))) \geq r.$$

(4) ⇒ (1). Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ with $x_t \in F^l(\mu)$. then we have by (4), that $F^l(I_\eta(C_\eta(\mu, r), r) = \lambda$ (say) and $\tau(\lambda) \geq r$. Since $\mu \leq I_\eta(C_\eta(\mu, r), r))$ then $x_t \in F^l(\mu) \leq F^l(I_\eta(C_\eta(\mu, r), r) = \lambda$ and (1) follows. □

Theorem 2.4. Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η) and let $\mu \in L^Y$. Then the following are equivalent:

(1) F is FLA-continuous.
(2) $F^u(\mu) \geq C_\tau(F^u(C_\eta(I_\eta(\mu, r), r)), r)$, if $\eta(\mu^c) \geq r$.
(3) $\tau((F^u(\mu))^c) \geq r$, if μ is r-frc set.

Proof. (1) ⇒ (2). Let $\mu \in L^Y$, $\eta(\mu^c) \geq r$ and let F be FLA-continuous. Then by Theorem 2.3(2), $F^l(\mu^c) \leq I_\tau(F^l(I_\eta(C_\eta(\mu^c, r), r)), r) = I_\tau(F^l(I_\eta([I_\eta(\mu, r)]^c, r)), r) = I_\tau(F^l([C_\eta(I_\eta(\mu, r), r)]^c), r)$.

Since,
$[F^u(\mu)]^c = F^l(\mu^c) \leq I_\tau(F^l([C_\eta(I_\eta(\mu, r), r)]^c), r) = I_\tau([F^u(C_\eta(I_\eta(\mu, r), r))]^c, r) = [C_\tau(F^u(C_\eta(I_\eta(\mu, r), r)), r)]^c$, we obtain $F^u(\mu) \geq C_\tau(F^u(C_\eta(I_\eta(\mu, r), r)), r)$.

(2) ⇒ (3). Let $\mu \in L^Y$ and let μ be an r-frc set, we have by (2), that

$$F^u(\mu) \geq C_\tau(F^u(C_\eta(I_\eta(\mu, r), r)), r) = C_\tau(F^u(\mu), r).$$

Then $\tau((F^u(\mu))^c) \geq r$.

(3) ⇒ (1). Let μ be an r-fro set, then μ^c is an r-frc and hence by (3),

$$\tau((F^u(\mu^c))^c) \geq r, \ i.e., \ \tau(F^l(\mu)) \geq r.$$

Then by Theorem 2.3(3), F is FLA-continuous. □

Theorem 2.5. Let $F : X \multimap Y$ be a FM and normalized between (X, τ), and (Y, η) and let $\mu \in L^Y$, then the following are equivalent:

(1) F is FUA-continuous.
(2) $F^u(\mu) \leq I_\tau(F^u(I_\eta(C_\eta(\mu, r), r)), r)$, if $\eta(\mu) \geq r$.
(3) $\tau(F^u(\mu)) \geq r$, if μ is r-fro set.
(4) $\tau(F^u(I_\eta(C_\eta(\mu, r), r))) \geq r$, if $\eta(\mu) \geq r$.

Proof. This can be proved in a similar way as Theorem 2.3. □

Theorem 2.6. Let $F : X \multimap Y$ be a FM and normalized between (X, τ), and (Y, η) and let $\mu \in L^Y$. Then the following are equivalent:

(1) F is FUA-continuous.
(2) $F^l(\mu) \geq C_\tau(F^l(C_\eta(I_\eta(\mu, r), r)), r)$, if $\eta(\mu^c) \geq r$.
(3) $\tau((F^l(\mu))^c) \geq r$, if μ is r-frc set.

Proof. This can be proved in a similar way as Theorem 2.4. □

Example 2.7. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. We assume that $\top = 1$ and $\bot = 0$. Define fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if } \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.5}, \\ \bot, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} \top, & \text{if } \mu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \mu = \underline{0.5}, \\ \frac{1}{3}, & \text{if } \mu = \underline{0.4}, \\ \bot, & \text{otherwise}. \end{cases}$$

(1) Since $\underline{0.5}$ is $\frac{1}{2}$-fro, $F^l(\underline{0.5}) = \underline{0.5}$ and $\tau(F^l(\underline{0.5})) = \frac{1}{2}$. Then, F is FLA-continuous.

