ON PRIME FUZZY BI-IDEALS OF SEMIGROUPS

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Abstract. In this paper, we introduce and study the prime, strongly prime, semiprime and irreducible fuzzy bi-ideals of a semigroup. We characterize those semigroups for which each fuzzy bi-ideal is semiprime. We also characterize those semigroups for which each fuzzy bi-ideal is strongly prime.

1. **Introduction**

The concept of the fuzzy set was first introduced by Zadeh in [26]. Since then, fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathematics. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. For example, Rosenfeld [20] applied this concept to generalize some of the basic concepts of groupoids. Another example is the generalization of classical semigroups by Kuroki [11].

A survey of selected papers by Zadeh has been published in [28], where his main field of interest is centered around linguistic modeling of complex systems, approximate reasoning and the use of fuzzy logical and natural language tools in expert systems, and other areas of man-machine interaction. Selected papers by Zadeh have been recently published in [30] , which contains his deeply influential contributions to the field of fuzzy sets and their applications to approximate reasoning and modeling. New tools are needed for solving more difficult social and biological problems. This type of mathematics will be capable of handling uncertainties, making decisions and modeling very large systems and networks which are complex, non-linear and distributive. For an important article by Zadeh towards a generalized theory of uncertainty we refer to [31]. One of the areas in which fuzzy sets have been applied most extensively is in modeling managerial decision making. The fuzzy set theory can be applied to the area of human decision making, as well [32]. Interesting results on fuzzy set theory and its applications have been also published by Zimmermann in [33], where one can see the application of fuzzy technology in information processing which is already important. In [33], one can also find new methodological development in dynamic fuzzy data analysis, which will also be of growing importance in the future.

A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [18], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. Fuzziness has a natural

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place in the field of formal languages. The monograph by Mordeson and Malik [17] deals with the application of a fuzzy approach to the concepts of automata and formal languages.

In [1], Ahsan et al, introduced the notion of prime fuzzy ideals of a semigroup S, and proved that a semigroup S is semisimple if and only if every fuzzy ideal of S is the intersection of all prime fuzzy ideals of S containing it. In [21], Shabir introduced the notion of prime fuzzy right ideals and proved that a semigroup S is right weakly regular if and only if each fuzzy right ideal is the intersection of those fuzzy prime right ideals of S which contain it. In [4], fuzzy prime and fuzzy semiprime S-subacts are defined and studied.

In this paper, we introduce the notions of prime, strongly prime, semiprime and irreducible fuzzy bi-ideals of a semigroup. We characterize those semigroups for which each fuzzy bi-ideal is semiprime. We also characterize those semigroups for which each fuzzy bi-ideal is strongly prime. In this paper we show that a semigroup S in which every bi-ideal is prime (strongly prime) does not necessarily have every fuzzy bi-ideal prime (strongly prime).

2. **Preliminaries**

A semigroup is a non-empty set S together with an associative binary operation "$\cdot$" . An element 0 of a semigroup S with at least two elements is called a zero element of S if $x0 = 0x = 0$ for all x in S. A semigroup which contains a zero element is called a semigroup with zero. If a semigroup S has no zero element then it is easy to adjoin a zero element 0 to the set by defining $0x = x0 = 0 = 00$ for all x in S. We shall use the notation S^0 with the following meanings:

$$S^0 = \begin{cases} S, & \text{if } S \text{ has a zero element} \\ S \cup \{0\}, & \text{otherwise.} \end{cases}$$

An element 1 of a semigroup S is called an identity element of S if $x1 = 1x = x$ for all x in S. A semigroup which contains an identity element is called a semigroup with identity or a monoid. If a semigroup S has no identity element then it is easy to adjoin an element 1 to the set by defining $1x = x1 = x$ and $11 = 1$ for all x in S. We shall use the notation S^1 with the following meanings:

$$S^1 = \begin{cases} S, & \text{if } S \text{ has an identity element} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$$

A non empty subset A of a semigroup S is called a subsemigroup if $ab \in A$ for all $a, b \in A$. A subsemigroup B of a semigroup S is called a bi-ideal of S if $BSB \subseteq B$. It is well known that the intersection of any number of bi-ideals of a semigroup S is either empty or a bi-ideal of S. Also the product of two bi-ideals of a semigroup S is a bi-ideal of S.

