# FUZZY BASIS OF FUZZY HYPERVECTOR SPACES

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Abstract. The aim of this paper is the study of fuzzy basis and dimension of fuzzy hypervector spaces. In this regard, first the notions of fuzzy linear independence and fuzzy basis are introduced and then some related results are obtained. In particular, it is shown that for a large class of fuzzy hypervector space the fuzzy basis exist. Finally, dimension of a fuzzy hypervector space is defined and the basic properties of that are investigated.

## 1. Introduction

The notion of a hypergroup was introduced by F. Marty in 1934 [17]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more see [8], [9], [24]). In 1990, M. Scafati Tallini introduced the notion of hypervector spaces ([22], [23]) and studied basic properties of them.

The concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [25] as a function from a nonempty set $X$ into the unit real interval $I = [0, 1]$. Rosenfeld [21] applied this to the theory of groups and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [19]. In 1977, Katsaras and Liu [15] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers.

Recently, fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. (for example see [1], [2], [3], [6], [7], [10], [11], [12], [13]). In [1] the first author introduced and studied the notion of fuzzy hypervector space over valued fields. In this paper we follow [1] and [5] and study more properties of fuzzy hypervector spaces. In this regards we study the algebraic properties of fuzzy hypervector spaces. We define the concept of fuzzy basis and show that a very wide class of fuzzy hypervector spaces possess it. We define fuzzy dimension for all fuzzy hypervector spaces as a non-negative real number or infinity. Finally we investigate the properties of the introduced concepts.

## 2. Preliminaries

In this section we present some definitions and simple properties of hypervector spaces and fuzzy subsets and fuzzy hypervector spaces, that we shall use in later.

Received: December 2008; Revised: November 2009; Accepted: November 2009

*Key words and phrases:* Fuzzy hypervector space, Fuzzy linear independence, Fuzzy basis, Dimension.

A map $\circ : H \times H \longrightarrow P_*(H)$ is called a hyperoperation or join operation, where $P_*(H)$ is the set of all non-empty subsets of $H$. The join operation is extended to subsets of $H$ in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\ \}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element $a$.

**Definition 2.1.** [22] Let $K$ be a field and $(V, +)$ be an Abelian group. We define a hypervector space over $K$ to be the quadruple $(V, +, \circ, K)$, where "$\circ$" is a mapping

$$\circ : K \times V \quad \longrightarrow \quad P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:
($H_1$) $a \circ (x + y) \subseteq a \circ x + a \circ y$,
($H_2$) $(a + b) \circ x \subseteq a \circ x + b \circ x$,
($H_3$) $a \circ (b \circ x) = (ab) \circ x$,
($H_4$) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
($H_5$) $x \in 1 \circ x$.

**Remark 2.2.** (i) In the right hand side of the right distributive law ($H_1$) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly it is in the left distributive law ($H_2$).

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hypervector spaces, respectively. The hypervector space $(V, +, \circ, K)$ is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of the associative law ($H_3$) means the set-theoretical union of all the sets $a \circ y$, where $y$ runs over the set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. In [22] it is shown that if $V$ is either strongly right or strongly left distributive, then $\Omega_V$ is a subgroup of $(V, +)$.

**Example 2.3.** In $(\mathbb{R}^2, +)$ we define the product times a scalar in $\mathbb{R}$ by setting:

$$\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^2 : a \circ x = \begin{cases} \overline{ox} & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases}$$

where $\overline{ox}$ is the line through the point $x$ and $\underline{0} = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

In the sequel of this note, unless otherwise specified, we assume that $V$ is a hypervector space over the field $K$.

**Definition 2.4.** [1] A nonempty subset $W$ of $V$ is a subhypervector space if $W$ is itself a hypervector space with the hyperoperation on $V$, i.e.

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ \forall a \in K, \ \forall x \in W \Longrightarrow a \circ x \subseteq W. \end{cases}$$

In this case we write $W \leqslant V$.

**Proposition 2.5.** [1] *The intersection of a family of subhypervector spaces is a subhypervector space.*

**Definition 2.6.** [1] If $S$ is a nonempty subset of $V$, then the linear span of $S$ is the smallest subhypervector space of $V$ containing $S$, i.e.

$$L(S) = \langle S \rangle = \bigcap_{S \subseteq W \leqslant V} W.$$

**Lemma 2.7.** [3] *If $S$ is a nonempty subset of $V$, then*

$$L(S) = \left\{ t \in \sum_{i=1}^{n} a_i \circ s_i, a_i \in K, s_i \in S \ , n \in \ \mathbb{N} \right\}.$$

**Definition 2.8.** [1] A subset $S$ of $V$ is called linearly independent if for every vectors $v_1, v_2, \ldots, v_n$ in $S$, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, implies that $c_1 = c_2 = \cdots = c_n = 0$. A subset $S$ of $V$ is called linearly dependent if it is not linearly independent.

**Definition 2.9.** [1] A basis for $V$ is a linearly independent subset $\beta$ of $V$ such that linearly spans $V$, i,e, $L(\beta) = V$. We say that $V$ is finite dimensional if it has a finite basis.

**Remark 2.10.** Note that some hypervector spaces $V$ (some set $W$ of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. Clearly if $V$ is independentless, then $V$ has not any basis and for such hypervector spaces dimension is not defined. In this case we say that V is dimensionless. The hypervector space $(\mathbb{R}^2, +, \circ, \mathbb{R})$ in Example 2.3 is an nontrivial example of an independentless hypervector space, since $\underline{0}$ belongs to every line through the $\underline{0}$.

**Definition 2.11.** [5] A hypervector space $V$ over $K$ is said to be $K$-invertible or shortly invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$, for $u, v \in V$, $a \in K \setminus \{0\}$.

