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SOME PROPERTIES OF FUZZY HILBERT SPACES AND NORM OF OPERATORS

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Abstract. In the present paper we define the notion of fuzzy inner product and study the properties of the corresponding fuzzy norm. In particular, it is shown that the Cauchy-Schwarz inequality holds. Moreover, it is proved that every such fuzzy inner product space can be imbedded in a complete one and that every subspace of a fuzzy Hilbert space has a complementary subspace. Finally, the notions of fuzzy boundedness and operator norm are introduced and the relationship between continuity and boundedness are investigated. It is shown also that the space of all fuzzy bounded operators is complete.

1. Introduction

The concept of fuzzy metric spaces was initially introduced by O. Kaleva and S. Seikkla [5], who proved a fixed point theorem for such spaces. C. Felbin [3] introduced the concept of fuzzy norm and showed that every finite dimensional normed linear space has a completion. J. Xiao and X. Zhu [11] modified the definition of fuzzy norm and studied the topological properties of fuzzy normed linear spaces. we introduce the concept of a fuzzy inner product and show that the resulting norm satisfies the Cauchy-Schwarz inequality. Moreover, every fuzzy inner product space can be imbedded in a complete fuzzy inner product space and that every subspace of a fuzzy Hilbert space has a complementary subspace.

Finally, We note that the definition of the fuzzy norm of an operator was given in [1] and [10]; do not satisfy the basic properties

$$\|Tx\| \leq \|T\|\|x\| \quad and \quad \|TS\| \leq \|T\|\|S\|$$

for operators T, S and vector x in general. the notions of fuzzy boundedness and operator norm are introduced and it is shown that any operator on a finite dimensional fuzzy normed linear spaces is fuzzy bounded. Moreover, the relationship between continuity and boundedness of this are investigated. It is shown also that the space of all fuzzy bounded operators with the new norm is complete.

2. Preliminaries

Definition 2.1. [11] A mapping $\eta : \mathbf{R} \longrightarrow [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0 \in \mathbf{R}$ such that $\eta(t_0) = 1$.

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(N2) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.
The set of all fuzzy real numbers is denoted by $F(\mathbf{R})$. Since each $r \in \mathbf{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbf{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1 & , \quad t = r \\ 0 & , \quad t \neq r, \end{cases}$$

it follows that $\mathbf{R}$ can be embedded in $F(\mathbf{R})$.

Definition 2.2. [11] A mapping $\eta : \mathbf{R} \longrightarrow [0,1]$ is called convex if $\eta(t) \geq \min(\eta(s), \eta(r))$ where $s \leq t \leq r$. If there exists a $t_0 \in \mathbf{R}$ such that $\eta(t_0) = 1$, then η is called normal.

Remark 2.3. A mapping η is convex if and only if each of its α-level sets $[\eta]_\alpha$, $0 < \alpha \leq 1$, is a convex set in $\mathbf{R}$.

Lemma 2.4. *$\eta \in F(\mathbf{R})$ if and only if η satisfies:*
(1) η is normal, convex and upper semicontinuous.
(2) $\lim_{t \longrightarrow -\infty} \eta(t) = \lim_{t \longrightarrow +\infty} \eta(t) = 0$.

Proof. Lemma 2.1 [11]. □

Definition 2.5. [5] The arithmetic operations $+$, $-$, $\times$ and $/$ on $F(\mathbf{R}) \times F(\mathbf{R})$ are defined by

$$\begin{aligned} (\eta + \gamma)(t) &= \sup_{t=x+y} (\min(\eta(x), \gamma(y))), \\ (\eta - \gamma)(t) &= \sup_{t=x-y} (\min(\eta(x), \gamma(y))), \\ (\eta \times \gamma)(t) &= \sup_{t=xy} (\min(\eta(x), \gamma(y))), \\ (\eta / \gamma)(t) &= \sup_{t=x/y} (\min(\eta(x), \gamma(y))), \end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 2.6. [5] The absolute value $|\eta|$ of $\eta \in F(\mathbf{R})$ is defined by

$$|\eta|(t) = \begin{cases} \max(\eta(t), \eta(-t)) & , \quad t \geq 0 \\ 0 & , \quad t < 0. \end{cases}$$

Definition 2.7. [5] Let $\eta \in F(\mathbf{R})$. If $\eta(t) = 0$ for all $t < 0$, then η is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $F^+(\mathbf{R})$.