A bi-ideal B of a semigroup S is called prime (strongly prime) if $B_1B_2 \subseteq B$ (or $B_1B_2 \cap B_2B_1 \subseteq B$, resp.) implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of

S (see [22]). A bi-ideal B of a semigroup S is called semiprime if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S [22].

An element a of a semigroup S is called a regular element if there exists an element x in S such that $axa = a$. A semigroup S is called regular if every element of S is regular. An element a of a semigroup S is called intra-regular if there exist elements x and y in S such that $xa^2y = a$. A semigroup S is called intra-regular if every element of S is intra-regular.

A function f from a non-empty set A to the unit interval $[0, 1]$ of real numbers is called a fuzzy subset of A [26]. A fuzzy subset f of a semigroup S is called a fuzzy subsemigroup of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$. A fuzzy subsemigroup of S is called a fuzzy bi-ideal of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$ (see [11]). Equivalently a fuzzy subset f of a semigroup S is a fuzzy bi-ideal of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, z \in S$ and $y \in S^1$. Let f and g be two fuzzy subsets of S. We define the relation $\leq$ between f and g, the union, intersection and product of f and g, respectively, as follows:

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \in S,$$
$$(f \vee g)(x) = \max\{f(x), g(x)\},$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ and}$$
$$(f \circ g)(x) = \begin{cases} \sup_{x=yz}[\min\{f(y), g(z)\}] & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

We give the operation $\circ$ precedence over $\vee$ and $\wedge$. So, for example, $f \circ g \vee h$ means $(f \circ g) \vee h$. It is easily verified that the "product" of fuzzy sets is associative. The following Lemma is due to Kuroki [14].

Lemma 2.1. *Let f be a fuzzy subset of S. Then the following holds:*
(i) f is a fuzzy subsemigroup of S if and only if $f \circ f \leq f$.
(ii) f is a fuzzy bi-ideal of S if and only if $f \circ f \leq f$ and $f \circ \mathcal{S} \circ f \leq f$ where $\mathcal{S}$ is the fuzzy subset of S which maps every element of S onto 1.

The following lemma is trivial.

Lemma 2.2. *The intersection of any family of fuzzy bi-ideals of a semigroup S is a fuzzy bi-ideal of S.*

Lemma 2.3. *[2] If f, g, h are fuzzy subsets of a semigroup S and $f \leq g$ then $f \circ h \leq g \circ h$ and $h \circ f \leq h \circ g$.*

The following lemma is trivial.

Lemma 2.4. *Let A be a non-empty subset of a semigroup S, then C_A, the characteristic function of A, is the fuzzy bi-ideal of S if and only if A is a bi-ideal of S.*

Lemma 2.5. *If A and B are subsets of a semigroup S. Then $C_A \circ C_B = C_{AB}$.*

Definition 2.6. Let $a \in S$ and $t \in (0, 1]$, then the fuzzy subset a_t of S, defined by

$$a_t(x) = \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } x \in S$$

is called a fuzzy point.

For any fuzzy subset f of S, it is clear that $f = \vee_{a_t \leq f} a_t$. Let a_t and b_r be fuzzy points of S, then $a_t \circ b_r = (ab)_{t \wedge r}$.

Definition 2.7. Let f be a fuzzy subset of a semigroup S. Let $B(f)$ denotes the intersection of all fuzzy bi-ideals of S which contain f. Then $B(f)$ is a fuzzy bi-ideal of S, called the fuzzy bi-ideal generated by f.

Lemma 2.8. *Let a_t be a fuzzy point of a semigroup S. Then the fuzzy bi-ideal generated by a_t denoted by $B(a_t)$ is defined at each $x \in S$ by*

$$B(a_t)(x) = \begin{cases} t & \text{if } x \in B(a) \\ 0 & \text{otherwise.} \end{cases}$$

Where $B(a) = \{a\} \cup aS^1a$ is a bi-ideal of S generated by a.

Proof. Let $x \in S$. If $y, z \in B(a)$ such that $x = yz$, then

$$\begin{aligned} (B(a_t))^2(x) &= \vee_{x=yz}\{B(a_t)(y) \wedge B(a_t)(z)\} = t \\ &= B(a_t)(x). \end{aligned}$$

Otherwise

$$(B(a_t))^2(x) = 0 \leq B(a_t)(x).$$

Thus

$$(B(a_t))^2 \leq B(a_t).$$

Hence by Lemma 1 (i), $B(a_t)$ is a fuzzy subsemigroup of S.