**Theorem 2.12.** [4] *Let $V$ be strongly left distributive and $v_1, v_2, ..., v_n$ be linearly independent in $V$. Then every element in their linear span belongs to a unique sum in the form $c_1 \circ v_1 + c_2 \circ v_2 + ... + c_n \circ v_n$, with $c_i \in K$.*

**Theorem 2.13.** [4] *Let $V$ be invertible. Then for every $v_1, ..., v_n$ in $V$, either $v_1, ..., v_n$ are linearly independent or for some $1 \leq j \leq n$, $v_j$ is in a linear combination of the others.*

**Theorem 2.14.** [4] *Let $V$ be strongly left distributive and invertible. If $V$ has a finite basis with $n$ elements, then every linearly independent subset of $V$ has no more than $n$ elements.*

**Definition 2.15.** [1] Let $V$ and $W$ be hypervector spaces over $K$. A mapping $T : V \longrightarrow W$ is called

(i) weak linear transformation iff

$$T(x+y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \varnothing,$$

(ii) (inclusion) linear transformation iff

$$T(x+y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x),$$

(iii) good transformation iff

$$T(x+y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x).$$

**Definition 2.16.** [18] (i) For a fuzzy subset $\mu$ of $X$, $\mu \in FS(X)$, the level subset $\mu_t$ is defined by

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad t \in [0,1].$$

(ii) The image of $\mu$ is denoted by $Im(\mu)$ and is defined by

$$Im(\mu) = \mu(X) = \{\mu(x) : x \in X\}.$$

(iii) If $\mu \in FS(X)$ and $A \subseteq X$, then by $\underline{\mu}(A)$ and $\bar{\mu}(A)$ we mean

$$\underline{\mu}(A) = \bigwedge_{a \in A} \mu(a) \qquad \text{and} \qquad \bar{\mu}(A) = \bigvee_{a \in A} \mu(a).$$

(iv) (*Extension principle*) Let $f : X \longrightarrow Y$ be a mapping, $\mu \in FS(X)$ and $\nu \in FS(Y)$. Then we define $f(\mu) \in FS(Y)$ and $f^{-1}(\nu) \in FS(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X.$$

## 3. **Fuzzy Hypervector Spaces**

**Definition 3.1.** [20] Let $K$ be a field and $\nu \in FS(K)$. Suppose the following conditions hold:

(i) $\nu(a+b) \geq \nu(a) \wedge \nu(b), \quad \forall a, b \in K,$
(ii) $\nu(-a) \geq \nu(a), \quad \forall a \in K,$
(iii) $\nu(ab) \geq \nu(a) \wedge \nu(b), \quad \forall a, b \in K,$
(iv) $\nu(a^{-1}) \geq \nu(a), \quad \forall a \in K \setminus \{0\},$
(v) $\nu(1) = \nu(0) = 1.$

Then we call $\nu$ a fuzzy field in $K$ and denote it by $\nu_K$.

Obviously, Definition 3.1 is a generalization of the classical field notion.

**Definition 3.2.** [1] Let $\mu$ be a fuzzy subset of $V$ and $\nu$ be a fuzzy field of $K$. Then the pair $\tilde{V} = (V, \mu)$ is said to be a fuzzy hypervector space of $V$ over fuzzy field $\nu_K$, if for all $x, y \in V$ and all $a \in K$, the following conditions are satisfied:

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(-x) \geq \mu(x)$,

(iii) $\bigwedge\limits_{y \in a \circ x} \mu(y) \geq \nu(a) \wedge \mu(x)$, $(\underline{\mu}(a \circ x) \geq \nu(a) \wedge \mu(x))$,

(iv) $\nu(1) \geq \mu(\underline{0})$.

If we consider $\nu = \chi_K$, the characteristic function of $K$, then $\tilde{V} = (V, \mu)$ is called a fuzzy subhyperspace of $V$.

**Proposition 3.3.** [1] *Let $W$ be a proper subhypervector space of $V$. Then the fuzzy subset $\mu$ of $V$ is defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in W, \\ t & \text{otherwise}, \end{cases}$$

*where $t \in [0, 1)$, is a fuzzy subhyperspace of $V$.*

**Example 3.4.** In Example 2.3, set $W = \{(b, 0) : b \in \mathbb{R}\}$. Then $(W, +, \circ, R)$ is a subhypervector space of $V = (R^2, +, \circ, R)$, such that $\forall a, b \in R$,

$$a \circ (b, 0) = \begin{cases} W & \text{if } b \neq 0, \\ \{\underline{0}\} & \text{if } b = 0. \end{cases}$$

Choose numbers $t_1, t_2 \in [0, 1]$, such that $t_1 > t_2$. Define fuzzy subset $\mu$ by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in W, \\ t_2 & \text{otherwise}, \end{cases}$$

Then $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace of $V$.

**Definition 3.5.** [1] Let $\{\mu_i\}_{i \in I}$ be a nonempty collection of fuzzy subhyperspaces of $V$. Then the fuzzy subset $\bigcap\limits_{i \in I} \mu_i$ of $V$ is defined by the following:

$$\left( \bigcap_{i \in I} \mu_i \right)(x) = \bigwedge_{i \in I} \mu_i(x).$$

**Proposition 3.6.** [1] *The intersection of a family of fuzzy subhyperspaces is a fuzzy subhyperspace.*

**Proposition 3.7.** [1] *Let $V$ be strongly left distributive, $\nu_K$ be a fuzzy field and $\mu \in FS(V)$. Then $\mu$ is a fuzzy hypervector space over $\nu_K$ iff*

$$\bigwedge_{z \in a \circ x + b \circ y} \mu(z) \geq (\nu(a) \wedge \mu(x)) \wedge (\nu(b) \wedge \mu(y)),$$

*and $\nu(1) \geq \mu(x)$, for all $a, b \in K$ and $x, y \in V$.*

**Corollary 3.8.** *If $V$ is strongly left distributive and $\mu \in FS(V)$. Then $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace iff*

$$\underline{\mu}(a \circ x + b \circ y) \geq \mu(x) \wedge \mu(y).$$

*Proof.* It follows from Proposition 3.7, by setting $\nu_K = \chi_K$, the characteristic function of $K$. □

**Proposition 3.9.** [1] *Let $\mu \in FS(V)$ and $\nu \in FS(K)$. Then $\mu$ is a fuzzy hypervector space over $\nu_K$ if and only if $\mu_\alpha$ is a hypervector space over the field $\nu_\alpha$, for all $\alpha \in Im(\nu)$ and $\nu(1) \geq \mu(\underline{0})$. $\mu_\alpha$ is called a level subhyperspace of $V$.*

**Definition 3.10.** If $\tilde{V} = (V, \mu)$ is a fuzzy hypervector space then we define

$$T_\mu^\alpha = \mu^{-1}(\alpha) \text{ and } H_\mu^\alpha = \mu^{-1}((\alpha, 1]).$$

