

A RELATED FIXED POINT THEOREM IN n FUZZY METRIC SPACES

F. MERGHADI AND A. ALIOUCHE

ABSTRACT. We prove a related fixed point theorem for n mappings which are not necessarily continuous in n fuzzy metric spaces using an implicit relation one of them is a sequentially compact fuzzy metric space which generalize results of Aliouche, et al. [2], Rao et al. [14] and [15].

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [20] in 1965. George and Veeramani [5] modified the concept of fuzzy metric space introduced by [7] and defined the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and E -infinity theory which were given and studied by El- Naschie [8, 9, 10, 11, 12] and [18]. They showed also that every metric induces a fuzzy metric.

Recently, Fisher [4], Aliouche and Fisher [1], Aliouche et.al [2], Rao et.al [14], [15], and [19] proved some related fixed point theorems in compact metric spaces and sequentially compact fuzzy metric spaces. Motivated by a work due to Popa [13], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition.

Definition 1.1. [17] A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t -norm if it satisfies the following conditions:

- (1) $*$ is associative and commutative,
- (2) $*$ is continuous,
- (3) $a * 1 = a$ for all $a \in [0, 1]$,
- (4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t -norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. [5] The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t -norm, and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

$$(FM-1) \quad M(x, y, t) > 0,$$

Received: December 2008; Revised: November 2009; Accepted: December 2009

Key words and phrases: Fuzzy metric space, Implicit relation, Sequentially compact fuzzy metric space, Related fixed point.

- (FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
 (FM-3) $M(x, y, t) = M(y, x, t)$,
 (FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
 (FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X . Then τ is called the topology on X induced by the fuzzy metric M . This topology is Hausdorff and first countable.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Definition 1.4. [5] Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \rightarrow 1$ as $n, m \rightarrow \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5. [6] For all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \rightarrow \infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7. [6] M is a continuous function on $X^2 \times (0, \infty)$.

Definition 1.8. $(X, M, *)$ is said to be sequentially compact fuzzy metric space if every sequence in X has a convergent sub- sequence in it.

Let Φ be the set of all functions $\phi : [0, 1]^6 \rightarrow [0, 1]$ such that if either $\phi(u, 1, u, v, v, 1) > 0$ or $\phi(u, u, 1, v, 1, v) > 0$ for all $u, v \in [0, 1]$, then $u > v$.

Example 1.9. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$. Then $\phi \in \Phi$.

2. Main Results

Theorem 2.1. Let (X_i, M_i, θ_i) , $i = \overline{1, n}$ be n fuzzy metric spaces, $A_i : X_i \rightarrow X_{i+1}$, $i = \overline{1, n-1}$ and $A_n : X_n \rightarrow X_1$ be n mappings satisfying

$$\phi_1 \left(\begin{array}{c} M_1(A_n A_{n-1} \dots A_2 x_2, A_n A_{n-1} \dots A_2 A_1 x_1, t), M_1(x_1, A_n A_{n-1} \dots A_2 x_2, t), \\ M_1(x_1, A_n A_{n-1} \dots A_2 A_1 x_1, t), M_2(x_2, A_1 x_1, t), \\ M_2(x_2, A_1 A_n A_{n-1} \dots A_2 x_2, t), M_2(A_1 x_1, A_1 A_n A_{n-1} \dots A_2 x_2, t) \end{array} \right) > 0 \quad (2.1)$$

for all $x_1 \in X_1$, $x_2 \in X_2$ with $x_2 \neq A_1 x_1$ and for all $t > 0$,

$$\phi_2 \left(\begin{array}{c} M_2(A_1 A_n A_{n-1} \dots A_3 x_3, A_1 A_n A_{n-1} \dots A_3 A_2 x_2, t), \\ M_2(x_2, A_1 A_n A_{n-1} \dots A_3 x_3, t), \\ M_2(x_2, A_1 A_n A_{n-1} \dots A_2 x_2, t), M_3(x_3, A_2 x_2, t), \\ M_3(x_3, A_2 A_1 A_n A_{n-1} \dots A_3 x_3, t), \\ M_3(A_2 x_2, A_2 A_1 A_n A_{n-1} \dots A_3 x_3, t) \end{array} \right) > 0 \quad (2.2)$$

for all $x_2 \in X_2$, $x_3 \in X_3$ with $x_3 \neq A_2 x_2$ and for all $t > 0$,

$$\phi_3 \left(\begin{array}{c} M_3(A_2 A_1 A_n A_{n-1} \dots A_4 x_4, A_2 A_1 A_n A_{n-1} \dots A_3 x_3, t), \\ M_3(x_3, A_2 A_1 A_n A_{n-1} \dots A_4 x_4, t), \\ M_3(x_3, A_2 A_1 A_n A_{n-1} \dots A_3 x_3, t), M_4(x_4, A_3 x_3, t), \\ M_4(x_4, A_3 A_2 A_1 A_n A_{n-1} \dots A_4 x_4, t), \\ M_4(A_3 x_3, A_3 A_2 A_1 A_n A_{n-1} \dots A_4 x_4, t) \end{array} \right) > 0$$