Let $x, y, z \in S$. If $x, z \in B(a)$ then $xyz \in B(a)$ and so

$$\begin{aligned} B(a_t)(xyz) &= t \\ &= B(a_t)(x) \wedge B(a_t)(z). \end{aligned}$$

If one of x, z does not belong to $B(a)$ then

$$B(a_t)(x) \wedge B(a_t)(z) = 0 \leq B(a_t)(xyz).$$

Thus

$$B(a_t)(xyz) \geq B(a_t)(x) \wedge B(a_t)(y).$$

Hence $B(a_t)$ is a fuzzy bi-ideal of S. Clearly $a_t \leq B(a_t)$. Let δ be a fuzzy bi-ideal of S containing a_t and let $x \in S$. If $x \notin B(a)$, then

$$B(a_t)(x) = 0 \leq \delta(x).$$

If $x \in B(a) = \{a\} \cup aS^1a$, then $B(a_t)(x) = t$.

If $x = a$, then $a_t(x) = t \leq \delta(x)$.

If $x = aya$ for some $y \in S^1$, then

$$\begin{aligned} \delta(x) &= \delta(aya) \\ &\geq \delta(a) \wedge \delta(a) \\ &\geq a_t(a) \wedge a_t(a) \\ &= t \\ &= B(a_t)(x). \end{aligned}$$

Hence
$$\delta(x) \geq B(a_t)(x).$$
Thus $B(a_t)$ is the smallest fuzzy bi-ideal of S containing a_t. □

3. Prime Fuzzy Bi-ideals

Definition 3.1. A fuzzy bi-ideal f of a semigroup S is called a prime fuzzy bi-ideal of S if for any fuzzy bi-ideals g, h of S, $g \circ h \leq f$ implies $g \leq f$ or $h \leq f$.

Theorem 3.2. *Let I be a subset of a semigroup S. Then I is a prime bi-ideal of S if and only if C_I (the characteristic function of I) is a prime fuzzy bi-ideal of S.*

Proof. Let I be a bi-ideal of S. Then C_I is a fuzzy bi-ideal of S by Lemma 2.4. Let f and g be fuzzy bi ideals of S such that $f \circ g \leq C_I$. If $f \nleq C_I$, then there exists a fuzzy point $x_t \leq f$ $(t > 0)$ such that $x_t \nleq C_I$. For any $y_r \leq g$ $(r \neq 0)$, since
$$B(x_t) \circ B(y_r) \leq f \circ g \leq C_I \tag{1}$$

We have for all $z \in S$
$$B(x_t) \circ B(y_r)(z) = \begin{cases} t \wedge r > 0 & \text{if } z \in B(x) \cdot B(y) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$
Thus by (1) and (2),
$$B(x) \cdot B(y) \subseteq I.$$
Now from hypothesis, it follows that
$$B(x) \subseteq I \text{ or } B(y) \subseteq I.$$
Since $x_t \nleq C_I$, we have $g = \vee_{y_r \leq g} y_r \leq C_I$.

Conversely, let A and B be bi-ideals of S. Suppose $AB \subseteq I$. Then C_A, C_B are fuzzy bi-ideals of S and $C_A \circ C_B = C_{AB} \leq C_I$. From hypothesis, we have $C_A \leq C_I$ or $C_B \leq C_I$. Thus $A \subseteq I$ or $B \subseteq I$. □

Definition 3.3. A fuzzy bi-ideal f of a semigroup S is called a strongly prime fuzzy bi-ideal if $g \circ h \wedge h \circ g \leq f$ implies either $g \leq f$ or $h \leq f$ for any fuzzy bi-ideals g and h of S.

Theorem 3.4. *Let I be a subset of a semigroup S. Then I is a strongly prime bi-ideal of S if and only if C_I is a strongly prime fuzzy bi-ideal of S.*