**Lemma 3.11.** *Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace of $V$. Then*
*(i) For any subset $A \subseteq V$, $\underline{\mu}(1 \circ A) \leq \underline{\mu}(A)$,*
*(ii) For any subset $A \subseteq V$, $\underline{\mu}(a \circ A) \geq \underline{\mu}(A)$, $\forall a \in K$,*
*(iii) For any subset $A \subseteq V$, $\underline{\mu}(1 \circ A) = \underline{\mu}(A)$,*
*(iv) $\forall a \in K \backslash \{0\}$, $\underline{\mu}(a \circ x) = \mu(x)$,*
*(v) If $x, y \in V$ and $\mu(x) > \mu(y)$, then $\mu(x + y) = \mu(y)$,*
*(vi) If $x, y \in V$ and $\mu(x) \neq \mu(y)$, then $\mu(x + y) = \mu(x) \wedge \mu(y)$,*
*(vii) $\mu(\underline{0}) = \bar{\mu}(V)$.*

*Proof.* (i) and (ii) are clear, because for any subset $A \subseteq V$, $A \subseteq 1 \circ A$ and by Definition 3.2

$$\begin{aligned} \underline{\mu}(a \circ A) &= \underline{\mu}\left(\bigcup_{t \in A} a \circ t\right) \\ &= \bigwedge_{r \in \bigcup\limits_{t \in A} a \circ t} \mu(r) \\ &\geq \bigwedge_{t \in A} \mu(t) \\ &= \underline{\mu}(A). \end{aligned}$$

(iii) It follows from (i) and (ii).
(iv) From (ii) we have $\underline{\mu}(a \circ x) \geq \mu(x)$. On the other hand $x \in 1 \circ x$, so

$$\begin{aligned} \mu(x) &\geq \bigwedge_{t \in 1 \circ x} \mu(t) \\ &= \bigwedge_{t \in a^{-1} \circ (a \circ x)} \mu(t) \\ &= \underline{\mu}(a^{-1} \circ (a \circ x)) \\ &\geq \underline{\mu}(a \circ x). \qquad \text{by (ii)} \end{aligned}$$

Thus $\underline{\mu}(a \circ x) = \mu(x)$.

(v) Since $\mu(x) > \mu(y)$, so $\mu(x+y) \geq \mu(x) \wedge \mu(y) > \mu(y)$. Also

$$\mu(y) = \mu(x+y-x) \geq \mu(x+y) \wedge \mu(x),$$

thus $\mu(y) \geq \mu(x+y)$, because $\mu(x) > \mu(y)$. Consequently $\mu(x+y) = \mu(y)$.

(vi) Apply (v).

(vii) For all $x \in V$, $\mu(\underline{0}) = \mu(x-x) \geq \mu(x) \wedge \mu(-x) \geq \mu(x) \wedge \mu(x) = \mu(x)$. Thus $\mu(\underline{0}) = \bar{\mu}(V)$. □

## 4. **Fuzzy Linear Independence**

**Definition 4.1.** Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace of $V$. We say that a finite set of vectors $\{x_1, x_2, \ldots, x_n\}$ is fuzzy linearly independent in $\tilde{V}$ if and only if $\{x_1, x_2, \ldots, x_n\}$ is linearly independent in $V$ and for all $a_1, a_2, \ldots, a_n \in K$,

$$\bigwedge_{t_i \in a_i \circ x_i} \mu(t_1 + t_2 + \cdots + t_n) = \bigwedge_{t_i \in a_i \circ x_i} (\mu(t_1), \mu(t_2), \ldots, \mu(t_n)).$$

In other words:

$$\underline{\mu}\left(\sum_{i=1}^{n} a_i \circ x_i\right) = \bigwedge_{t \in \sum_{i=1}^{n} a_i \circ x_i} \mu(t) = \bigwedge_{t_i \in a_i \circ x_i} \mu(t_i) = \bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i).$$

A set of vectors is fuzzy linearly independent in $\tilde{V}$ if all finite subsets of it are fuzzy linearly independent in $\tilde{V}$.

**Lemma 4.2.** *Let $V$ be invertible and strongly left distributive hypervector space. Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace of $V$, and let $S = \{x_1, \ldots, x_n\} \subseteq V$ has distinct $\mu$-values and $x \in \sum_{i=1}^{n} a_i \circ x_i$, $a_i \in K \backslash \{0\}$. Then $\mu(x) = \underline{\mu}\,(S)$.*

*Proof.* Let $\underline{\mu}\,(S) = \mu(x_1)$ and let $x \in \sum_{i=1}^{n} a_i \circ x_i$. Then by Corollary 3.8

$$\begin{aligned} \mu(x) &\geq \underline{\mu}\left(\sum_{i=1}^{n} a_i \circ x_i\right) \\ &\geq \bigwedge_{i=1}^{n} \mu(x_i) \\ &= \underline{\mu}\,(S). \end{aligned}$$

On the other hand, $x \in \sum_{i=1}^{n} a_i \circ x_i$, so $x = t_1 + \cdots + t_n$, such that $t_i \in a_i \circ x_i$, $i = 1, \ldots, n$. Thus

$$\begin{aligned} x_1 &\in a_1^{-1} \circ t_1 \\ &\subseteq a_1^{-1} \circ (x - t_2 - \cdots - t_n) \\ &\subseteq a_1^{-1} \circ x - a_1^{-1} \circ t_2 - \cdots - a_1^{-1} \circ t_n \\ &\subseteq a_1^{-1} \circ x - a_1^{-1} \circ (a_2 \circ x_2) - \cdots - a_1^{-1} \circ (a_n \circ x_n), \end{aligned}$$

therefore

$$\begin{aligned}\mu(x_1) &\geq \underline{\mu}\left(a_1^{-1} \circ x - (a_1^{-1}a_2) \circ x_2 - \cdots - (a_1^{-1}a_n) \circ x_n\right) \\ &\geq \mu(x) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n).\end{aligned}$$

Thus $\mu(x_1) \geq \mu(x)$, since $\mu(x_1) = \bigwedge_{i=1}^{n} \mu(x_i)$. Consequently, $\mu(x) = \mu(x_1) = \underline{\mu}(S)$. □