for all $x_3 \in X_2$, $x_4 \in X_3$ with $x_4 \neq A_3 x_3$ and for all $t > 0$

$$\phi_i \left(\begin{array}{c} M_i(A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i x_i, t), \\ M_i(x_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, t), \\ M_i(x_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i x_i, t), M_{i+1}(x_{i+1}, A_i x_i, t), \\ M_{i+1}(x_{i+1}, A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, t), \\ M_{i+1}(A_i x_i, A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} x_{i+1}, t) \end{array} \right) > 0 \quad (2.i)$$

for all $x_i \in X_i$, $x_{i+1} \in X_{i+1}$ with $x_{i+1} \neq A_i x_i$, $4 \leq i \leq n-1$ and for all $t > 0$,

$$\phi_n \left(\begin{array}{c} M_n(A_{n-1} A_{n-2} \dots A_1 x_1, A_{n-1} A_{n-2} \dots A_1 A_n x_n, t), \\ M_n(x_n, A_{n-1} A_{n-2} \dots A_1 x_1, t), \\ M_n(x_n, A_{n-1} A_{n-2} \dots A_1 A_n x_n, t), M_1(x_1, A_n x_n, t), \\ M_1(x_1, A_n A_{n-1} A_{n-2} \dots A_1 x_1, t), \\ M_1(A_n x_n, A_n A_{n-1} A_{n-2} \dots A_1 x_1, t) \end{array} \right) > 0 \quad (2.n)$$

for all $x_1 \in X_1$, $x_n \in X_n$ with $x_1 \neq A_n x_n$ and for all $t > 0$, where $\phi_i \in \Phi$, $i = 1, \dots, n$.

Suppose that one of the following is true

(a₁) (X_1, M_1, θ_1) is sequentially compact and $A_n A_{n-1} \dots A_1$ is continuous on X_1

(a₂) (X_2, M_2, θ_2) is sequentially compact and $A_1 A_n A_{n-1} \dots A_2$ is continuous on

X_2 ,

(a_i) (X_i, M_i, θ_i) is sequentially compact and $A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_i$ is continuous on X_i , $i \geq 3$,

(a_n) (X_n, M_n, θ_n) is sequentially compact and $A_{n-1}A_{n-2}\dots A_1A_n$ is continuous on X_n .

Then $A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_i$ has a unique fixed point $w_i \in X_i$, for all $i = \overline{1, n}$.

Further, $A_i w_i = w_{i+1}$ for all $i = \overline{1, n-1}$ and $A_n w_n = w_1$.

Proof. Suppose that (X_1, M_1, θ_1) is sequentially compact and $A_n A_{n-1} \dots A_1$ is continuous on X_1 .

For every $t > 0$, define

$$\phi(x_1) = M_1(x_1, A_n A_{n-1} \dots A_1 x_1, t) \text{ for all } x_1 \in X_1.$$

Then, there exists $p_1 \in X_1$ such that

$$\phi(p_1) = M_1(p_1, A_n A_{n-1} \dots A_1 p_1, t) = \max \{ \phi(x_1) : x_1 \in X_1 \},$$

there exists $p_2 \in X_2$ such that $p_2 = A_1 p_1$, $p_3 \in X_3$ such that $p_3 = A_2 p_2 = A_2 A_1 p_1$.

In the same manner, there exists $p_{i-1} \in X_{i-1}$ and $p_i \in X_i$ such that

$$p_i = A_{i-1} p_{i-1} = A_{i-1} A_{i-2} p_{i-2} = \dots = A_{i-1} A_{i-2} \dots A_2 A_1 p_1, i = \overline{2..n}$$

for $i = n$ we get

$$p_n = A_{n-1} p_{n-1} = A_{n-1} A_{n-2} p_{n-2} = \dots = A_{n-1} A_{n-2} \dots A_2 A_1 p_1$$

Therefore, $\phi(p_1) = M_1(p_1, A_n p_n, t) = \max \{ \phi(x_1) : x_1 \in X_1 \}$.

Suppose that

$$A_1 (A_n A_{n-1} \dots A_1)^{n-2} (p_1) \neq A_1 (A_n A_{n-1} \dots A_1)^{n-1} (p_1), \text{ i.e.,}$$

$$(A_1 A_n A_{n-1} \dots A_2)^{n-2} A_1 (p_1) \neq (A_1 A_n A_{n-1} \dots A_2)^{n-1} A_1 (p_1)$$

as $A_1(p_1) = p_2$. Then

$$\left\{ \begin{array}{l} (A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2) \neq (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2) \\ (A_1 A_n A_{n-1} \dots A_2)^{n-3} (p_1) \neq (A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2) \\ \vdots \\ (A_1 A_n A_{n-1} \dots A_2) (p_1) \neq (A_1 A_n A_{n-1} \dots A_2)^2 (p_2) \\ (A_n A_{n-1} \dots A_2 p_2) \neq (A_n A_{n-1} \dots A_2) (A_1 A_n A_{n-1} \dots A_2 p_2) \\ (A_{n-1} \dots A_2 p_2) \neq (A_{n-1} \dots A_2) (A_1 A_n A_{n-1} \dots A_2 p_2) \\ \vdots \\ p_2 \neq A_1 A_n A_{n-1} \dots A_2 p_2 \\ p_1 \neq A_n A_{n-1} \dots A_1 p_1 \end{array} \right.$$