Proof. Let I be a bi-ideal of S. Then C_I is a fuzzy bi-ideal of S. Let f and g be fuzzy bi-ideals of S such that $f \circ g \wedge g \circ f \leq C_I$. If $f \nleq C_I$, then there exists a fuzzy point $x_t \leq f$ $(t > 0)$ such that $x_t \nleq C_I$. Now for any $y_r \leq g$ $(r \neq 0)$, we have
$$B(x_t) \circ B(y_r) \wedge B(y_r) \circ B(x_t) \leq f \circ g \wedge g \circ f \leq C_I.$$
and $(\forall z \in S)$,

$$(B(x_t) \circ B(y_r) \wedge B(y_r) \circ B(x_t))(z) = \begin{cases} t \wedge r & \text{if } z \in B(x)\,B(y) \cap B(y)\,B(x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$B(x)\,B(y) \cap B(y)\,B(x) \subseteq I$$

and from hypothesis

$$B(x) \subseteq I \text{ or } B(y) \subseteq I.$$

Since $x_t \nleq C_I$, we have $B(x) \nsubseteq I$. Hence $B(y) \subseteq I$ which implies $y_r \leq C_I$. Thus $g = \vee_{yr \leq g} y_r \leq C_I$.

Conversely, let A and B be bi-ideals of S. Suppose $AB \cap BA \subseteq I$. Then C_A, C_B are fuzzy bi-ideals of S and

$$C_A \circ C_B \wedge C_B \circ C_A = C_{AB \cap BA} \leq C_I.$$

From hypothesis, we have

$$C_A \leq C_I \text{ or } C_B \leq C_I.$$

Thus $A \subseteq I$ or $B \subseteq I$.

Clearly every strongly prime fuzzy bi-ideal is a fuzzy prime bi-ideal but the converse is not true. □

Definition 3.5. A fuzzy bi-ideal g of S is said to be idempotent if $g = g^2 = g \circ g$.

Definition 3.6. A fuzzy bi-ideal f of a semigroup S is called a fuzzy semiprime bi-ideal if

$g \circ g = g^2 \leq f$ implies that $g \leq f$ for every fuzzy bi-ideal g of S.

Theorem 3.7. *Let A be a subset of a semigroup S. Then A is a semiprime bi-ideal of S if and only if C_A is a semiprime fuzzy bi-ideal of S.*

Proof. Let A be a semiprime bi-ideal of S. Then C_A is a fuzzy bi-ideal of S. Let g be a fuzzy bi-ideal of S such that $g \circ g \leq C_I$. If $g \nleq C_I$ then there exists a fuzzy point $x_t \leq g$ $(t > 0)$ such that $x_t \nleq C_I$. Now

$$B(x_t) \circ B(x_t) \leq g \circ g \leq C_I.$$

Next for all $z \in S$:

$$B(x_t) \circ B(x_t)(z) = \begin{cases} t & \text{if } z \in B(x)\,B(x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$B(x)\,B(x) \subseteq I.$$

From hypothesis, we have

$$B(x) \subseteq I.$$

Hence

$$B(x_t) \leq C_I.$$

But $x_t \leq B(x_t)$ so $x_t \leq C_I$ which is a contradiction. Hence $g \leq C_I$.

Conversely, suppose that C_I is a semiprime fuzzy bi-ideal of S, then I is a bi-ideal of S. Let B be a bi-ideal of S such that $B^2 \subseteq I$. Then

$$C_B \circ C_B = C_{B^2} \leq C_I.$$

By hypothesis $C_B \leq C_I$, that is $B \subseteq I$. □

Lemma 3.8. *The intersection of any family of prime fuzzy bi-ideals of a semigroup S is a semiprime fuzzy bi-ideal of S.*

Proof. Let $\{f_i : i \in I\}$ be any collection of fuzzy prime bi-ideals of a semigroup S. Then $\wedge_{i\in I} f_i$ is a fuzzy bi-ideal of S by Lemma 2.2. Let δ be any fuzzy bi-ideal of S such that $\delta^2 \leq \wedge_{i\in I} f_i$. Then $\delta^2 \leq f_i$ for all $i \in I$. Since each f_i is a fuzzy prime bi-ideal of S, so $\delta \leq f_i$ for all $i \in I$. Hence $\delta \leq \wedge_{i\in I} f_i$. Thus $\wedge_{i\in I} f_i$ is a fuzzy semiprime bi-ideal of S.

Definition 3.9. A fuzzy bi-ideal f of a semigroup S is called an irreducible (strongly irreducible) fuzzy bi-ideal if for any fuzzy bi-ideals g and h of S, such that $g \wedge h = f$ ($g \wedge h \leq f$) implies either $g = f$ or $h = f$ ($g \leq f$ or $h \leq f$).