**Proposition 4.3.** *Let $V$ be an invertible hypervector space and let singletons of $V$ be linearly independent. Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace of $V$. Then any set of vectors $\{x_1, x_2, \ldots, x_n\} \subseteq V \backslash \{\underline{0}\}$ which has distinct $\mu$-values is linearly and fuzzy linearly independent.*

*Proof.* We prove the proposition by induction on $n$. In case $n = 1$, clearly the statement is true. Now suppose that the proposition is true for $n$. Let $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ be a set of vectors in $V \backslash \{\underline{0}\}$ with distinct $\mu$-values. By inductive hypothesis we have: $\{x_1, x_2, \ldots, x_n\}$ is fuzzy linearly independent. If $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ is not linearly independent, then by Theorem 2.13,

$$x_{n+1} \in a_1 \circ x_1 + a_2 \circ x_2 + \cdots + a_n \circ x_n.$$

Thus by Lemma 4.2, $\mu(x_{n+1}) \in \{\mu(x_i)\}_{i=1}^{n}$ and this contradicts the fact that

$$\{x_1, x_2, \ldots, x_n, x_{n+1}\}$$

has distinct $\mu$-values. Therefore $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ is linearly independent. Finally Lemmas 3.11 and 4.2, clearly show that $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ is fuzzy linearly independent, since $\underline{\mu}\left(\sum_{i=1}^{n} a_i \circ x_i\right) = \bigwedge_{i=1}^{n} \mu(x_i) = \bigwedge_{i=1}^{n} \underline{\mu}(a_i \circ x_i)$. □

**Corollary 4.4.** *Let $V$ be a strongly left distributive and invertible hypervector space such that $\dim V = n$, and let singletons of $V$ be linearly independent. If $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace of $V$ then $|Im(\mu)| \leq n + 1$, where $|Im(\mu)|$ denotes the cardinality of $Im(\mu)$.*

*Proof.* Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for $V$. Then $\{\mu(x_1), \ldots, \mu(x_n), \mu(\underline{0})\} \subseteq Im(\mu)$. Suppose that $|Im(\mu)| \geq n + 2$. Then by Proposition 4.3, $V$ has a linearly independent subset with $n + 1$ elements, which contradict with Theorem 2.14. □

## 5. **Fuzzy Basis**

**Definition 5.1.** A fuzzy basis for a fuzzy subhyperspace $\tilde{V} = (V, \mu)$ is a fuzzy linearly independent basis for $V$.

**Example 5.2.** Consider classical vector space $(\mathbb{R}^3, +, ., \mathbb{R})$ with standard basis $\{i, j, k\}$. Define the mapping

$$\begin{cases} \circ : \quad \mathbb{R} \times \quad \mathbb{R}^3 \quad \longrightarrow \quad P_*(\mathbb{R}^3) \\ \qquad a \circ (x_0, y_0, z_0) = L, \end{cases}$$

where $L$ is the line with the parametric equations:

$$L : \begin{cases} x = ax_0 \\ y = ay_0 \\ z = t \end{cases}$$

It is easy to verify that $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space with basis $\{i, j\}$. Define fuzzy subset $\mu$ on $V$ by $\mu((0,0,0)) = 1$, $\mu(\mathbb{R} \times \{0\} \times \{0\} \backslash (0,0,0)) = \frac{1}{3}$, $\mu\left(\mathbb{R}^3 \backslash \mathbb{R} \times \{0\} \times \{0\}\right) = \frac{1}{5}$. Then it is routine to see that $\mu$ is a fuzzy subhyperspace with fuzzy basis $\{i, j\}$.

The next theorem shows that the class of fuzzy subhyperspaces with fuzzy basis is as enough as rich.

**Theorem 5.3.** *Let $V$ be invertible and strongly left distributive hypervector space with basis $\beta = \{v_\alpha\}_{\alpha \in I}$ and let $\mu_0 \in (0,1]$ be constant and $\{\mu_\alpha\}_{\alpha \in I} \subseteq (0,1]$ be any set of constants such that $\mu_0 \geq \mu_\alpha$ for all $\alpha \in I$. Now construct a function $\mu : V \longrightarrow [0,1]$ in the following way: By Theorem 2.12 any $x \neq \underline{0}$, $x \in V$ is contained in a unique combination $\sum\limits_{i=1}^{n} a_i \circ v_{\alpha_i}$, with $a_i \neq 0$. Define*

$$\mu(x) = \bigwedge_{i=1}^{n} \mu(v_{\alpha_i}) = \bigwedge_{i=1}^{n} \mu_{\alpha_i},$$

*and $\mu(\underline{0}) = \mu_0$. Then $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace with fuzzy basis $\beta$.*

*Proof.* Let $x, y \in V \backslash \{\underline{0}\}$. Then by Theorem 2.12 $x$ and $y$ can be uniquely written in the following way:

$$x \in \sum_{i \in C \cup D_x} a_i \circ v_{\alpha_i} \qquad \text{and} \qquad y \in \sum_{i \in C \cup D_y} b_i \circ v_{\alpha_i},$$

such that $C \cap D_x = \emptyset$, $C \cap D_y = \emptyset$, $D_x \cap D_y = \emptyset$, $C \cup D_x$ and $C \cup D_y$ are finite and non-empty and for all $i \in C \cup D_x$, $a_i \in K \backslash \{0\}$ and for all $i \in C \cup D_y$, $b_i \in K \backslash \{0\}$. Suppose $a, b \neq 0$, $a, b \in K$ and $a \circ x + b \circ y \neq \{\underline{0}\}$. Let $Z = \{i \in C : aa_i + bb_i = 0\}$ and $N = C \backslash Z$. At this stage suppose that $C, D_x, D_y, Z$ and $N$ are all non-empty. In case some of these sets are empty the proof of the theorem is almost identical. Now let $w \in a \circ x + b \circ y$, then

$$\begin{aligned} w &\in a \circ \left( \sum_{i \in C \cup D_x} a_i \circ v_{\alpha_i} \right) + b \circ \left( \sum_{i \in C \cup D_y} b_i \circ v_{\alpha_i} \right) \\ &= \sum_{i \in C \cup D_x} (aa_i) \circ v_{\alpha_i} + \sum_{i \in C \cup D_y} (bb_i) \circ v_{\alpha_i} \\ &= \sum_{i \in C} (aa_i + bb_i) \circ v_{\alpha_i} + \sum_{i \in D_x} (aa_i) \circ v_{\alpha_i} + \sum_{i \in D_y} (bb_i) \circ v_{\alpha_i} \\ &= \sum_{i \in N} (aa_i + bb_i) \circ v_{\alpha_i} + \sum_{i \in D_x} (aa_i) \circ v_{\alpha_i} + \sum_{i \in D_y} (bb_i) \circ v_{\alpha_i}, \end{aligned}$$

so

$$\mu(w) \geq \underline{\mu}\left(\sum_{i\in N}(aa_i+bb_i)\circ v_{\alpha_i}+\sum_{i\in D_x}(aa_i)\circ v_{\alpha_i}+\sum_{i\in D_y}(bb_i)\circ v_{\alpha_i}\right).$$