(2) Since $F^l(\underline{0.4}) = \underline{0.4}$, $\tau(F^l(\underline{0.4})) = 0$ and $\eta(\underline{0.4}) = \frac{1}{3}$. Hence, $\tau(F^l(\underline{0.4})) \not\geq \eta(\underline{0.4})$. Then, F is not FLS-continuous.

(3) Since $\underline{0.5}$ is $\frac{1}{2}$-fro, $F^u(\underline{0.5}) = \underline{0.5}$ and $\tau(F^u(\underline{0.5})) = \frac{1}{2}$. Then, F is FUA-continuous.

(4) Since $F^u(\underline{0.4}) = \underline{0.4}$, $\tau(F^u(\underline{0.4})) = 0$ and $\eta(\underline{0.4}) = \frac{1}{3}$. Hence, $\tau(F^u(\underline{0.4})) \not\geq \eta(\underline{0.4})$. Thus, F is not FUS-continuous.

Theorem 2.8. Let $F : X \multimap Y$ be a FM between (X, τ), and (Y, η). Then F is FLA-continuous iff $C_\tau(F^u(\mu), r) \leq F^u(C_\eta(\mu, r))$, for any r-fso set $\mu \in L^Y$.

Proof. Let F be FLA-continuous and $\mu \in L^Y$ be r-fso. Then $\mu \leq C_\eta(I_\eta(\mu, r), r) = \nu$ (say), where ν is r-frc. By Theorem 2.4(3), $\tau((F^u(\nu))^c) \geq r$ and thus $C_\tau(F^u(\mu), r) \leq C_\tau(F^u(\nu), r) = F^u(\nu) = F^u(C_\eta(I_\eta(\mu, r), r)) \leq F^u(C_\eta(\mu, r))$.

$\underline{Conversely}$, since every r-frc set is r-fso, for any r-frc set $\mu \in L^Y$ we have $\overline{C_\tau(F^u(\mu), r)} \leq F^u(C_\eta(\mu, r)) = F^u(\mu)$. Thus $\tau((F^u(\mu))^c) \geq r$, and hence by Theorem 2.4(3), F is FLA-continuous. □

Theorem 2.9. Let $F : X \multimap Y$ be a FM and normalized between (X, τ), (Y, η). Then F is FUA-continuous iff $C_\tau(F^l(\mu), r) \leq F^l(C_\eta(\mu, r))$ for any r-fso set $\mu \in L^Y$.

Proof. This can be proved in a similar way as Theorem 2.8. □

Theorem 2.10. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be FMs and let (X, τ), (Y, η) and (Z, δ) be L-fts. If F is FLS-continuous and H is FLA-continuous, then $H \circ F$ is FLA-continuous.

Proof. Let F be FLS-continuous, H be FLA-continuous and $\gamma \in L^Z$ be r-fro set. Then from Theorem 1.15(2) and Theorem 2.3(3) we have $\tau((H \circ F)^l(\gamma)) = \tau((F^l(H^l(\gamma))) \geq \eta(H^l(\gamma)) \geq r$. Thus $H \circ F$ is FLA-continuous. □

Theorem 2.11. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be FMs and let (X, τ), (Y, η) and (Z, δ) be L-ftss. If F and H are normalized, F is FUS-continuous and H is FUA-continuous, then $H \circ F$ is FUA-continuous.

Proof. This can be proved in a similar way as Theorem 2.10. □

Theorem 2.12. Let $\{F_i\}_{i\in\Gamma}$ be a family of FLA-continuous between two L-ftss (X,τ) and (Y,η). Then $\bigcup_{i\in\Gamma} F_i$ is FLA-continuous.

Proof. Let $\mu \in L^Y$, then $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma}(F_i^l(\mu))$ by Theorem 1.7(2). Since $\{F_i\}_{i\in\Gamma}$ is a family of FLA-continuous between tow L-ftss (X,τ) and (Y,η), then $\tau(F_i^l(\mu)) \geq r$ for any r-fro set μ and $i \in \Gamma$. Then we have $\tau(\bigcup_{i\in\Gamma} F_i)^l(\mu)) = \tau(\bigvee_{i\in\Gamma}(F_i^l(\mu)) \geq \bigwedge_{i\in\Gamma} \tau(F_i^l(\mu)) \geq r$ for each r-fro set μ. Hence $\bigcup_{i\in\Gamma} F_i$ is FLA-continuous. □

Theorem 2.13. Let F_1 and F_2 be two normalized FUA-continuous between L-ftss (X,τ) and (Y,η). Then $F_1 \bigcup F_2$ is FUA-continuous.