Note that every strongly irreducible fuzzy bi-ideal is an irreducible fuzzy bi-ideal. □

Proposition 3.10. *Every strongly irreducible, semiprime fuzzy bi-ideal of a semigroup S is a strongly prime fuzzy bi-ideal.*

Proof. Let f be a strongly irreducible, semiprime fuzzy bi-ideal of a semigroup S. Let g, h be any fuzzy bi-ideals of S such that $g \circ h \wedge h \circ g \leq f$. As $(g \wedge h)^2 \leq g \circ h$ and $(g \wedge h)^2 \leq h \circ g$. Thus $(g \wedge h)^2 \leq g \circ h \wedge h \circ g$ and so $(g \wedge h)^2 \leq f$. Since f is a semiprime fuzzy bi-ideal of S, therefore $g \wedge h \leq f$. As f is strongly irreducible fuzzy bi-ideal of S, so either $g \leq f$ or $h \leq f$. Thus f is a strongly prime fuzzy bi-ideal of S.

□

4. **Semigroups in which each Fuzzy Bi-ideal is Strongly Prime**

In this section we study those semigroups in which each fuzzy bi-ideal is semiprime. We, also study those semigroups in which each fuzzy bi-ideal is strongly prime.

Proposition 4.1. *Let f be a fuzzy bi-ideal of a semigroup S with $f(a) = \alpha$, where a is any element of S and $\alpha \in (0,1]$, then there exists an irreducible fuzzy bi-ideal g of S such that $f \leq g$ and $g(a) = \alpha$.*

Proof. Let $X = \{h : h \text{ is a fuzzy bi-ideal of } S, \, h(a) = \alpha \text{ and } f \leq h\}$. Then $X \neq \varnothing$, since $f \in X$. The collection X is a partially ordered set under inclusion. If Y is any totally ordered subset of X, say $Y = \{h_i : i \in I\}$. Then $\vee_{i\in I} h_i$ is a fuzzy bi-ideal of S containing f. Indeed, if a, $b \in S$ and $x \in S^1$ then,

$$\begin{aligned}(\vee_{i\in I}h_i)(axb) &= \vee_{i\in I}(h_i(axb)) \\ &\geq \vee_{i\in I}(h_i(a)\wedge h_i(b)) \\ &= (\vee_{i\in I}h_i(a))\wedge(\vee_{i\in I}h_i(b)) \\ &= (\vee_{i\in I}h_i)(a)\wedge(\vee_{i\in I}h_i)(b).\end{aligned}$$

Hence $\vee_{i\in I}h_i$ is a fuzzy bi-ideal of S.

As $f \leq h_i$ for each $i \in I$, so $f \leq \vee_{i\in I}h_i$.

Also $(\vee_{i\in I}h_i)(a) = \vee_{i\in I}h_i(a) = \alpha$.

Thus $\vee_{i\in I}h_i$ is the supremum of Y.

By Zorn's Lemma, there exists a fuzzy bi-ideal g of S which is maximal with respect to the property that $f \leq g$ and $g(a) = \alpha$. We now show that g is an irreducible fuzzy bi-ideal of S. Suppose $g = g_1 \wedge g_2$ where g_1 and g_2 are fuzzy bi-ideals of S. Then $g \leq g_1$ and $g \leq g_2$. We claim that $g = g_1$ or $g = g_2$. Suppose on the contrary that, $g \neq g_1$ and $g \neq g_2$. Since g is maximal with respect to the property that $g(a) = \alpha$ and since $g \lneqq g_1$ and $g \lneqq g_2$, it follows that $g_1(a) \neq \alpha$ and $g_2(a) \neq \alpha$. Hence $\alpha = g(a) = (g_1 \wedge g_2)(a) \neq \alpha$, which is a contradiction. Hence either $g = g_1$ or $g = g_2$. Thus g is an irreducible fuzzy bi-ideal of S.

Next Theorem characterizes those semigroups for which each fuzzy bi-ideal is semiprime. □

Theorem 4.2. *For a semigroup S, the following assertions are equivalent.*

(i) S is both regular and intra-regular.

(ii) $f \circ f = f$ for every fuzzy bi-ideal f of S.

(iii) $g \wedge h = g \circ h \wedge h \circ g$ for all fuzzy bi-ideals g and h of S.

(iv) Each fuzzy bi-ideal of S is fuzzy semiprime.

(v) Each proper fuzzy bi-ideal of S is the intersection of irreducible semiprime fuzzy bi-ideals of S which contain it.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ (see [14]).