All coefficients in the above linear combination are non-zero and thus by definition of $\mu$ we have:

$$\begin{aligned}
\mu(w) &\geq \left(\bigwedge_{i\in N}\mu(v_{\alpha_i})\right)\wedge\left(\bigwedge_{i\in D_x}\mu(v_{\alpha_i})\right)\wedge\left(\bigwedge_{i\in D_y}\mu(v_{\alpha_i})\right)\\
&= \left(\bigwedge_{i\in N}\mu_{\alpha_i}\right)\wedge\left(\bigwedge_{i\in D_x}\mu_{\alpha_i}\right)\wedge\left(\bigwedge_{i\in D_y}\mu_{\alpha_i}\right)\\
&= \bigwedge_{i\in N\cup D_x\cup D_y}\mu_{\alpha_i}\\
&\geq \bigwedge_{i\in C\cup D_x\cup D_y}\mu_{\alpha_i}\\
&= \left(\bigwedge_{i\in C\cup D_x}\mu_{\alpha_i}\right)\wedge\left(\bigwedge_{i\in C\cup D_y}\mu_{\alpha_i}\right).
\end{aligned}$$

Therefore if $a, b \neq 0$ and $a\circ x+b\circ y\neq\{\underline{0}\}$, then

$$\bigwedge_{w\in a\circ x+b\circ y}\mu(w)\geq\mu(x)\wedge\mu(y).$$

Thus by Theorem 3.7 $\mu$ is a fuzzy subhyperspace of $V$. In the case where $a\circ x+b\circ y = \{\underline{0}\}$, since $\mu(\underline{0})=\mu_{\underline{0}}\geq\bar{\mu}(\beta)$, we must have $\underline{\mu}(a\circ x+b\circ y)=\mu(\underline{0})\geq\mu(x)\wedge\mu(y)$. In the case where $a$ or $b$ is zero, without loss of generality we may say $a=0$, then

$$\begin{aligned}
\underline{\mu}(0\circ x+b\circ y) &= \bigwedge_{r\in 0\circ x,\ s\in b\circ y}\mu(r+s)\\
&\geq \left(\bigwedge_{r\in 0\circ x}\mu(r)\right)\wedge\left(\bigwedge_{s\in b\circ y}\mu(s)\right)\\
&\geq \mu(x)\wedge\mu(y).
\end{aligned}$$

□

**Remark 5.4.** Fuzzy subhyperspace in Example 3.4 is a fuzzy subhyperspace with out fuzzy basis.

In the following we give a simple condition under which a fuzzy subhyperspace has a fuzzy basis.

**Definition 5.5.** [16] A set $B$ is said to be *upper well ordered* if for all non-empty subsets $C \subseteq B$, $\sup C \in C$.

Let us investigate the upper well order subsets of $[0, 1]$.

**Definition 5.6.** [16] A subset $B \subseteq [0,1]$ is said to have an *increasing monotonic limit* $x \in [0,1]$ if and only if $x$ is a limit of a monotonically increasing sequence in $B$.

**Proposition 5.7.** [16] *A set* $B \subseteq [0,1]$ *is without any increasing monotonic limits if and only if it is upper well ordered.*

**Proposition 5.8.** [16] *All upper well ordered subsets of* $[0,1]$ *are countable.*

**Remark 5.9.** We can construct $B \subseteq [0,1]$ upper well ordered with an infinite number of decreasing limit points. For example consider

$$B = \left\{ \frac{1}{n} + \frac{1}{m} : n, m \in \{2,3,4,\ldots\} \right\}.$$

This concludes a study of upper well ordered subsets of $[0,1]$.

**Lemma 5.10.** *If* $\tilde{V} = (V, \mu)$ *is a fuzzy subhyperspace such that* $\mu(V)$ *is upper well ordered, and* $W$ *is a proper subhypervector space of* $V$*, then there exists* $v \in V \backslash W$ *such that*

$$\forall w \in W, \quad \mu(v+w) = \mu(v) \wedge \mu(w).$$

*Proof.* Since $\mu(V)$ is upper well ordered we can find $v \in V\backslash W$ such that $\mu(V) = \bar{\mu}[\mu(V\backslash W)]$. Let $w \in W$. If $\mu(w) \neq \mu(v)$, then by Lemma 3.11, $\mu(v+w) = \mu(v) \wedge \mu(w)$. If $\mu(w) = \mu(v)$, then by Definition 3.2, $\mu(v+w) \geq \mu(v) \wedge \mu(w)$. Since $v + w \in V\backslash W$ and $\mu(V) = \bar{\mu}[\mu(V\backslash W)]$, we must have $\mu(v+w) \leq \mu(v) = \mu(w)$ and thus $\mu(v+w) = \mu(v) \wedge \mu(w)$. □

**Lemma 5.11.** *If* $\tilde{V} = (V, \mu)$ *is a fuzzy subhyperspace such that* $\mu(V)$ *is upper well ordered, and if* $\acute{\beta}$ *is a fuzzy basis for* $\tilde{W} = (W, \mu\,|_W)$*, where* $W$ *is a proper subhypervector space of* $V$*, then there exists* $v \in V\backslash W$ *such that* $\overset{*}{\beta} = \acute{\beta} \cup \{v\}$ *is a fuzzy basis for* $\tilde{U} = \left(U = \left\langle \overset{*}{\beta} \right\rangle, \mu\,|_U\right)$*, where* $\left\langle \overset{*}{\beta} \right\rangle$ *is the hypervector space spanned by* $\overset{*}{\beta}$.