Proof. Let $\mu \in L^Y$, then $(F_1 \bigcup F_2)^u(\mu) = F_1^u(\mu) \wedge F_2^u(\mu)$ by Theorem 1.7(3). Since F_1 and F_2 are two normalized FUA-continuous between L-ftss (X,τ) and (Y,η), then $\tau(F_i^u(\mu)) \geq r$, for any r-fro set μ and $i \in \{1,2\}$. Then we have $\tau((F_1 \bigcup F_2)^u(\mu)) = \tau(F_1^u(\mu) \wedge F_2^u(\mu)) \geq \tau(F_1^u(\mu)) \wedge \tau(F_2^u(\mu)) \geq r$ for each r-fro set μ. Hence $F_1 \bigcup F_2$ is FUA-continuous. □

3. **Weakly Continuous Fuzzy Multifunctions**

Definition 3.1. Let $F : X \multimap Y$ be a FM between L-ftss (X,τ), and (Y,η) and let $r \in L_\circ$. Then F is called:

(1) Fuzzy upper weakly continuous (FUW-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(C_\eta(\mu,r))$.

(2) Fuzzy lower weakly continuous (FLW-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu,r))$.

(3) FUW-continuous (resp. FLW-continuous) iff it is FUW-continuous (resp. FLW-continuous) at every $x_t \in dom(F)$.

Proposition 3.2. If F is normalized, then F is FUW-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu,r))$.

Then the following implications hold:

1. FUA-continuous $\Rightarrow$ FUW-continuous.
2. FLA-continuous $\Rightarrow$ FLW-continuous.

In general the converses are not true.

Theorem 3.3. Let $F : X \multimap Y$ be a FM between L-ftss (X,τ), and (Y,η). Then F is FLW-continuous iff $F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu,r)),r)$, for any $\mu \in L^Y$ and $\eta(\mu) \geq r$.

Proof. Let F be FLW-continuous, $\mu \in L^Y$ and $\eta(\mu) \geq r$. If $x_t \in F^l(\mu)$, then there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu,r))$ and hence $\lambda \leq I_\tau(F^l(C_\eta(\mu,r)),r)$. Thus $F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu,r)),r)$.

Conversely, let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then $x_t \in F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu, r)), r) = \lambda$ (say). Thus $x_t \in \lambda$ and $\tau(\lambda) \geq r$ such that $\lambda = I_\tau(F^l(C_\eta(\mu, r)), r) \leq F^l(C_\eta(\mu, r))$, then F is FLW-continuous. □

Theorem 3.4. Let $F : X \multimap Y$ be a FM and normalized between L-ftss (X, τ), (Y, η). Then F is FUW-continuous iff $F^u(\mu) \leq I_\tau(F^u(C_\eta(\mu, r)), r)$ for any $\mu \in L^Y$ and $\eta(\mu) \geq r$.

Proof. This can be proved in a similar way as Theorem 3.3. □

Example 3.5. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. Define $\mu(y_1) = 0.4$, $\mu(y_2) = 0.1$, $\mu(y_3) = 0.2$. We assume that $\top = 1$ and $\bot = 0$. Define fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if} \quad \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{0.5}, \\ \bot, & \text{otherwise}, \end{cases}$$

$$\eta(\nu) = \begin{cases} \top, & \text{if} \quad \nu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \nu = \mu, \\ \bot, & \text{otherwise}. \end{cases}$$

(1) Since $\eta(\mu) = \frac{1}{2}$, $C_\eta(\mu, \frac{1}{2}) = \mu^c$, $F^u(\mu)(x_1) = 0.1$, $F^u(\mu)(x_2) = 0.2$,

$$F^u(C_\eta(\mu, \frac{1}{2}))(x_1) = 0.9,\ F^u(C_\eta(\mu, \frac{1}{2}))(x_2) = 0.6,$$

$$I_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2}) = \underline{0.5}.$$

Hence, $F^u(\mu) \leq I_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2})$. Then, F is FUW-continuous.

(2) Since μ is $\frac{1}{2}$-fro and $\tau(F^u(\mu)) = \bot$, i.e., $\tau(F^u(\mu)) \ngeq \frac{1}{2}$.
Then, F is not FUA-continuous.