Note: real number $\eta_\alpha^- \geq 0$ for all $\eta \in F^+(\mathbf{R})$ and all $\alpha \in (0,1]$.

Lemma 2.8. *Let $\eta, \gamma \in F(\mathbf{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. Then*
i) $[\eta + \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+]$
ii) $[\eta - \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-]$
iii) $[\eta \times \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+]$ for $\eta, \gamma \in F^+(\mathbf{R})$

iv) $[\tilde{1}/\eta]_\alpha = [\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}]$ *if* $\eta_\alpha^- > 0$
v) $[|\eta|]_\alpha = [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)]$.

Proof. Lemma 2.1 [5]. □

Lemma 2.9. *Let* $[a^\alpha, b^\alpha]$, $0 < \alpha \le 1$, *be a given family of non-empty intervals. Assume*
a)$[a^{\alpha_1}, b^{\alpha_1}] \supset [a^{\alpha_2}, b^{\alpha_2}]$ *for all* $0 < \alpha_1 \le \alpha_2$,
b)$[\lim_{k\to-\infty} a^{\alpha_k}, \lim_{k\to\infty} b^{\alpha_k}] = [a^\alpha, b^\alpha]$ *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0,1]$ *converging to* α,
c)$-\infty < a^\alpha \le b^\alpha < +\infty$, *for all* $\alpha \in (0,1]$.
Then the family $[a^\alpha, b^\alpha]$ *represents the* α*-level sets of a fuzzy real number* $\eta \in F(\mathbf{R})$.

Conversely if $[a^\alpha, b^\alpha]$, $0 < \alpha \le 1$, *are the* α*-level sets of a fuzzy number* $\eta \in F(\mathbf{R})$, *then the conditions (a),(b) and (c) are satisfied.*

Proof. Lemma 2.2 [5]. □

Definition 2.10. [5] Let $\eta, \gamma \in F(\mathbf{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \le \gamma$ if and only if $\eta_\alpha^- \le \gamma_\alpha^-$ and $\eta_\alpha^+ \le \gamma_\alpha^+$, for all $\alpha \in (0,1]$. Strict inequality in $F(\mathbf{R})$ is defined by $\eta < \gamma$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0,1]$.

Lemma 2.11. *Let* $\eta \in F(\mathbf{R})$. *Then* $\eta \in F^+(\mathbf{R})$ *if and only if* $\tilde{0} \le \eta$.

Proof. The proof follows immediately from Definition 2.10. □

Theorem 2.12. *Let* $\eta \in F^+(\mathbf{R})$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, *for all* $\alpha \in (0,1]$. *Furthermore, let* $\left[\sqrt{\eta_\alpha^-}, \sqrt{\eta_\alpha^+}\right]$, $0 < \alpha \le 1$, *be a family of non-empty intervals. Then the conditions (a),(b) and (c) of Lemma 2.9 are satisfied.*

Remark 2.13. Let $\eta \in F^+(\mathbf{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, for all $\alpha \in (0,1]$. Then by Theorem 2.12 and Lemma 2.9, the family $\left[\sqrt{\eta_\alpha^-}, \sqrt{\eta_\alpha^-}\right]$, $\alpha \in (0,1]$, represents the α-level sets of a fuzzy number γ in $F^+(\mathbf{R})$. Thus we conclude the following definition.

Definition 2.14. For a positive fuzzy real number η we define $\sqrt{\eta} = \gamma$, where $[\gamma]_\alpha = \left[\sqrt{\eta_\alpha^-}, \sqrt{\eta_\alpha^+}\right]$, $\alpha \in (0,1]$.