*Proof.* Let $v \in V\backslash W$ such that $\mu(V) = \bar{\mu}[\mu(V\backslash W)]$, then clearly by Lemma 5.10, $v$ is fuzzy linearly independent from $\acute{\beta}$. Let $\overset{*}{\beta} = \acute{\beta} \cup \{v\}$. Clearly $\overset{*}{\beta}$ is a fuzzy basis for $\tilde{U} = \left(U = \left\langle \overset{*}{\beta} \right\rangle, \mu\,|_U\right)$. □

**Theorem 5.12.** *All fuzzy subhyperspaces* $\tilde{V} = (V, \mu)$ *for which* $\mu(V)$ *is upper well ordered have a fuzzy basis.*

*Proof.* Let $\tilde{V} = (V, \mu)$ be any fuzzy subhyperspace for which $\mu(V)$ is upper well ordered. Let $\Gamma = \{\beta \subseteq V : \beta \text{ is fuzzy linear independent}\}$. Partial order $\Gamma$ by set inclusion. Let $\Lambda$ be a totally ordered subset of $\Gamma$ and let $\bar{\beta} = \bigcup_{\beta \in \Lambda} \beta$. Then $\bar{\beta}$ is an upper bound for $\Lambda$. Suppose $x_1, x_2, \ldots, x_n \in \bar{\beta}$. Then there exist $\beta_{\alpha(1)}, \ldots, \beta_{\alpha(n)} \in \Lambda$ such that $x_i \in \beta_{\alpha(i)}$. Since $\Lambda$ is totally ordered, one of the sets, say $\beta_{\alpha(k)}$, is a superset of the others. Hence $x_1, x_2, \ldots, x_n \in \beta_{\alpha(k)}$. Since $\beta_{\alpha(k)}$ is fuzzy linearly independent, $x_1, x_2, \ldots, x_n$ are fuzzy linearly independent. Thus $\bar{\beta}$ is upper bound of $\Lambda$ in $\Gamma$. By Zorn's Lemma there exists a maximal element $\overset{*}{\beta}$ in $\Gamma$. Suppose $\left\langle \overset{*}{\beta} \right\rangle = W$ is a proper subhypervector space of $V$, then by Lemma 5.11, there exists $v \in V \backslash W$ such that $\overset{*}{\beta} \cup \{v\} = \hat{\beta}$ is a fuzzy basis for $\tilde{U} = (U, \mu\,|_U\,)$. This contradicts the fact that $\overset{*}{\beta}$ is a maximal element in $\Gamma$. Thus we must have $\left\langle \overset{*}{\beta} \right\rangle = V$ and $\hat{\beta}$ is a fuzzy basis for $V$. □

**Corollary 5.13.** *If $V$ is finite dimensional, then $\tilde{V} = (V, \mu)$ has a fuzzy basis.*

*Proof.* Since $V$ is finite dimensional, $\mu(V)$ is finite and upper well ordered. Thus by Theorem 5.12, $\tilde{V}$ has a fuzzy basis. □

**Definition 5.14.** Let $\tilde{V_1} = (V, \mu_1)$ and $\tilde{V_2} = (V, \mu_2)$ be two fuzzy subhyperspaces of $V$. Define the intersection of $\tilde{V_1}$ and $\tilde{V_2}$ to be

$$\tilde{V_1} \cap \tilde{V_2} = (V,\ \mu_1 \wedge \mu_2).$$

Define the sum of $\tilde{V_1}$ and $\tilde{V_2}$ to be

$$\tilde{V_1} + \tilde{V_2} = (V,\ \mu_1 + \mu_2),$$

where $\mu_1 + \mu_2$ is

$$\begin{aligned} (\mu_1 + \mu_2)(x) &= \bigvee_{\substack{x = x_1 + x_2 \\ x_1, x_2 \in V}} (\mu_1(x_1) \wedge \mu_2(x_2)) \\ &= \bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(x - x_1))\,. \end{aligned}$$

**Proposition 5.15.** *Let $V$ be invertible and strongly left distributive and let $\tilde{V_1} = (V, \mu_1)$ and $\tilde{V_2} = (V, \mu_2)$ be two fuzzy subhyperspaces of $V$. Then the following hold:*

*(i) $\tilde{V_1} \cap \tilde{V_2}$ is a fuzzy subhyperspace of $V$,*

*(ii) $\tilde{V_1} + \tilde{V_2}$ is a fuzzy subhyperspace of $V$,*

*(iii) If $\mu_1(V)$ and $\mu_2(V)$ are upper well ordered, then $\tilde{V_1} \cap \tilde{V_2}$ and $\tilde{V_1} + \tilde{V_2}$ have fuzzy basis.*

*Proof.* (i) Apply 3.6.

(ii) Suppose $(\mu_1 + \mu_2)(x + y) < (\mu_1 + \mu_2)(x) \wedge (\mu_1 + \mu_2)(y)$. Then there exist $x_1$ and $x_2$, such that for all $x_3$ we have:

(1) $\quad \mu_1(x_3) \wedge \mu_2(x + y - x_3) < [\mu_1(x_1) \wedge \mu_2(x - x_1)] \wedge [\mu_1(x_2) \wedge \mu_2(y - x_2)]\,.$

But

$$\begin{aligned} [\mu_1(x_1) \wedge \mu_2(x - x_1)] \wedge [\mu_1(x_2) \wedge \mu_2(y - x_2)] &= \\ &= \mu_1(x_1) \wedge \mu_1(x_2) \wedge \mu_2(x - x_1) \wedge \mu_2(y - x_2) \\ &\leq \mu_1(x_1 + x_2) \wedge \mu_2(x + y - x_1 - x_2). \end{aligned}$$

Therefore there exists $x_3 = x_1 + x_2$ for which (1) is false. Thus we have a contradiction and

$$(\mu_1 + \mu_2)(x + y) \geq (\mu_1 + \mu_2)(x) \wedge (\mu_1 + \mu_2)(y).$$

Also

$$\begin{aligned}
(\mu_1 + \mu_2)(-x) &= \bigvee_{-x = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \\
&= \bigvee_{x = -x_1 - x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \\
&= \bigvee_{x = -x_1 - x_2} (\mu_1(-x_1) \wedge \mu_2(-x_2)) \\
&= (\mu_1 + \mu_2)(x).
\end{aligned}$$