(3) Since $\eta(\mu) = \frac{1}{2}$, $C_\eta(\mu, \frac{1}{2}) = \mu^c$, $F^l(\mu)(x_1) = 0.1$, $F^l(\mu)(x_2) = 0.4$,

$$F^l(C_\eta(\mu, \frac{1}{2}))(x_1) = 0.9,\ F^l(C_\eta(\mu, \frac{1}{2}))(x_2) = 0.8,$$

$$I_\tau(F^l(C_\eta(\mu, \frac{1}{2})), \frac{1}{2}) = \underline{0.5}.$$

Hence, $F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu, \frac{1}{2})), \frac{1}{2})$. , F is FLW-continuous.

(4) Since μ is $\frac{1}{2}$-fro and $\tau(F^l(\mu)) = \bot$, i.e., $\tau(F^l(\mu)) \ngeq \frac{1}{2}$ Then, F is not FLA-continuous.

Theorem 3.6. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η). If F is FLW-continuous, then $F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu, r)), r)$ for any r-fpo set $\mu \in L^Y$.

Proof. Let F be FLW-continuous and $\mu \in L^Y$ be r-fpo set. If $x_t \in F^l(\mu) \leq F^l(I_\eta(C_\eta(\mu, r), r))$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that

$$\lambda \leq F^l(C_\eta(I_\eta(C_\eta(\mu, r), r), r)) \leq F^l(C_\eta(\mu, r)).$$

Thus $\lambda \leq I_\tau(F^l(C_\eta(\mu, r)), r)$ and hence $F^l(\mu) \leq I_\tau(F^l(C_\eta(\mu, r)), r)$. □

Theorem 3.7. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X, τ), (Y, η). If F is FUW-continuous, then $F^u(\mu) \leq I_\tau(F^u(C_\eta(\mu, r)), r)$ for any r-fpo set $\mu \in L^Y$.

Proof. This can be proved in a similar way as Theorem 3.6. □

Definition 3.8. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η). F is said to be fuzzy preopen iff $F(\lambda)$ is r-fpo for every $\lambda \in L^X$ and $\tau(\lambda) \geq r$.

Theorem 3.9. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η). If F is FUW-continuous, fuzzy preopen and a crisp, then F is FUA-continuous.

Proof. Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^u(\mu)$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$, by Theorem 1.4(5) $F(\lambda) \leq F(F^u(C_\eta(\mu, r))) \leq C_\eta(\mu, r)$. Since F is fuzzy preopen, $F(\lambda)$ is r-fpo so

$$F(\lambda) \leq I_\eta(C_\eta(F(\lambda), r), r) \leq I_\eta(C_\eta(\mu, r), r)$$

and by Theorem 1.4(6), $\lambda \leq F^u(F(\lambda)) \leq F^u(I_\eta(C_\eta(\mu, r), r))$. Then F is FUA-continuous. □

Definition 3.10. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η). F is said to be pointwise fuzzy preopen iff $F(x_t)$ is r-fpo for every $x_t \in \lambda$ and $\tau(\lambda) \geq r$.

Theorem 3.11. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η). If F is FLW-continuous and pointwise fuzzy preopen, then F is FLA-continuous.

Proof. This can be proved in a similar way as Theorem 3.9. □

4. **Almost Weakly Continuous Fuzzy Multifunctions**

Definition 4.1. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η) and let $r \in L_\circ$. Then F is called:

(1) Fuzzy upper almost weakly continuous ($FUAW$-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, then $x_t \in I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r)$.

(2) Fuzzy lower almost weakly continuous ($FLAW$-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, then $x_t \in I_\tau(C_\tau(F^l(C_\eta(\mu, r)), r), r)$.

(3) $FUAW$-continuous (resp. $FLAW$-continuous) iff it is $FUAW$-continuous (resp. $FLAW$-continuous) at every $x_t \in dom(F)$.

The following implications hold:
1. FLW-continuous $\Rightarrow$ $FLAW$-continuous.
2. FUW-continuous (if F is normalized) $\Rightarrow$ $FUAW$-continuous.
In general the converses are not true.