$(iii) \Rightarrow (iv)$

Let f, g be fuzzy bi-ideals of S such that $f^2 \leq g$. By hypothesis,

$$\begin{aligned}f &= f \wedge f \\ &= f \circ f \wedge f \circ f \\ &= f \circ f \\ &= f^2.\end{aligned}$$

Thus $f \leq g$. Hence every fuzzy bi-ideal of S is semiprime.

$(iv) \Rightarrow (v)$

Let f be a proper fuzzy bi-ideal of S and $\{f_i : i \in I\}$ be the collection of all irreducible fuzzy bi-ideals of S which contain f. By Proposition 4.1, this collection is non-empty.

Hence

$$f \leq \wedge_{i\in I}f_i.$$

Let $a \in S$. There exists an irreducible fuzzy bi-ideal f_α of S such that $f \leq f_\alpha$ and $f(a) = f_\alpha(a)$. Thus $f_\alpha \in \{f_i : i \in I\}$. Hence

$$\wedge_{i \in I} f_i \leq f_\alpha.$$

So

$$\wedge_{i \in I} f_i(a) \leq f_\alpha(a) = f(a).$$

By hypothesis, each fuzzy bi-ideal of S is semiprime. So each fuzzy bi-ideal of S is the intersection of all irreducible semiprime fuzzy bi-ideals of S which contain it.

$(v) \Rightarrow (ii)$

Let f be a fuzzy bi-ideal of S. Then f^2 is also a fuzzy bi-ideal of S. Also $f^2 \leq f$ because f is a fuzzy subsemigroup of S (see Lemma 2.1).

By hypothesis $f^2 = \wedge_{i \in I} f_i$ where f_i are irreducible semiprime fuzzy bi-ideals of S. Thus $f^2 \leq f_i$ for all $i \in I$.

Hence $f \leq f_i$ for all $i \in I$, because f_i are semiprime. Thus $f \leq \wedge_{i \in I} f_i = f^2$, so that $f^2 = f$. □

Proposition 4.3. *Let S be a regular and intra-regular semigroup. Then the following assertions for a fuzzy bi-ideal f of S are equivalent.*

(i) f is strongly irreducible.
(ii) f is strongly prime.

Proof. In view of Theorem 4.2 (ii), we have (for all fuzzy bi-ideals g, h) the equivalence of $g \wedge h \leq f$ with $g \circ h \wedge h \circ g \leq f$. Then the equivalence of (i) with (ii) follows immediately from the definition of strongly irreducible and strongly prime.

Each fuzzy bi-ideal of a semigroup S is strongly prime if and only if S is regular, intra-regular and the set of fuzzy bi-ideals of S is totally ordered by inclusion. □

Proof. Suppose that each fuzzy bi-ideal of S is strongly prime, then each fuzzy bi-ideal of S is semiprime. Thus by Theorem 4.2, S is both regular and intra-regular. We show that the set of fuzzy bi-ideals of S is totally ordered. Let g and h be any two fuzzy bi-ideals of S so that $g \wedge h = (g \circ h) \wedge (h \circ g)$ by Theorem 4.2. As each fuzzy bi-ideal is strongly prime, so $g \wedge h$ is strongly prime. Hence either $g \leq g \wedge h$ or $h \leq g \wedge h$. If $g \leq g \wedge h$ then $g \leq h$ and if $h \leq g \wedge h$, then $h \leq g$.

Conversely, assume that S is regular, intra-regular and the set of fuzzy bi-ideals of S is totally ordered under inclusion. Then we show that each fuzzy bi-ideal of S is strongly prime. Let f be an arbitrary fuzzy bi-ideal of S and g, h are fuzzy bi-ideals of S such that $g \circ h \wedge h \circ g \leq f$.

Now S is both regular and intra-regular, so by Theorem 4.2, we have $g \circ h \wedge h \circ g = g \wedge h$. Thus $g \wedge h \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered, so either $g \leq h$ or $h \leq g$, that is either $g \wedge h = g$ or $g \wedge h = h$. Thus either $g \leq f$ or $h \leq f$. Hence f is strongly prime. □

Theorem 4.4. *If the set of fuzzy bi-ideals of a semigroup S is totally ordered under inclusion, then S is simultaneously regular and intra-regular if and only if each fuzzy bi-ideal of S is prime.*

Proof. Suppose S is both regular and intra-regular. Let f be any fuzzy bi-ideal of S and g, h are fuzzy bi-ideals of S such that $g \circ h \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered, either $g \leq h$ or $h \leq g$. Suppose $g \leq h$ then $g \circ g \leq g \circ h \leq f$. By Theorem 4.2, f is semiprime, so $g \leq f$. Hence f is a prime fuzzy bi-ideal of S.