Thus $(\mu_1 + \mu_2)(-x) = (\mu_1 + \mu_2)(x)$. Now let $x \in V,\ a \in K,\ a \neq 0$ and $t \in a \circ x$. Then

$$(\mu_1 + \mu_2)(t) = \bigvee_{t = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \,.$$

$V$ is invertible, so from $r = x_1 + x_2 \in a \circ x$ it follows that:

$$\begin{aligned}
x &\in a^{-1} \circ t \\
&= a^{-1} \circ (x_1 + x_2) \\
&\subseteq a^{-1} \circ x_1 + a^{-1} \circ x_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\mu_1 + \mu_2)(t) &= \bigvee_{x \in a^{-1} \circ (x_1 + x_2)} (\mu_1(x_1) \wedge \mu_2(x_2)) \\
&= \bigvee_{x = t + s,\ t \in a^{-1} \circ x_1,\ s \in a^{-1} \circ x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \\
&= \bigvee_{x = t + s,\ x_1 \in a \circ t,\ x_2 \in a \circ s} (\mu_1(x_1) \wedge \mu_2(x_2)) \\
&\geq \bigvee_{x = t + s} \left(\underline{\mu_1}(a \circ t) \wedge \underline{\mu_2}(a \circ s)\right) \\
&\geq \bigvee_{x = t + s} (\mu_1(t) \wedge \mu_2(s)) \\
&= (\mu_1 + \mu_2)(x).
\end{aligned}$$

So if $a \neq 0$, then

$$\begin{aligned}
\left(\underline{\mu_1 + \mu_2}\right)(a \circ x) &= \bigwedge_{t \in a \circ x} (\mu_1 + \mu_2)(t) \\
&\geq (\mu_1 + \mu_2)(x).
\end{aligned}$$

If $a = 0$, then

$$\begin{aligned}
\underline{(\mu_1 + \mu_2)}\,(0 \circ x) &= \underline{(\mu_1 + \mu_2)}\,((a - a) \circ x) \\
&= \underline{(\mu_1 + \mu_2)}\,(a \circ x - a \circ x) \\
&= \bigwedge_{t \in a \circ x,\ s \in -a \circ x} (\mu_1 + \mu_2)(t + s) \\
&\geq \left( \bigwedge_{t \in a \circ x} (\mu_1 + \mu_2)(t) \right) \wedge \left( \bigwedge_{s \in -a \circ x} (\mu_1 + \mu_2)(s) \right) \\
&\geq \left( \underline{(\mu_1 + \mu_2)}\,(a \circ x) \right) \wedge \left( \underline{(\mu_1 + \mu_2)}\,(-a \circ x) \right) \\
&\geq (\mu_1 + \mu_2)(x) \wedge (\mu_1 + \mu_2)(x) \\
&= (\mu_1 + \mu_2)(x).
\end{aligned}$$

Consequently $\tilde{V}_1 + \tilde{V}_2$ is a fuzzy subhyperspace of $V$.

(iii) Clearly $(\mu_1 \cap \mu_2)(V) \subseteq \mu_1(V) \cup \mu_2(V)$ and $(\mu_1 + \mu_2)(V) \subseteq \mu_1(V) \cup \mu_2(V)$. Since $\mu_1(V)$ and $\mu_2(V)$ are upper well ordered, $\mu_1(V) \cup \mu_2(V)$ is upper well ordered and thus $(\mu_1 \cap \mu_2)(V)$ and $(\mu_1 + \mu_2)(V)$ are upper well ordered. Hence by Theorem 5.12, $\tilde{V}_1 \cap \tilde{V}_2$ and $\tilde{V}_1 + \tilde{V}_2$ have a fuzzy basis. □

## 6. **Dimension of Fuzzy Hypervector Spaces**

**Definition 6.1.** We define the dimension of a fuzzy hypervector space $\tilde{V} = (V, \mu)$ to be

$$\dim \tilde{V} = \bigvee \left\{ \sum_{v \in \beta} \mu(v) : \beta \text{ is a basis for } V \right\}.$$

Clearly dim is a function from the class of all fuzzy hypervector spaces to $[0, \infty]$. A fuzzy hypervector space $\tilde{V}$ is finite dimensional if and only if $\dim \tilde{V} < \infty$.

**Proposition 6.2.** *Let $V$ be invertible and let singletons of $V$ be linearly independent. Let $\tilde{V} = (V, \mu)$ be a finite dimensional fuzzy subhyperspace of $V$. Then $\tilde{V} = (V, \mu)$ has a fuzzy basis.*

*Proof.* We shall first show that $\mu(V)$ is upper well ordered. Suppose $\mu(V) \subseteq [0, 1]$ has an increasing monotonic limit. Then there exists a sequence $\{x_i\}_{i=1}^{\infty} \subseteq V$ such that $\{\mu(x_i)\}_{i=1}^{\infty}$ is a strictly increasing sequence with limit $\alpha$. we may suppose that $\mu(x_1) = \alpha > 0$. By Proposition 4.3, $\{x_i\}_{i=1}^{\infty}$ is linearly independent. Consider $H_n$ as the extension of the linearly independent set $\{x_i\}_{i=1}^{n}$ to basis for $V$. Then we have a sequence of bases for $V$, such that $\sum_{x \in H_n} \mu(x) > n\alpha$, which implies that $\dim \tilde{V} = \infty$. This is a contradiction. Therefore by Proposition 5.7, $\mu(V)$ is upper well ordered. Accordingly, by Theorem 5.12, $\tilde{V}$ has a fuzzy basis. □