Putting $x_2 = A_1 (A_n A_{n-1} \dots A_1)^{n-2} (p_1)$ and $x_1 = (A_n A_{n-1} \dots A_1)^{n-1} (p_1)$ in (2.1) we have

$$\phi_1 \left(\begin{array}{l} M_1 \left((A_n A_{n-1} \dots A_1)^{n-1} (p_1), (A_n A_{n-1} \dots A_2 A_1)^n (p_1), t \right), \\ M_1 \left((A_n A_{n-1} \dots A_1)^{n-1} (p_1), (A_n A_{n-1} \dots A_1)^{n-1} (p_1), t \right), \\ M_1 \left((A_n A_{n-1} \dots A_1)^{n-1} (p_1), (A_n A_{n-1} \dots A_1)^n (p_1), t \right), \\ M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right) \end{array} \right) > 0 \quad (3.1)$$

and so

$$\begin{aligned} & \phi \left((A_n A_{n-1} \dots A_1)^{n-1} (p_1) \right) \\ &= M_1 \left((A_n A_{n-1} \dots A_1)^{n-1} (p_1), (A_n A_{n-1} \dots A_2 A_1)^n (p_1), t \right) \\ &> M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right) \end{aligned} \quad (3.2)$$

Putting $x_2 = (A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2)$ and $x_3 = A_2 (A_1 A_n A_{n-1} \dots A_2)^{n-3} (p_2)$ in (2.2) we get

$$\phi_2 \left(\begin{array}{l} M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1} \dots A_3)^{n-3} (p_3), (A_2 A_1 A_n A_{n-1} \dots A_3)^{n-2} (p_3), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1} \dots A_3)^{n-3} (p_3), (A_2 A_1 A_n A_{n-1} \dots A_3)^{n-2} (p_3), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1} \dots A_3)^{n-2} (p_3), (A_2 A_1 A_n A_{n-1} \dots A_3)^{n-2} (p_3), t \right) \end{array} \right) > 0$$

and so

$$\begin{aligned} & M_2 \left(\begin{array}{l} (A_1 A_n A_{n-1} \dots A_2)^{n-2} (p_2), \\ (A_1 A_n A_{n-1} \dots A_2)^{n-1} (p_2), t \end{array} \right) \\ &> M_3 \left(\begin{array}{l} (A_2 A_1 A_n A_{n-1} \dots A_3)^{n-3} (p_3), \\ (A_2 A_1 A_n A_{n-1} \dots A_3)^{n-2} (p_3), t \end{array} \right) \end{aligned} \quad (3.3)$$

By the same manner we get for all $i = \overline{1, n-1}$

$$\begin{aligned} & M_i \left(\begin{array}{l} (A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i)^{n-i} (p_i), \\ (A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i)^{n-i+1} (p_i), t \end{array} \right) \\ &> M_{i+1} \left(\begin{array}{l} (A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1})^{n-i-1} (p_{i+1}), \\ ((A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1}))^{n-i} (p_{i+1}), t \end{array} \right). \end{aligned} \quad (3.1)$$

For example if $i = n - 1$ we have

$$\begin{aligned} & M_{n-1} \left(\begin{array}{l} (A_{n-2}A_{n-3}\dots A_1A_nA_{n-1})(p_{n-1}), \\ (A_{n-2}A_{n-3}\dots A_1A_nA_{n-1})^2(p_{n-1}), t \end{array} \right) \\ & > M_n(p_n, (A_{n-1}A_{n-2}\dots A_1A_n)(p_n), t) \end{aligned} \quad (3.n-1)$$

Now, putting $x_1 = p_1$ and $x_n = A_{n-1}A_{n-2}\dots A_1(p_1) = p_n$ in (2.n) we obtain

$$\phi_n \left(\begin{array}{l} M_n(p_n, A_{n-1}A_{n-2}\dots A_1A_n(p_n), t), \\ M_n(p_n, p_n, t), \\ M_n(p_n, A_{n-1}A_{n-2}\dots A_1A_n(p_n), t), \\ M_1(p_1, A_n p_n, t), \\ M_1(p_1, A_n p_n, t), \\ M_1(A_n p_n, A_n p_n, t) \end{array} \right) > 0$$

and so

$$M_n(p_n, A_{n-1}A_{n-2}\dots A_1A_n(p_n), t) > M_1(p_1, A_n A_{n-1}A_{n-2}\dots A_1, t) \quad (3.n)$$

From (3.1), (3.2), (3.3), ..., (3.i), ..., (3.n - 1) and (3.n) we get

$$\phi \left((A_n A_{n-1} \dots A_1)^{n-1}(p_1) \right) > \phi(p_1) \quad (4.1)$$

which is a contradiction. Therefore

$$(A_1 A_n A_{n-1} \dots A_2)^{n-2}(p_2) = (A_1 A_n A_{n-1} \dots A_2)^{n-1}(p_2). \quad (4.2)$$

Using (4.2), we have

$$\begin{aligned} p_2 &= A_1 A_n A_{n-1} \dots A_3 A_2(p_2) = A_1 A_n A_{n-1} \dots A_3(p_3) = \dots \\ &= A_1 A_n A_{n-1} \dots A_{i+1} A_i(p_i) = \dots = A_1 A_n(p_n) = A_1 p_1. \end{aligned}$$