Lemma 2.15. *Let* $\eta \in F^+(\mathbf{R})$ *and* $\gamma \in F(\mathbf{R})$. *Then*
(i) $(\sqrt{\eta})^2 = \eta$,
(ii) $\gamma \le |\gamma|$.

Definition 2.16. The sequence $\{\eta_n\}$ in $F(\mathbf{R})$ converges to η in $F(\mathbf{R})$ ($\lim_{n\to\infty} \eta_n = \eta$), if $\lim_{n\to\infty} |\eta_n - \eta|_\alpha^+ = 0$, for all $\alpha \in (0,1]$.

Definition 2.17. [11] Let X be a vector space over $\mathbf{R}$. Assume the mappings $L, R : [0,1] \times [0,1] \longrightarrow [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|.\| : X \longrightarrow F^+(\mathbf{R})$. The quadruple

$(X, \|.\|, L, R)$ is called a fuzzy normed linear space (briefly, FNS) with the fuzzy norm $\|.\|$, if the following conditions are satisfied:
(F_1) if $x \neq 0$ then $\inf_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$,
(F_2) $\|x\| = \tilde{0}$ if and only if $x = 0$,
(F_3) $\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbf{R}$,
(F_4) for all $x, y \in X$,
$(F_4L)\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,
$(F_4R)\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$.

Lemma 2.18. *Let $(X, \|.\|, L, R)$ be an FNS.*
(1) If $L \leq \min$, then (F_4L) holds whenever $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $\alpha \in (0, 1]$ and $x, y \in X$.
(2) If $L \geq \min$, then $\|x + y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $\alpha \in (0, 1]$ and $x, y \in X$ whenever (F_4L) holds.
(3) If $R \geq \max$, then (F_4R) holds whenever $\|x + y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $\alpha \in (0, 1]$ and $x, y \in X$.
(4) If $R \leq \max$, then $\|x, y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $\alpha \in (0, 1]$ and $x, y \in X$ whenever (F_4R) holds.

In the sequel we fix $L(s, t) = \min(s, t)$ and $R(s, t) = \max(s, t)$ for all $s, t \in [0, 1]$ and we write $(X, \|.\|)$ or simply X when L and R are as indicated above.

The following result is an analogue of the triangle inequality.

Theorem 2.19. *In a fuzzy normed linear space $(X, \|.\|)$, the condition (F_4) is equivalent to*

$$\|x + y\| \leq \|x\| \oplus \|y\|.$$

Definition 2.20. [11] Let $(X, \|.\|)$ be a FNS.
i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0, 1]$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{m,n\to\infty} \|x_n - x_m\|_\alpha^+ = 0$, for all $\alpha \in (0, 1]$.

Definition 2.21. [11] Let $(X, \|.\|)$ be a FNS. A subset A of X is said to be complete, if every Cauchy sequence in A converges in A.

Definition 2.22. [11] Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. A function $\varphi : X \longrightarrow Y$ is said to be continuous at $x \in X$, if $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$ whenever $\{x_n\} \subseteq X$ and $\lim_{n\to\infty} x_n = x$.

Theorem 2.23. *Let $(X, \|.\|)$ be a fuzzy normed linear space. Then, for all $\alpha \in (0, 1]$, $(X, \|.\|_\alpha^-)$ and $(X, \|.\|_\alpha^+)$ are normed linear spaces.*

Proof. Let $\alpha \in (0, 1]$ and $x, y \in X$.

(1) By Definition 2.17 (F_1), $\|x\|_\alpha^- \geq 0$.