Conversely, assume that every fuzzy bi-ideal of S is prime. Since every prime fuzzy bi-ideal is semiprime, by Theorem 4.2, S is both regular and intra-regular.

For a semigroup S, the following assertions are equivalent.

(i) The set of fuzzy bi-ideals of S is totally ordered under inclusion.

(ii) Each fuzzy bi-ideal of S is strongly irreducible.

(iii) Each fuzzy bi-ideal of S is irreducible. □

Proof. $(i) \Rightarrow (ii)$

Let f be an arbitrary fuzzy bi-ideal of S and g, h are two fuzzy bi-ideals of S such that $g \wedge h \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered, either $g \leq h$ or $h \leq g$. Thus either $g \wedge h = h$ or $g \wedge h = g$. Hence $g \wedge h \leq f$ implies either $h \leq f$ or $g \leq f$. Hence f is strongly irreducible.

$(ii) \Rightarrow (iii)$ trivial.

$(iii) \Rightarrow (i)$

Let g, h be any two fuzzy bi-ideals of S. Then $g \wedge h$ is a fuzzy bi-ideal of S. Also $g \wedge h = g \wedge h$. So by hypothesis, either $g = g \wedge h$ or $h = g \wedge h$, that is either $g \leq h$ or $h \leq g$. Hence the set of fuzzy bi-ideals of S is totally ordered.

Theorem 4.2 and Theorem 2.10 of [25] show that, every bi-ideal of a semigroup S is semiprime if and only if every fuzzy bi-ideal of S is semiprime. However the following examples show that if every bi-ideal of S is prime (strongly prime), then it is not necessarily true that every fuzzy bi-ideal of S is prime (strongly prime). □

Example 4.5. Consider the semigroup $S = \{0, a, b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

It is evident that S is both regular and intra-regular.

Bi-ideals in S are $\{0\}, \{0, a\}, \{o, b\}$ and S.

All bi-ideals are prime bi-ideals and hence semiprime bi-ideals.

Fact: 1. Every fuzzy subset of S is a fuzzy subsemigroup of S.

Fact: 2. A fuzzy subset f of S is a fuzzy bi-ideal of S if and only if $f(0) \geq f(x)$ for all $x \in S$.

Proof. If f is a fuzzy bi-ideal of S then by definition $f(0) = f(x0x) \geq f(x) \wedge f(x) = f(x)$ for all $x \in S$.

Conversely, assume that f satisfies $f(0) \geq f(x)$ for all $x \in S$. Then $xyz = x$ if $x, y, z \in \{a, b\}$ and $xyz = 0$ if one of them is zero. Thus $f(xyz) \geq f(x) \wedge f(z)$. □

Fact: 3. Since S is regular and intra-regular so every fuzzy bi-ideal is semiprime.

Consider the fuzzy bi-ideal f, g, h of S given by

$f(0) = .7, \;\; f(a) = .6, \;\; f(b) = .4,$

$g(0) = 1,\ \ g(a) = .5,\ \ g(b) = .4,$
$h(0) = .7,\ \ h(a) = .65,\ \ h(b) = .3,$
and $g \circ h(0) = .7,\ \ g \circ h(a) = .5,\ \ g \circ h(b) = .4.$

Thus $g \circ h \leq f$ but neither $g \leq f$ nor $h \leq f$. Hence f is not a prime fuzzy bi-ideal of S.

Example 4.6. Consider the semigroup $S = \{0, x, 1\}$

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

It is evident that S is commutative and both regular and intra-regular.

Bi-ideals in S are $\{0\}, \{0, x\}$ and S. All bi-ideals are strongly prime.

Fact: 1. Every fuzzy subset of S is a fuzzy subsemigroup of S.

Fact: 2. A fuzzy subset f of S is a fuzzy bi-ideal of S if and only if $f(0) \geq f(x) \geq f(1)$.