Also

$$\begin{aligned} p_1 &= A_n A_{n-1} \dots A_2 A_1(p_1) = A_n A_{n-1} \dots A_2(p_2) = \dots \\ &= A_n A_{n-1} \dots A_{i+1} A_i(p_i) = \dots = A_n(p_n). \end{aligned}$$

For all $\overline{i = 1, n}$, there exists p_i in X_i such that

$$\begin{aligned} p_i &= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i(p_i) \\ &= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1}(p_{i+1}) \\ &= \dots \\ &= A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1}(p_{n-1}) \\ &= A_{i-1} A_{i-2} \dots A_1 A_n(p_n) \\ &= A_{i-1} A_{i-2} \dots A_1(p_1). \end{aligned}$$

For the uniqueness of p_1 , suppose that there exists z_1 in X_1 such that $A_n A_{n-1} \dots A_2 A_1(z_1) = z_1$ with $z_1 \neq p_1$ and for all $i = \overline{2, n}$, $z_i = A_{i-1} z_{i-1}$ in X_i . Then,

$$\begin{aligned} A_n A_{n-1} \dots A_2 A_1(z_1) &\neq A_n A_{n-1} \dots A_2 A_1(p_1) \\ A_{n-1} \dots A_2 A_1(z_1) &\neq A_{n-1} \dots A_2 A_1(p_1) \\ &\vdots \\ A_2 A_1(z_1) &\neq A_2 A_1(p_1) \\ A_1(z_1) &\neq A_1(p_1) \end{aligned}$$

Putting $x_1 = p_1$ and $x_2 = A_1 z_1 = z_2$ in (2.1) we have

$$\phi_1 \left(\begin{array}{c} M_1(A_n A_{n-1} \dots A_2 A_1 z_1, A_n A_{n-1} \dots A_2 A_1 p_1, t), \\ M_1(p_1, A_n A_{n-1} \dots A_2 A_1 z_1, t), \\ M_1(p_1, A_n A_{n-1} \dots A_2 A_1 p_1, t), M_2(A_1 z_1, A_1 p_1, t), \\ M_2(A_1 z_1, A_1 A_n A_{n-1} \dots A_2 A_1 z_1, t), \\ M_2(A_1 p_1, A_1 A_n A_{n-1} \dots A_2 A_1 z_1, t) \end{array} \right) > 0$$

and so

$$M_1(z_1, p_1, t) > M_2(z_2, p_2, t) \quad (4.3)$$

Putting $x_3 = A_2 A_1 p_1 = A_2 p_2 = p_3$, $x_2 = A_1 z_1 = z_2$ in (2.2) we get

$$\phi_2 \left(\begin{array}{c} M_2(A_1 A_n A_{n-1} \dots A_3 A_2 p_2, A_1 A_n A_{n-1} \dots A_3 A_2 z_2, t), \\ M_2(z_2, A_1 A_n A_{n-1} \dots A_3 A_2 p_2, t), \\ M_2(z_2, A_1 A_n A_{n-1} \dots A_2 z_2, t), M_3(p_3, A_2 z_2, t), \\ M_3(p_3, A_2 A_1 A_n A_{n-1} \dots A_3 p_3, t), \\ M_3(A_2 z_2, A_2 A_1 A_n A_{n-1} \dots A_3 p_3, t) \end{array} \right) > 0$$

and so

$$M_2(z_2, p_2, t) > M_3(z_3, p_3, t) \quad (4.4)$$

By the same manner, putting for all $i = \overline{3, n-1}$, $x_{i+1} = A_i A_{i-1} \dots A_2 A_1 p_1 = A_i p_i = p_{i+1}$ and $x_i = A_{i-1} z_{i-1} = z_i$ in (2.i) we obtain

$$\phi_i \left(\begin{array}{c} M_i(A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} A_i p_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i z_i, t), \\ M_i(z_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_{i+1} A_i p_i, t), \\ M_i(z_i, A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i z_i, t), M_{i+1}(p_{i+1}, A_i z_i, t), \\ M_{i+1}(p_{i+1}, A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} p_{i+1}, t), \\ M_{i+1}(A_i z_i, A_i A_{i-1} \dots A_1 A_n A_{n-1} \dots A_{i+1} p_{i+1}, t) \end{array} \right) > 0$$

and so

$$M_i(z_i, p_i, t) > M_{i+1}(z_{i+1}, p_{i+1}, t) \quad (4.i)$$

Putting $x_1 = A_n A_{n-1} \dots A_2 A_1 p_1 = A_n p_n = p_1$ and $x_n = A_{n-1} \dots A_2 A_1 A_n z_n = z_n$ in (2.n) we get:

$$\phi_n(M_n(p_n, z_n, t), M_n(z_n, p_n, t), 1, M_1(p_1, z_1, t), 1, M_1(z_1, p_1, t)) > 0$$

and so

$$M_n(p_n, z_n, t) > M_1(p_1, z_1, t) \quad (4.n)$$

Using (4.3), (4.4), (4.i) and (4.n) we get

$$M_1(p_1, z_1, t) > M_1(p_1, z_1, t)$$

which is a contradiction. Hence, p_1 is the unique fixed point of $A_n A_{n-1} \dots A_2 A_1$. Similarly, we can prove the uniqueness of fixed points of $A_{i-1} A_{i-2} \dots A_1 A_n A_{n-1} \dots A_i$ for all $i = \overline{2, n}$.