(2) If $\|x\|_\alpha^- = 0$ then by Definition 2.17 (F_1), $x = 0$.
If $x = 0$ then by Definition 2.17 (F_2), $\|x\| = 0$ and hence $\|x\|_\alpha^- = 0$.
(3) Let $r \in \mathbf{R}$. By Definition 2.17 (F_3), we have $\|rx\| = |r|\|x\|$. Hence

$$[\|rx\|_\beta^-, \|rx\|_\beta^+] = [\|rx\|]_\beta = [|r|\|x\|]_\beta = [|r|\|x\|_\beta^-, |r|\|x\|_\beta^+], \text{ for all } \beta \in (0,1].$$

And thus $\|rx\|_\alpha^- = |r|\|x\|_\alpha^-$.
(4) By Theorem 2.19 we have

$$\|x+y\| \leq \|x\| \oplus \|y\|$$

hence

$$\|x+y\|_\beta^- \leq \|x\|_\beta^- + \|y\|_\beta^- \text{ and } \|x+y\|_\beta^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+ \text{ for all } \beta \in (0,1].$$

And so $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$.
By (1) to (4) $(X, \|.\|_\alpha^-)$ is a normed linear space.
(5) Since $\|x\|_\alpha^- \leq \|x\|_\alpha^+$, By Definition 2.17 ($F_1$), $\|x\|_\alpha^+ \geq 0$.
(6) Let $\|x\|_\alpha^+ = 0$. Since $\|x\|_\alpha^- \leq \|x\|_\alpha^+$ it follows that $\|x\|_\alpha^- = 0$. Then by Definition 2.17 (F_1), $x = 0$.
If $x = 0$ then by Definition 2.19 (F_2), $\|x\| = 0$ and hence $\|x\|_\alpha^+ = 0$.
(7) Let $r \in \mathbf{R}$. By Definition 2.17 (F_3), we have $\|rx\| = |r|\|x\|$. Hence

$$[\|rx\|_\beta^-, \|rx\|_\beta^+] = [\|rx\|]_\beta = [|r|\|x\|]_\beta = [|r|\|x\|_\beta^-, |r|\|x\|_\beta^+], \text{ for all } \beta \in (0,1].$$

And thus $\|rx\|_\alpha^+ = |r|\|x\|_\alpha^+$.
(8) By Theorem 2.19 we have

$$\|x+y\| \leq \|x\| \oplus \|y\|$$

hence

$$\|x+y\|_\beta^- \leq \|x\|_\beta^- + \|y\|_\beta^- \text{ and } \|x+y\|_\beta^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+ \text{ for all } \beta \in (0,1].$$

And so $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$.
By (5) to (8) $(X, \|.\|_\alpha^+)$ is a normed linear space. □

3. **Fuzzy Hilbert Space**

Definition 3.1. Let X be a vector space over $\mathbf{R}$. A fuzzy inner product on X is a mapping $\langle .,. \rangle : X \times X \longrightarrow F(\mathbf{R})$ such that for all vectors $x, y, z \in X$ and all $r \in \mathbf{R}$, we have:
(IP_1) $\langle x+y, z\rangle = \langle x, z\rangle \oplus \langle y, z\rangle$
(IP_2) $\langle rx, y\rangle = \tilde{r}\langle x, y\rangle$
(IP_3) $\langle x, y\rangle = \langle y, x\rangle$
(IP_4) $\langle x, x\rangle \geq \tilde{0}$
(IP_5) $\inf_{\alpha\in(0,1]} \langle x, x\rangle_\alpha^- > 0$ if $x \neq 0$
(IP_6) $\langle x, x\rangle = \tilde{0}$ *if and only if* $x = 0$.
The vector space X equipped with a fuzzy inner product is called a fuzzy inner product space.

A fuzzy inner product on X defines a fuzzy number

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad for \quad all \quad x \in X. \tag{1}$$

The following Lemma shows that the function $\|.\|$ is a well-defined fuzzy norm.