**Proposition 6.3.** *Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace such that* $\dim V = n <$ $\infty$. *Then if $\beta^*$ is a fuzzy basis for $V$ and $\beta$ is any basis for $V$, then*

$$\sum_{v \in \beta} \mu(v) \leq \sum_{v \in \beta^*} \mu(v).$$

*Proof.* Since $V$ is finite dimensional, $|\mu(V\setminus\{\underline{0}\})| = k \leq \dim V$. Let $\mu(V\setminus\{\underline{0}\}) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, such that $\alpha_i > \alpha_{i+1}$. Since $\beta^*$ is a fuzzy basis for $\tilde{V}$, $\beta^* \cap \mu_{\alpha_i}$ is a basis for hypervector space $\mu_{\alpha_i}$, and $\beta \cap \mu_{\alpha_i}$ is independent subset of $\mu_{\alpha_i}$. Thus $|\beta \cap \mu_{\alpha_i}| \leq |\beta^* \cap \mu_{\alpha_i}|$ for all $i \in \{1, \ldots, k\}$. Define recursively a set of injections $\{f_1, \ldots, f_k\}$ as follows: Let $f_1$ be any injection from $\beta \cap \mu_{\alpha_1}$ to $\beta^* \cap \mu_{\alpha_1}$. Such $f_1$ exists since $|\beta \cap \mu_{\alpha_1}| \leq |\beta^* \cap \mu_{\alpha_1}|$ and $\mu(v) \leq \mu(f_1(v))$, for all $v \in \beta \cap \mu_{\alpha_1}$. Given $f_{n-1}$ to be an injection from $\beta \cap \mu_{\alpha_{n-1}}$ to $\beta^* \cap \mu_{\alpha_{n-1}}$, such that $\mu(v) \leq \mu(f_{n-1}(v))$ for all $v \in \beta \cap \mu_{\alpha_{n-1}}$. Let $g_n$ be any injection from $\beta \cap \mu^{-1}(\alpha_n)$ to $(\beta^* \cap \mu_{\alpha_n})\setminus f_{n-1}(\beta \cap \mu_{\alpha_{n-1}})$. Such $g_n$ exists since

$$\begin{aligned}
\left|\beta \cap \mu^{-1}(\alpha_n)\right| &= \left|\beta \cap \mu_{\alpha_n}\right| - \left|\beta \cap \mu_{\alpha_{n-1}}\right| \\
&\leq \left|\beta^* \cap \mu_{\alpha_n}\right| - \left|\beta \cap \mu_{\alpha_{n-1}}\right| \\
&= \left|\beta^* \cap \mu_{\alpha_n}\right| - \left|f_{n-1}(\beta \cap \mu_{\alpha_{n-1}})\right| \\
&= \left|(\beta^* \cap \mu_{\alpha_n})\setminus f_{n-1}(\beta \cap \mu_{\alpha_{n-1}})\right|.
\end{aligned}$$

Define

$$\begin{cases} f_n : \beta \cap \mu_{\alpha_n} \longrightarrow \beta^* \cap \mu_{\alpha_n} \\ f_n(v) = \begin{cases} f_{n-1}(v) & v \in \beta \cap \mu_{\alpha_{n-1}}, \\ g_n(v) & \text{otherwise.} \end{cases} \end{cases}$$

Then $f_n$ is an injection and since $g_n(\beta \cap \mu^{-1}(\alpha_n) \subseteq \mu_{\alpha_n},\ n \in \{2, \ldots, k\}$, it follows that $\mu(v) \leq \mu(f_n(v))$ for all $v \in \beta \cap \mu_{\alpha_n}$. Since $\mu_{\alpha_k} = V$ and $|\beta| = |\beta^*|$, then $f_k$ is a bijection between $\beta$ and $\beta^*$. Now

$$\sum_{v\in\beta^*} \mu(v) = \sum_{v\in\beta} \mu(f_k(v)) \geq \sum_{v\in\beta} \mu(v).$$

□

**Lemma 6.4.** *If $\tilde{V} = (V, \mu)$ is a finite dimensional fuzzy subhyperspace, then for all $\alpha \in Im(\mu)\setminus\{0\}$, $\mu_\alpha$ is finite dimensional subhypervector space of $V$.*

*Proof.* If $\mu_\alpha$ is infinite dimensional and $\beta^*$ is a fuzzy basis for $\tilde{V}$, then $|\beta^* \cap \mu_\alpha|$ is infinite, since $\beta^* \cap \mu_\alpha$ is a basis for $\mu_\alpha$. Thus

$$\sum_{v\in\beta^*} \mu(v) \geq \sum_{v\in\beta^*\cap\mu_\alpha} \mu(v) \geq \sum_{v\in\beta^*\cap\mu_\alpha} \alpha = \infty.$$

Therefore $\dim \tilde{V} = \infty$, which is a contradiction. Thus $\mu_\alpha$ must be finite dimensional. □

**Theorem 6.5.** *If $\tilde{V} = (V, \mu)$ is finite dimensional, then*

$$\dim(\tilde{V}) = \sum_{v\in\beta^*} \mu(v),$$

*where $\beta^*$ is any fuzzy basis for $\tilde{V}$.*

*Proof.* It is sufficient to show that

$$\sum_{v\in\beta}\mu(v)\leq\sum_{v\in\beta^*}\mu(v),$$

where $\beta$ is any basis for $V$. By Lemma 6.4, for all $\alpha>0$, $\mu_\alpha$ is finite dimensional and $\beta^*\cap\mu_\alpha$ is a fuzzy basis for $\tilde{V}_\alpha=(\mu_\alpha,\mu\,|_{\mu_\alpha})$. As $\beta\cap\mu_\alpha$ is an independent subset of $\mu_\alpha$, Proposition 6.3, implies that

$$\sum_{v\in\beta\cap\mu_\alpha}\mu(v)\leq\sum_{v\in\beta^*\cap\mu_\alpha}\mu(v).$$

This is true for all $\alpha>0$, and thus we must have

$$\sum_{v\in\beta}\mu(v)\leq\sum_{v\in\beta^*}\mu(v).$$

□

**Remark 6.6.** De Luca and Termini [14] proposed the following definition of cardinality of a fuzzy set:

If $\tilde{A}=(A,\mu_A)$, where $\mu_A:A\longrightarrow[0,1]$, is a fuzzy set, then $card(\tilde{A})=\sum\limits_{a\in A}\mu_A(a)$.

Let $\tilde{V}=(V,\mu)$ be any finite dimensional fuzzy hypervector space with fuzzy basis $\beta$. Let $\tilde{\beta}=(\beta,\mu\,|_\beta)$. In view of Theorem 6.5, we have $\dim(\tilde{V})=card(\tilde{\beta})$, and hence the dimension of a fuzzy hypervector space and the cardinality of it's basis are coincide. It is clearly in agreement with the definition of dimension in the crisp case.

**Acknowledgements.** This research is partially supported by the "Fuzzy Systems and Its Applications Center of Excellence, Shahid Bahonar University of Kerman, Iran" and "Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran".

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