In a similar manner, the Theorem 2.1 holds if one of (a_i) , $i \geq 2$ is satisfied. \square

The following example illustrates our Theorem 2.1.

Example 2.2. Let (M_i, X_i, θ_i) , $i = \overline{1, n}$, be n fuzzy metric spaces such that $M_i(x_i, y_i, t) = \frac{t}{t + |x_i - y_i|}$ for $i = \overline{1, n}$ and $X_1 = [0, 1]$, $X_i =]i - 1, i[$ for all $i \geq 2$.

Define $A_i : X_i \rightarrow X_{i+1}$ for $i = \overline{1, n-1}$ and $A_n : X_n \rightarrow X_1$ by

$$\begin{aligned} A_1 x_1 &= \begin{cases} \frac{5}{4} & \text{if } x_1 \in [0, \frac{1}{2}[\\ \frac{3}{2} & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases}, \\ A_i x_i &= \begin{cases} i + \frac{1}{4} & \text{if } x_i \in]i - 1, i - \frac{3}{4}[\\ i + \frac{1}{2} & \text{if } x_i \in [i - \frac{3}{4}, i[\end{cases} \quad \text{for all } i = \overline{2, n-1}, \\ A_n x_n &= \begin{cases} \frac{3}{4} & \text{if } x_n \in]n - 1, n - \frac{3}{4}[\\ 1 & \text{if } x_n \in [n - \frac{3}{4}, n[\end{cases} \end{aligned}$$

Let $\phi_1 = \phi_2 = \dots = \phi_n = \phi$ and $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$.

Note that there exists w_i in X_i such that $(A_{i-1} A_{i-2} \dots A_1 A_n \dots A_i) w_i = w_i$, $\forall i = \overline{1, n}$.

(a) For $i = n$ we get $(A_{n-1} A_{n-2} \dots A_1 A_n) w_n = w_n$ if $w_n = n - \frac{1}{2}$ because

$$\begin{aligned} & (A_{n-1} A_{n-2} \dots A_1 A_n) \left(n - \frac{1}{2} \right) \\ &= A_{n-1} A_{n-2} \dots A_1 (1) = A_{n-1} A_{n-2} \dots A_2 \left(\frac{3}{2} \right) \\ &= A_{n-1} A_{n-2} \dots A_{i+1} \left(i + \frac{1}{2} \right) = \dots \\ &= A_{n-1} A_{n-2} \left(n - \frac{5}{2} \right) = A_{n-1} \left(n - \frac{3}{2} \right) \\ &= n - \frac{1}{2} \end{aligned}$$

since $n - \frac{3}{2} \in [n - \frac{7}{4}, n - 1[$.

(b) Remark that for all $i = \overline{1, n-1}$ and $x_i \in \left[i - \frac{3}{4}, i \right]$; $A_i x_i \in \left[(i+1) - \frac{3}{4}, i+1 \right]$ and $A_n x_n \in \left[\frac{1}{2}, 1 \right]$ for all $x_n \in \left(n - \frac{3}{4}, n \right)$, then there exists $w_i = i + \frac{1}{2}$ such that

$$(A_{i-1}A_{i-2}\dots A_1A_n\dots A_i) \left(i + \frac{1}{2} \right) = i + \frac{1}{2} \text{ for all } i = 1, 2, \dots, n-1.$$

Further, $A_nA_{n-1}\dots A_3A_2A_1$ is continuous in X_1 because if $x = \frac{1}{2}$ is the point of discontinuity for A_1 we have

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}^-} A_nA_{n-1}\dots A_3A_2A_1x &= A_nA_{n-1}\dots A_3A_2 \left(\frac{3}{2} \right) = \\ A_nA_{n-1}\dots A_3 \left(2 + \frac{1}{2} \right) &= A_nA_{n-1}\dots A_4 \left(3 + \frac{1}{2} \right) = \\ A_nA_{n-1}\dots A_{i+1}A_i \left(i - \frac{1}{2} \right) &= \dots = \\ &= A_nA_{n-1} \left(n - \frac{3}{2} \right) = A_n \left(n - \frac{1}{2} \right) = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}^+} A_nA_{n-1}\dots A_3A_2A_1x &= A_nA_{n-1}\dots A_3A_2 \left(\frac{3}{2} \right) = \\ A_nA_{n-1}\dots A_3 \left(2 + \frac{1}{2} \right) &= A_nA_{n-1}\dots A_4 \left(3 + \frac{1}{2} \right) = \\ A_nA_{n-1}\dots A_{i+1}A_i \left(i - \frac{1}{2} \right) &= \dots = A_nA_{n-1} \left(n - \frac{3}{2} \right) = A_n \left(n - \frac{1}{2} \right) = 1 \end{aligned}$$