Theorem 4.2. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η) and let $\mu \in L^Y$ the following are equivalent:

(1) F is $FUAW$-continuous.
(2) $F^u(\mu) \leq I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r)$, if $\eta(\mu) \geq r$.
(3) $C_\tau(I_\tau(F^l(\mu), r), r) \leq F^l(C_\eta(\mu, r))$, if $\eta(\mu) \geq r$.
(4) $PC_\tau(F^l(\mu), r) \leq F^l(C_\eta(\mu, r))$, if $\eta(\mu) \geq r$.
(5) $F^u(\mu) \leq PI_\tau(F^u(C_\eta(\mu, r)), r)$, if $\eta(\mu) \geq r$.
(6) For each $x_t \in dom(F)$ and each $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^u(\mu)$ there exists an r-fpo set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$.

Proof. (1) $\Rightarrow$ (2). Let $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^u(\mu)$, then

$$x_t \in I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r).$$

Therefore, $F^u(\mu) \leq I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r)$.

(2) $\Rightarrow$ (3). Let $\mu \in L^Y$, $\eta(\mu) \geq r$. Since $\eta([C_\eta(\mu, r)]^c) \geq r$, then $[F^l(C_\eta(\mu, r))]^c = F^u([C_\eta(\mu, r)]^c) \leq I_\tau(C_\tau(F^u(C_\eta([C_\eta(\mu, r)]^c, r)), r), r) \leq I_\tau(C_\tau(F^u(\mu^c), r), r)$ $= I_\tau(C_\tau([F^l(\mu)]^c, r), r) = [C_\tau(I_\tau(F^l(\mu), r), r)]^c$. Thus, we obtain

$$C_\tau(I_\tau(F^l(\mu), r), r) \leq F^l(C_\eta(\mu, r)).$$

(3) $\Rightarrow$ (4). Let $\mu \in L^Y$, $\eta(\mu) \geq r$. By Lemma 1.12, we have

$$PC_\tau(F^l(\mu), r) = F^l(\mu) \vee C_\tau(I_\tau(F^l(\mu), r), r) \leq F^l(C_\eta(\mu, r)).$$

(4) $\Rightarrow$ (5). Let $\mu \in L^Y$ and $\eta(\mu) \geq r$. Since $\eta([C_\eta(\mu, r)]^c) \geq r$, then $[PI_\tau(F^u(C_\eta(\mu, r)), r)]^c = PC_\tau([F^u(C_\eta(\mu, r))]^c, r) = PC_\tau(F^l([C_\eta(\mu, r)]^c), r) \leq F^l(C_\eta([C_\eta(\mu, r)]^c, r)) \leq F^l(\mu^c) = [F^u(\mu)]^c$. Thus, we obtain

$$F^u(\mu) \leq PI_\tau(F^u(C_\eta(\mu, r)), r).$$

(5) $\Rightarrow$ (6). Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^u(\mu)$, then

$$x_t \in F^u(\mu) \leq PI_\tau(F^u(C_\eta(\mu, r)), r).$$

Thus, there exists an r-fpo set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$.

(6) $\Rightarrow$ (1) Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^u(\mu)$. There exists an r-fpo set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$. Thus, we obtain $x_t \in \lambda \leq I_\tau(C_\tau(\lambda, r), r) \leq I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r)$., and so F is $FUAW$-continuous. □

Theorem 4.3. Let $F : X \multimap Y$ be a FM between L-ftss (X, τ), and (Y, η) and let $\mu \in L^Y$. Then the following are equivalent:

(1) F is $FLAW$-continuous.
(2) $F^l(\mu) \leq I_\tau(C_\tau(F^l(C_\eta(\mu, r)), r), r)$, if $\eta(\mu) \geq r$.
(3) $C_\tau(I_\tau(F^u(\mu), r), r) \leq F^u(C_\eta(\mu, r))$, if $\eta(\mu) \geq r$.

(4) $PC_\tau(F^u(\mu), r) \leq F^u(C_\eta(\mu, r))$, if $\eta(\mu) \geq r$.

(5) $F^l(\mu) \leq PI_\tau(F^l(C_\eta(\mu, r)), r)$, if $\eta(\mu) \geq r$.

(6) For each $x_t \in dom(F)$ and each $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$ there exists an r-fpo $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu, r))$.