Lemma 3.2. *A fuzzy inner product space X together with its corresponding norm $\|.\|$ satisfy the Schwarz inequality*

$$|\langle x, y\rangle| \leq \|x\| \ \|y\| \quad for \quad all \quad x, y \in X.$$

Proof. Let $x, y \neq 0$, and $[\langle x, y\rangle]_\alpha = [\langle x, y\rangle^-_\alpha, \langle x, y\rangle^+_\alpha], \alpha \in (0, 1]$. Suppose that $m_\alpha = \max(0, \langle x, y\rangle^-_\alpha, -\langle x, y\rangle^+_\alpha)$ and $m'_\alpha = \max(|\langle x, y\rangle^-_\alpha|, |\langle x, y\rangle^+_\alpha|)$. Then, by Lemma 2.8,

$$[|\langle x, y\rangle|]_\alpha = [m_\alpha, m'_\alpha], \quad \alpha \in (0, 1].$$

Let $[\langle x, x\rangle]_\alpha = [\langle x, x\rangle^-_\alpha, \langle x, x\rangle^+_\alpha]$ and $[\langle y, y\rangle]_\alpha = [\langle y, y\rangle^-_\alpha, \langle y, y\rangle^+_\alpha], \quad \alpha \in (0, 1]$. By Definition 2.14, $[\|x\|]_\alpha = [\sqrt{\langle x, x\rangle^-_\alpha}, \sqrt{\langle x, x\rangle^+_\alpha}]$ and $[\|y\|]_\alpha = [\sqrt{\langle y, y\rangle^-_\alpha}, \sqrt{\langle y, y\rangle^+_\alpha}]$, $\alpha \in (0, 1]$.

We show that $m_\alpha \leq \sqrt{\langle x, x\rangle^-_\alpha}\sqrt{\langle y, y\rangle^-_\alpha}$. By Definition 3.1,

$$\tilde{0} \leq \langle x + ry, x + ry\rangle = \langle x, x\rangle + \tilde{2r}\langle x, y\rangle + \tilde{r^2}\langle y, y\rangle, \tag{2}$$

then

$$0 \leq \langle x + ry, x + ry\rangle^-_\alpha = \begin{cases} \langle x, x\rangle^-_\alpha + 2r\langle x, y\rangle^-_\alpha + r^2\langle y, y\rangle^-_\alpha, & r \geq 0 \\ \langle x, x\rangle^-_\alpha + 2r\langle x, y\rangle^+_\alpha + r^2\langle y, y\rangle^-_\alpha, & r < 0. \end{cases} \tag{3}$$

Case 1: Assume $\langle x, y\rangle^+_\alpha < 0$. Let $r < 0$. The following conditions are equivalent:
i) $2r\langle x, y\rangle^+_\alpha \leq -2rm_\alpha$,
ii) $2r(\langle x, y\rangle^+_\alpha + m_\alpha) \leq 0$,
iii) $\langle x, y\rangle^+_\alpha + m_\alpha \geq 0$.
Since $m_\alpha = \max(0, -\langle x, y\rangle^+_\alpha, \langle x, y\rangle^-_\alpha) = -\langle x, y\rangle^+_\alpha$, it follows that

$$\langle x, y\rangle^+_\alpha + m_\alpha = \langle x, y\rangle^+_\alpha - \langle x, y\rangle^+_\alpha \geq 0$$

and hence

$$2r\langle x, y\rangle^+_\alpha \leq -2rm_\alpha, \ \forall r < 0. \tag{4}$$

Next, let $r \geq 0$. The following condition are equivalent:
i) $2r\langle x, y\rangle^-_\alpha \leq -2rm_\alpha$,
ii) $2r(\langle x, y\rangle^-_\alpha + m_\alpha) \leq 0$,
iii) $\langle x, y\rangle^-_\alpha + m_\alpha \leq 0$,
iv) $\langle x, y\rangle^-_\alpha - \langle x, y\rangle^+_\alpha \leq 0$.
Hence

$$2r\langle x, y\rangle^-_\alpha \leq -2rm_\alpha, \quad \forall r \geq 0. \tag{5}$$