(c) The inequalities (2.i) for all $i = \overline{1, n}$ are satisfied since the value of the left hand side of each inequality is 1. In fact,

$$M_i(A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+1}x_{i+1}, A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_ix_i, t) = 1$$

for all $i = \overline{1, n-1}$ because

(1) if $x_i \in \left] i-1, i - \frac{3}{4} \right[$ we have

$$\begin{aligned} A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_ix_i &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+1} \left(i + \frac{1}{4} \right), \\ &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+2} \left((i+1) + \frac{1}{2} \right) \\ &\vdots \\ &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1} \left((n-2) + \frac{1}{2} \right), \\ &= A_{i-1}A_{i-2}\dots A_1A_n \left(n - \frac{1}{2} \right) \\ &= A_{i-1}A_{i-2}\dots A_2A_1(1), \\ &= A_{i-1}A_{i-2}\dots A_2 \left(1 + \frac{1}{2} \right) = \\ &\vdots \\ A_{i-1} \left((i-2) + \frac{1}{2} \right) &= (i-1) + \frac{1}{2} = i - \frac{1}{2}. \end{aligned}$$

(2) if $x_i \in [i - \frac{3}{4}, i[$ we have

$$\begin{aligned}
A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_ix_i &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+1}\left(i + \frac{1}{2}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\left((n-2) + \frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}\dots A_1A_n\left(n - \frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}\dots A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2) + \frac{1}{2}\right) &= (i-1) + \frac{1}{2} = i - \frac{1}{2}.
\end{aligned}$$

(3) if $x_{i+1} \in]i, i + \frac{1}{4}[$ we get

$$\begin{aligned}
A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+1}x_{i+1} &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+2}\left((i+1) + \frac{1}{4}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}\dots A_1A_n\left(n - \frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}\dots A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2) + \frac{1}{2}\right) &= (i-1) + \frac{1}{2} = i - \frac{1}{2}.
\end{aligned}$$

(4) if $x_{i+1} \in [i + \frac{1}{4}, i + 1[$ we obtain

$$\begin{aligned}
A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+1}x_{i+1} &= A_{i-1}A_{i-2}\dots A_1A_nA_{n-1}\dots A_{i+2}\left((i+1) + \frac{1}{2}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}\dots A_1A_n\left(n - \frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}\dots A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2) + \frac{1}{2}\right) &= (i-1) + \frac{1}{2} = i - \frac{1}{2}.
\end{aligned}$$

Hence, all conditions of Theorem 2.1 are satisfied.

If we take $n = 5$ in Theorem 2.1, we get the following Corollary 2.3.

Corollary 2.3. *Let (X_i, M_i, θ_i) , $i = \overline{1, 5}$ be 5 fuzzy metric spaces, $A_i : X_i \rightarrow X_{i+1}$, $i = \overline{1, 4}$ and $A_5 : X_5 \rightarrow X_1$ be 5 mappings satisfying*

$$\phi_1 \left(\begin{array}{c} M_1 (A_5 A_4 A_3 A_2 x_2, A_5 A_4 A_3 A_2 A_1 x_1, t), M_1 (x_1, A_5 A_4 A_3 A_2 x_2, t), \\ M_1 (x_1, A_5 A_4 A_3 A_2 A_1 x_1, t), M_2 (x_2, A_1 x_1, t), \\ M_2 (x_2, A_1 A_5 A_4 A_3 A_2 x_2, t), M_2 (A_1 x_1, A_1 A_5 A_4 A_3 A_2 x_2, t) \end{array} \right) > 0 \quad (5.1)$$

for all $x_1 \in X_1$, $x_2 \in X_2$ with $x_2 \neq A_1 x_1$ and for all $t > 0$,

$$\phi_2 \left(\begin{array}{c} M_2 (A_1 A_5 A_4 A_3 x_3, A_1 A_5 A_4 A_3 A_2 x_2, t), \\ M_2 (x_2, A_1 A_5 A_4 A_3 x_3, t), \\ M_2 (x_2, A_1 A_5 A_4 A_3 A_2 x_2, t), M_3 (x_3, A_2 x_2, t), \\ M_3 (x_3, A_2 A_1 A_5 A_4 A_3 x_3, t), \\ M_3 (A_2 x_2, A_2 A_1 A_5 A_4 A_3 x_3, t) \end{array} \right) > 0 \quad (5.2)$$

for all $x_2 \in X_2$, $x_3 \in X_3$ with $x_3 \neq A_2 x_2$ and for all $t > 0$,

$$\phi_3 \left(\begin{array}{c} M_3 (A_2 A_1 A_5 A_4 x_4, A_2 A_1 A_5 A_4 A_3 x_3, t), \\ M_3 (x_3, A_2 A_1 A_5 A_4 x_4, t), \\ M_3 (x_3, A_2 A_1 A_5 A_4 A_3 x_3, t), M_4 (x_4, A_3 x_3, t), \\ M_4 (x_4, A_3 A_2 A_1 A_5 A_4 x_4, t), \\ M_4 (A_3 x_3, A_3 A_2 A_1 A_5 A_4 x_4, t) \end{array} \right) > 0 \quad (5.3)$$

for all $x_3 \in X_3$, $x_4 \in X_4$ with $x_4 \neq A_3 x_3$ and for all $t > 0$

$$\phi_4 \left(\begin{array}{c} M_4 (A_3 A_2 A_1 A_5 x_5, A_3 A_2 A_1 A_5 A_4 x_4, t), \\ M_4 (x_4, A_3 A_2 A_1 A_5 x_5, t), \\ M_4 (x_4, A_3 A_2 A_1 A_5 A_4 x_4, t), M_5 (x_5, A_4 x_4, t), \\ M_5 (x_5, A_4 A_3 A_2 A_1 A_5 x_5, t), \\ M_5 (A_4 x_4, A_4 A_2 A_2 A_1 A_5 x_5, t) \end{array} \right) > 0 \quad (5.4)$$

for all $x_4 \in X_4$, $x_5 \in X_5$ with $x_5 \neq A_4 x_4$ and for all $t > 0$.