Proof. This can be proved in a similar way as Theorem 4.2. □

Example 4.4. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. Define $\mu(y_1) = 0.3$, $\mu(y_2) = \bot$, $\mu(y_3) = 0.5$. We assume that $\top = 1$ and $\bot = 0$. Define fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if } \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.4}, \\ \bot, & \text{otherwise,} \end{cases}$$

$$\eta(\nu) = \begin{cases} \top, & \text{if } \nu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \nu = \mu, \\ \bot, & \text{otherwise.} \end{cases}$$

(1) Since $\eta(\mu) = \frac{1}{2}$, $C_\eta(\mu, \frac{1}{2}) = \mu^c$,

$$F^u(\mu)(x_1) = \bot,\ F^u(\mu)(x_2) = 0.5,$$

$$F^u(C_\eta(\mu, \frac{1}{2}))(x_1) = 0.9,\ F^u(C_\eta(\mu, \frac{1}{2}))(x_2) = 0.5,$$

$$I_\tau(C_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2}), \frac{1}{2}) = \underline{\top}.$$

Hence, $F^u(\mu) \leq I_\tau(C_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2}), \frac{1}{2})$. Then, F is $FUAW$-continuous.

(2) Since $\eta(\mu) = \frac{1}{2}$ and $I_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2}) = \underline{0.4}$.
Hence, $F^u(\mu) \not\leq I_\tau(F^u(C_\eta(\mu, \frac{1}{2})), \frac{1}{2})$. Then, F is not FUW-continuous.

(3) Moreover, F is not FUA-continuous.

Example 4.5. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = 0.3$. Define $\mu(y_1) = 0.3$, $\mu(y_2) = \bot$, $\mu(y_3) = 0.5$. We assume that $\top = 1$ and $\bot = 0$. Define fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if } \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, \\ \bot, & \text{otherwise,} \end{cases}$$

$$\eta(\nu) = \begin{cases} \top, & \text{if } \nu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if } \nu = \mu, \\ \bot, & \text{otherwise.} \end{cases}$$

(1) Since $\eta(\mu) = \frac{1}{2}$, $C_\eta(\mu, \frac{1}{2}) = \mu^c$, $F^l(\mu)(x_1) = 0.1$, $F^l(\mu)(x_2) = 0.3$,

$$F^l(C_\eta(\mu, \frac{1}{2}))(x_1) = \top,\ F^l(C_\eta(\mu, \frac{1}{2}))(x_2) = 0.5,$$

$$I_\tau(C_\tau(F^l(C_\eta(\mu,\frac{1}{2})),\frac{1}{2}),\frac{1}{2}) = \underline{\top}.$$

Hence, $F^l(\mu) \leq I_\tau(C_\tau(F^l(C_\eta(\mu,\frac{1}{2})),\frac{1}{2}),\frac{1}{2})$. Then, F is $FLAW$-continuous.

(2) Since $\eta(\mu) = \frac{1}{2}$ and $I_\tau(F^l(C_\eta(\mu,\frac{1}{2})),\frac{1}{2}) = \underline{0.2}$.
Hence, $F^l(\mu) \nleq I_\tau(F^l(C_\eta(\mu,\frac{1}{2})),\frac{1}{2})$. Then, F is not FLW-continuous.

(3) Moreover, F is not FLA-continuous.

Theorem 4.6. Let $F : X \multimap Y$ be a FM between L-ftss (X,τ), and (Y,η) and let $\mu \in L^Y$. Then the following are equivalent:

(1) F is $FUAW$-continuous.
(2) $C_\tau(I_\tau(F^l(I_\eta(\mu,r)),r),r) \leq F^l(\mu)$, if $\eta(\mu^c) \geq r$.
(3) $PC_\tau(F^l(I_\eta(\mu,r)),r) \leq F^l(\mu)$, if $\eta(\mu^c) \geq r$.
(4) $PC_\tau(F^l(I_\eta(C_\eta(\nu,r),r)),r) \leq F^l(C_\eta(\nu,r))$, for each $\nu \in L^Y$.
(5) $F^u(I_\eta(\nu,r)) \leq PI_\tau(F^u(C_\eta(I_\eta(\nu,r),r)),r)$, for each $\nu \in L^Y$.

Proof. (1) $\Rightarrow$ (2) Let $\mu \in L^Y$, $\eta(\mu^c) \geq r$. By Theorem 4.2(2) we have $[F^l(\mu)]^c = F^u(\mu^c) \leq I_\tau(C_\tau(F^u(C_\eta(\mu^c,r)),r),r) = I_\tau(C_\tau(F^u([I_\eta(\mu,r)]^c),r),r)$ $= I_\tau(C_\tau([F^l(I_\eta(\mu,r))]^c,r),r) = [C_\tau(I_\tau(F^l(I_\eta(\mu,r)),r),r)]^c$. Therefore, we obtain $C_\tau(I_\tau(F^l(I_\eta(\mu,r)),r),r) \leq F^l(\mu)$.