Proof. If f is a fuzzy bi-ideal of S, then
$f(0) = f(a0a) \geq f(a) \wedge f(a) = f(a)$ for all $a \in S$.
Also $f(x) = f(1x1) \geq f(1) \wedge f(1) = f(1)$.
Hence $f(0) \geq f(x) \geq f(1)$.
Conversely, assume that f satisfies $f(0) \geq f(x) \geq f(1)$.
Then obviously f is a fuzzy subsemigroup of S. Also $abc = x$ if $a, b, c \in \{x, 1\}$ and one of a, b, c is x and $abc = 1$ if $a = b = c = 1$. Thus $f(abc) \geq f(a) \wedge f(c)$.
Hence f is a fuzzy bi-ideal of S. □

Consider the fuzzy bi-ideals f, g and h of S given by
$f(0) = .7,\quad f(x) = .6,\quad f(1) = .5,$
$g(0) = 1,\quad g(x) = .5,\quad g(1) = .4,$
$h(0) = .7,\quad h(x) = .65,\quad h(1) = .3.$
Then $g \circ h(0) = h \circ g(0) = .7,\ \ g \circ h(x) = h \circ g(x) = .5,\quad g \circ h(1) = h \circ g(1) = .3.$
Thus $g \circ h \wedge h \circ g \leq f$ but neither $g \leq f$ nor $h \leq f$. Hence f is not strongly a prime fuzzy bi-ideal of S.

Example 4.7. Let S be a Kronecker Semigroup, that is, S has a zero and

$$xy = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and assume that $|S| > 2$. Since $x^2 = x$ for all $x \in S$, S is both regular and intra-regular.

Fact: Every fuzzy subset f of S is a fuzzy bi-ideal of S if and only if $f(0) \geq f(x)$ for all $x \in S$.

Proof. If f is a fuzzy bi-ideal of S then by definition $f(0) = f(xyx) \geq f(x) \wedge f(x) = f(x)$ for all $x, y \in S$. Conversely, assume that f satisfies $f(0) \geq f(x)$ for all $x \in S$.
$f(xx) = f(x) \geq f(x) \wedge f(x)$ and

$f(xy) = f(0) \geq f(x) \wedge f(y)$ for all $x, y \in S$ and $x \neq y$.
Also
$f(xxx) = f(x) \geq f(x) \wedge f(x)$
and
$f(xyz) = f(0) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$ such that $|\{x, y, z\}| > 1$.

Since S is regular and intra-regular so every fuzzy bi-ideal of S is semiprime, that is, every fuzzy subset which satisfies $f(0) \geq f(x)$ is a semiprime fuzzy bi-ideal of S. Since S is commutative, so $f \circ g = g \circ f$ for all fuzzy subsets of S. Hence by Theorem 4.2, $f \circ g = f \wedge g$.

Let h be any non-constant fuzzy bi-ideal of S and a, $b \in S$ such that $a \neq b$ and both a, b are non-zero. Define $f : S \to [0,1]$ and $g : S \to [0,1]$ such that
$f(x) = h(x)$ for all $x \in S\backslash a$ and $f(0) \geq f(a) > h(a)$,
$g(x) = h(x)$ for all $x \in S\backslash b$ and $g(0) \geq g(b) > h(b)$.

Then f and g are fuzzy bi-ideals of S and $f \circ g = f \wedge g \leq h$ but neither $f \nleq h$ nor $g \nleq h$. Hence h is not prime. □

Example 4.8. Let S be a left zero semigroup, that is, $xy = x$, for all x, $y \in S$ and $|S| > 1$, where $|S|$ denotes the cardinality of S.

Then S is both regular and intra-regular.
Fact: Every fuzzy subset of S is a fuzzy bi-ideal of S.

Proof. Let f be a fuzzy subset of S. Then
$f(xy) = f(x) \geq f(x) \wedge f(y)$ for all x, $y \in S$.
Also
$f(xyz) = f(x) \geq f(x) \wedge f(z)$ for all x, y, $z \in S$.

Thus f is a fuzzy bi-ideal of S. Since S is both regular and intra-regular so every fuzzy subset of S is a fuzzy semiprime bi-ideal of S. □

Example 4.9. It is shown in [18, P. 51], that a semigroup S is a group if and only if every fuzzy bi-ideal of S is a constant function. Since every group is regular and intra-regular and the set of fuzzy bi-ideals of S is totally ordered, so every fuzzy bi-ideal of a group is strongly prime.

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