$$\phi_5 \left(\begin{array}{c} M_5 (A_4 A_3 A_2 A_1 x_1, A_4 A_3 A_2 A_1 A_5 x_5, t), \\ M_5 (x_5, A_4 A_3 A_2 A_1 x_1, t), \\ M_5 (x_5, A_4 A_3 A_2 A_1 A_5 x_5, t), M_1 (x_1, A_5 x_5, t), \\ M_1 (x_1, A_5 A_4 A_3 A_2 A_1 x_1, t), \\ M_1 (A_5 x_5, A_5 A_4 A_3 A_2 A_1 x_1, t) \end{array} \right) > 0 \quad (5.5)$$

for all $x_1 \in X_1$, $x_5 \in X_5$ with $x_1 \neq A_5 x_5$ and for all $t > 0$.

Suppose that one of the following is true

(a_1) (X_1, M_1, θ_1) is sequentially compact and $A_5 A_4 A_3 A_2 A_1$ is continuous on X_1 ,

- (a₂) (X_2, M_2, θ_2) is sequentially compact and $A_1 A_5 A_4 A_3 A_2$ is continuous on X_2 ,
(a₃) (X_3, M_3, θ_3) is sequentially compact and $A_2 A_1 A_5 A_4 A_3$ is continuous on X_3 ,
(a₄) (X_4, M_4, θ_4) is sequentially compact and $A_3 A_2 A_1 A_5 A_4$ is continuous on X_4 ,
(a₅) (X_5, M_5, θ_5) is sequentially compact and $A_4 A_3 A_2 A_1 A_5$ is continuous on X_5 .

Then

- (b₁) $A_5 A_4 A_3 A_2 A_1$ has a unique fixed point $w_1 \in X_1$,
(b₂) $A_1 A_5 A_4 A_3 A_2$ has a unique fixed point $w_2 \in X_2$,
(b₃) $A_2 A_1 A_5 A_4 A_3$ has a unique fixed point $w_3 \in X_3$,
(b₄) $A_3 A_2 A_1 A_5 A_4$ has a unique fixed point $w_4 \in X_4$,
(b₅) $A_4 A_3 A_2 A_1 A_5$ has a unique fixed point $w_5 \in X_5$,

Further, $A_1 w_1 = w_2$, $A_2 w_2 = w_3$, $A_3 w_3 = w_4$, $A_4 w_4 = w_5$ and $A_5 w_5 = w_1$.

If we take $n = 4$ in Theorem 2.1, we obtain Theorem 2.1 of [15]

If we take $n = 3$ in Theorem 2.1, we obtain Theorem 2.3 of [14].

If we take $n = 2$ in Theorem 2.1, we obtain Theorem 2.1 of [2].

The following example illustrates our Corollary 2.3.

Example 2.4. Let $X_1 = [0, 1]$, $X_2 = [1, 2[$, $X_3 =]2, 3]$, $X_4 = [3, 4[$, $X_5 =]4, 5]$ and let $M_i(x_i, x_{i+1}, t) = \frac{t}{t + |x_{i+1} - x_i|}$ for all $i = \overline{1, 4}$ and $M_5(x_5, x_1, t) = \frac{t}{t + |x_5 - x_1|}$. Define $A_1 : X_1 \rightarrow X_2$, $A_2 : X_2 \rightarrow X_3$, $A_3 : X_3 \rightarrow X_4$, $A_4 : X_4 \rightarrow X_5$ and $A_5 : X_5 \rightarrow X_1$ by

$$\begin{aligned}
 A_1 x_1 &= \begin{cases} 1 & \text{if } x_1 \in [0, \frac{3}{4}[\\ \frac{3}{2} & \text{if } x_1 \in [\frac{3}{4}, 1] \end{cases}, \quad A_2 x_2 = \begin{cases} \frac{5}{2} & \text{if } x_2 \in [1, \frac{3}{2}[\\ 3 & \text{if } x_2 \in [\frac{3}{2}, 2[\end{cases} \\
 A_3 x_3 &= \begin{cases} \frac{13}{4} & \text{if } x_3 \in]2, \frac{5}{2}[\\ \frac{7}{2} & \text{if } x_3 \in [\frac{5}{2}, 3] \end{cases}, \quad A_4 x_4 = \begin{cases} \frac{17}{4} & \text{if } x_4 \in [3, \frac{7}{2}[\\ \frac{9}{2} & \text{if } x_4 \in [\frac{7}{2}, 4[\end{cases} \\
 A_5 x_5 &= \begin{cases} \frac{3}{4} & \text{if } x_5 \in]4, \frac{9}{2}[\\ 1 & \text{if } x_5 \in [\frac{9}{2}, 5] \end{cases}
 \end{aligned}$$

Let $\phi_1(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$.