(2) $\Rightarrow$ (3) Let $\mu \in L^Y$, $\eta(\mu^c) \geq r$. By Lemma 1.12, we have

$$PC_\tau(F^l(I_\eta(\mu,r)),r) = F^l(I_\eta(\mu,r)) \vee C_\tau(I_\tau(F^l(I_\eta(\mu,r)),r),r) \leq F^l(\mu).$$

(3) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (5) Let $\nu \in L^Y$. Then we have $[PI_\tau(F^u(C_\eta(I_\eta(\nu,r),r)),r)]^c =$ $PC_\tau([F^u(C_\eta(I_\eta(\nu,r),r))]^c,r) = PC_\tau(F^l([C_\eta(I_\eta(\nu,r),r)]^c),r) \leq F^l(C_\eta(\nu^c,r))$ $= F^l([I_\eta(\nu,r)]^c) = [F^u(I_\eta(\nu,r))]^c$. Therefore, we obtain

$$F^u(I_\eta(\nu,r)) \leq PI_\tau(F^u(C_\eta(I_\eta(\nu,r),r)),r).$$

(5) $\Rightarrow$ (1) Let $\mu \in L^Y$ and $\eta(\mu) \geq r$. Then $F^u(\mu) \leq PI_\tau(F^u(C_\eta(\mu,r)),r)$ and hence F is $FUAW$-continuous (by Theorem 4.2(5)). □

Theorem 4.7. Let $F : X \multimap Y$ be a FM between L-ftss (X,τ), and (Y,η) and let $\mu \in L^Y$. Then the following are equivalent:

(1) F is $FLAW$-continuous.
(2) $C_\tau(I_\tau(F^u(I_\eta(\mu,r)),r),r) \leq F^u(\mu)$, if $\eta(\mu^c) \geq r$.
(3) $PC_\tau(F^u(I_\eta(\mu,r)),r) \leq F^u(\mu)$, if $\eta(\mu^c) \geq r$.
(4) $PC_\tau(F^u(I_\eta(C_\eta(\nu,r),r)),r) \leq F^u(C_\eta(\nu,r))$, for each $\nu \in L^Y$.
(5) $F^l(I_\eta(\nu,r)) \leq PI_\tau(F^l(C_\eta(I_\eta(\nu,r),r)),r)$, for each $\nu \in L^Y$.

Proof. This can be proved in a similar way as Theorem 4.6. □

Theorem 4.8. Let $F : X \multimap Y$ be a FM and normalized between L-ftss (X,τ), and (Y,η). If F is $FUAW$-continuous and FLA-continuous then F is FUW-continuous.

Proof. Let $\mu \in L^Y$, $\eta(\mu) \geq r$ and let F be $FUAW$-continuous. By Theorem 4.2(2) $F^u(\mu) \leq I_\tau(C_\tau(F^u(C_\eta(\mu, r)), r), r)$. Since $C_\eta(\mu, r)$ is r-frc, it follows from Theorem 2.4(3) that $\tau([F^u(C_\eta(\mu, r))]^c) \geq r$. Then $F^u(\mu) \leq I_\tau(F^u(C_\eta(\mu, r)), r)$ and hence it follows from Theorem 3.4 that F is FUW-continuous. □

Theorem 4.9. Let $F : X \multimap Y$ be a FM and normalized between L-ftss (X, τ), and (Y, η). If F is $FLAW$-continuous and FUA-continuous, then F is FLW-continuous.

Proof. This can be proved in a similar way as Theorem 4.8. □

5. **Conclusion**

It is well known that L-fuzzy set theory has been regarded as a generalization of classical set theory in one way. Furthermore, this is an important mathematical tool to deal with uncertainty. One of the main contributions of this paper is to introduce the concepts of fuzzy upper and fuzzy lower almost continuous, weakly continuous and almost weakly continuous multifunctions. Several characterizations and properties of these multifunctions along with their mutual relationships are established in L-fuzzy topological spaces.

Acknowledgements. The authors would like to express their sincere thanks to the editor and referees for their helpful suggestions which improved the presentation of the paper.