Here X_1 is compact, but the others spaces are not compact. Further the inequalities (5.1), (5.2), (5.3), (5.4), (5.5) are satisfied since the left hand side of each inequality is 1 and $A_5 A_4 A_3 A_2 A_1$ is continuous in X_1 because if $x = \frac{3}{4}$, the point

of discontinuity for A_1 , we get

$$\begin{aligned} \lim_{x \rightarrow \frac{3}{4}^-} A_5 A_4 A_3 A_2 A_1 x &= A_5 A_4 A_3 A_2 (1) = A_5 A_4 A_3 \left(\frac{5}{2}\right) \\ &= A_5 A_4 \left(\frac{7}{2}\right) = A_5 \left(\frac{9}{2}\right) = 1 \\ &\text{and} \\ \lim_{x \rightarrow \frac{3}{4}^+} A_5 A_4 A_3 A_2 A_1 x &= A_5 A_4 A_3 A_2 \left(\frac{3}{2}\right) = A_5 A_4 A_3 (3) \\ &= A_5 A_4 \left(\frac{7}{2}\right) = A_5 \left(\frac{9}{2}\right) = 1 \end{aligned}$$

We have

$$\begin{aligned} A_5 A_4 A_3 A_2 A_1 (1) &= 1 \\ A_1 A_5 A_4 A_3 A_2 \left(\frac{3}{2}\right) &= \frac{3}{2} \\ A_2 A_1 A_5 A_4 A_3 \left(\frac{5}{2}\right) &= \frac{5}{2} \\ A_3 A_2 A_1 A_5 A_4 \left(\frac{7}{2}\right) &= \frac{7}{2} \\ A_4 A_3 A_2 A_1 A_5 \left(\frac{9}{2}\right) &= \frac{9}{2}. \end{aligned}$$

REFERENCES

- [1] A. Aliouche and B. Fisher, *Fixed point theorems for mappings satisfying implicit relation on two complete and compact metric spaces*, Applied Mathematics and Mechanics., **27(9)** (2006), 1217-1222.
- [2] A. Aliouche, F. Merghadi and A. Djoudi, *A related fixed point theorem in two fuzzy metric spaces*, J. Nonlinear Sci. Appl., **2(1)** (2009), 19-24.
- [3] Y. J. Cho, *Fixed points in fuzzy metric spaces*, J. Fuzzy. Math., **5(4)** (1997), 949-962.
- [4] B. Fisher, *Fixed point on two metric spaces*, Glasnik Mat., **16(36)** (1981), 333-337.
- [5] A. George and P. Veeramani, *On some result in fuzzy metric space*, Fuzzy Sets and Systems, **64** (1994), 395-399.
- [6] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), 385-389.
- [7] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica., **11** (1975), 326-334.
- [8] M. S. El Naschie, *On the uncertainty of Cantorian geometry and two-slit experiment*, Chaos, Solitons and Fractals., **9** (1998), 517-29.
- [9] M. S. El Naschie, *A review of E-infinity theory and the mass spectrum of high energy particle physics*, Chaos, Solitons and Fractals., **19** (2004), 209-36.
- [10] M. S. El Naschie, *On a fuzzy Kahler-like manifold which is consistent with two-slit experiment*, Int. J. of Nonlinear Science and Numerical Simulation., **6** (2005), 95-98.
- [11] M. S. El Naschie, *The idealized quantum two-slit gedanken experiment revisited criticism and reinterpretation*, Chaos, Solitons and Fractals., **27** (2006), 9-13.
- [12] M. S. El Naschie *On two new fuzzy Kahler manifolds, Klein modular space and 't Hooft holographic principles*, Chaos, Solitons & Fractals., **29** (2006), 876-881.

- [13] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math., **32** (1999), 157-163.
- [14] K. P. R. Rao, N. Srinivasa Rao, T. Ranga Rao and J. Rajendra Prasad, *Fixed and related fixed point theorems in sequentially compact fuzzy metric spaces*, Int. Journal of Math. Analysis, **2(28)** (2008), 1353-1359
- [15] K. P. R. Rao, A. Aliouche and G. Ravi Babu, *Related fixed point theorems in fuzzy metric spaces*, J. Nonlinear Sci. Appl., **1(3)** (2008), 194-202
- [16] J. Rodríguez López and S. Ramaguera, *The hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, **147** (2004), 273-283.
- [17] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10** (1960), 313-334.
- [18] Y. Tanaka, Y. Mizno, T. Kado, *Chaotic dynamics in friedmann equation*, Chaos, Solitons and Fractals., **24** (2005), 407-422.
- [19] M. Telci, *Fixed points on two complete and compact metric spaces*, Applied Mathematics and Mechanics, **22(5)** (2001), 564-568.
- [20] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

FAYCEL MERGHADI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEBESSA, 12000, ALGERIA
E-mail address: faycel_mr@yahoo.fr

ABDELKRIM ALIOUCHE*, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LARBI BEN M'HIDI,
OUM-EL-BOUAGHI, 04000, ALGERIA
E-mail address: alioumath@yahoo.fr

*CORRESPONDING AUTHOR