References

[1] S. E. Abbas, M. A. Hebeshi and I. M. Taha, *On fuzzy upper and lower semi-continuous multifunctions*, Journal of Fuzzy Mathematics, **22(4)** (2014).
[2] K. M. A. Al-hamadi and S. B. Nimse, *On fuzzy α-continuous multifunctions*, Miskolc Mathematical Notes, **11(2)** (2010), 105-112.
[3] M. Alimohammady, E. Ekici, S. Jafari and M. Roohi, *On fuzzy upper and lower contra-continuous multifunctions*, Iranian Journal of Fuzzy Systems, **8(3)** (2011), 149-158.
[4] H. Aygün and S. E. Abbas, *Some good extensions of compactness in Šostak's L-fuzzy topology*, Hacettepe Journal of Mathematics and Statistics, **36(2)** (2007), 115-125.
[5] C. Berge, *Topological spaces, including a treatment of multi-valued functions, vector spaces and convexity*, Oliver, Boyd London, 1963.
[6] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.
[7] K. C. Chattopadhyay and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and Systems, **54(2)** (1993), 207-212.
[8] J. A. Goguen, *The Fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **43** (1993), 734-742.
[9] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anall. Appl., **78** (1980), 659-673.
[10] U. Höhle and A. P. Šostak, *A general theory of fuzzy topological spaces*, Fuzzy Sets and Systems, **73** (1995), 131-149.
[11] U. Höhle and A. P. Šostak, *Axiomatic Foundations of Fixed-Basis fuzzy topology*, The Handbooks of Fuzzy sets series, Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3), (1999).
[12] Y. C. Kim, A. A. Ramadan and S. E. Abbas, *Weaker forms of continuity in Šostaks fuzzy topology*, Indian J. Pure Appl. Math., **34(2)** (2003), 311-333.
[13] Y. C. Kim, *Initial L-fuzzy closure spaces*, Fuzzy Sets and Systems, **133** (2003), 277-297.
[14] T. Kubiak, *On fuzzy topologies*, Ph. D. Thesis, A. Mickiewicz, Poznan, 1985.

[15] T. Kubiak and A. P. Šostak, *Lower set-valued fuzzy topologies*, Quaestions Math., **20(3)** (1997), 423-429.
[16] Y. Liu and M. Luo, *Fuzzy topology*, World Scientific Publishing, Singapore, 1997.
[17] R. A. Mahmoud, *An application of continuous fuzzy multifunctions*, Chaos, Solitons and Fractals, **17** (2003), 833-841.
[18] M. N. Mukherjee and S. Malakar, *On almost continuous and weakly continuous fuzzy multifunctions*, Fuzzy Sets and Systems, **41** (1991), 113-125.
[19] T. Noiri and V. Popa, *Almost weakly continuous multifunctions*, Demonstratio Mathematica, **VI(2)** (1993), 363-480.
[20] N. S. Papageorgiou, *Fuzzy topology and fuzzy multifunctions*, J. Math. Anal. Appl., **109** (1985), 397-425.
[21] V. Popa, *On characterizations of irresolute multimapping*, J. Univ. Kuwait (Sci), **15** (1988), 21-25.
[22] V. Popa, *Irresolute multifunctions*, Internat. J. Math. and Math. Sci., **13(2)** (1990), 275-280.
[23] A. A. Ramadan, S. E. Abbas and Y. C. Kim, *Fuzzy irresolute mappings in smooth fuzzy topological spaces*, **9(4)** (2001), 865-877.
[24] A. P. Šostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palerms ser II, **11** (1985), 89-103.
[25] A. P. Šostak *Two decades of fuzzy topology: basic ideas, notion and results*, Russian Math. Surveys, **44(6)** (1989), 125-186.
[26] A. P. Šostak, *Basic structures of fuzzy topology*, J. Math. Sci., **78(6)** (1996), 662-701.
[27] E. Tsiporkova, B. De Baets and E. Kerre, *A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings*, Fuzzy Sets and Systems, **85** (1997), 93-108.
[28] E. Tsiporkova, B. De Baets and E. Kerre, *Continuity of fuzzy multivalued mappings*, Fuzzy Sets and Systems, **94** (1998), 335-348.

S. E. Abbas, Department of Mathematics, Faculty of Science, Jazan University, Saudi Arabia

E-mail address: sabbas73@yahoo.com

M. A. Hebeshi, Department of Mathematics, Faculty of Science, Sohag University, Egypt

E-mail address: mhebeshi@yahoo.com

I. M. Taha*, Department of Mathematics, Faculty of Science, Sohag University, Egypt

E-mail address: imtaha2010@yahoo.com

*Corresponding author