Then, by (3), (4) and (5),

$$0 \leq \langle x,x\rangle_\alpha^- - 2rm_\alpha + r^2\langle y,y\rangle_\alpha^-, \quad \forall r \in \mathbf{R}.$$

Let $r = m_\alpha/\langle y,y\rangle_\alpha^- \in \mathbf{R}$. Since $\langle y,y\rangle_\alpha^- \geq \inf\limits_{\alpha\in(0,1]} \langle y,y\rangle_\alpha^- > 0$, r is well defined and thus $m_\alpha^2 \leq \langle x,x\rangle_\alpha^- \langle y,y\rangle_\alpha^-$. Also since $m_\alpha \geq 0$, $m_\alpha \leq \sqrt{\langle x,x\rangle_\alpha^-}\sqrt{\langle y,y\rangle_\alpha^-}$.

Case 2: Assume $\langle x,y\rangle_\alpha^+ > 0$. Let $r < 0$. The following conditions are equivalent:

i) $2r\langle x,y\rangle_\alpha^+ \leq 2rm_\alpha$,
ii) $2r(\langle x,y\rangle_\alpha^+ - m_\alpha) \leq 0$,
iii) $\langle x,y\rangle_\alpha^+ - m_\alpha \geq 0$.

Since $m_\alpha = \max(0, -\langle x,y\rangle_\alpha^+, \langle x,y\rangle_\alpha^-) = \max(0, \langle x,y\rangle_\alpha^-)$,

$$2r\langle x,y\rangle_\alpha^+ \leq 2rm_\alpha, \ \forall r < 0. \tag{6}$$

Next, let $r \geq 0$. The following conditions are equivalent:

i) $2r\langle x,y\rangle_\alpha^- \leq 2rm_\alpha$,
ii) $\langle x,y\rangle_\alpha^- \leq m_\alpha$.

Hence

$$2r\langle x,y\rangle_\alpha^- \leq 2rm_\alpha, \quad \forall r \geq 0. \tag{7}$$

Then, by (3),(6) and (7),

$$0 \leq \langle x,x\rangle_\alpha^- + 2rm_\alpha + r^2\langle y,y\rangle_\alpha^-, \quad \forall r \in \mathbf{R}.$$

Let $r = -m_\alpha/\langle y,y\rangle_\alpha^-$. Since $\langle y,y\rangle_\alpha^- \geq \inf\limits_{\alpha\in(0,1]} \langle y,y\rangle_\alpha^- > 0$, r is well defined and thus $m_\alpha^2 \leq \langle x,x\rangle_\alpha^- \langle y,y\rangle_\alpha^-$. Also since $m_\alpha \geq 0$, $m_\alpha \leq \sqrt{\langle x,x\rangle_\alpha^-}\sqrt{\langle y,y\rangle_\alpha^-}$.

Now we show that $m'_\alpha \leq \sqrt{\langle x,x\rangle_\alpha^+}\sqrt{\langle y,y\rangle_\alpha^+}$. By (2),

$$0 \leq \langle x+ry, x+ry\rangle_\alpha^+ = \begin{cases} \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^+ + r^2\langle y,y\rangle_\alpha^+, & r \geq 0 \\ \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^- + r^2\langle y,y\rangle_\alpha^+, & r < 0. \end{cases} \tag{8}$$

From (3), we have

$$\begin{cases} 0 \leq \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^- + r^2\langle y,y\rangle_\alpha^+, & r \geq 0 \\ 0 \leq \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^+ + r^2\langle y,y\rangle_\alpha^+, & r < 0. \end{cases} \tag{9}$$

Then, by (8) and (9),

$$0 \leq \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^- + r^2\langle y,y\rangle_\alpha^+, \quad \forall r \in \mathbf{R}.$$

Let $r = -\langle x,y\rangle_\alpha^-/\langle y,y\rangle_\alpha^+$. Then $(\langle x,y\rangle_\alpha^-)^2 \leq \langle x,x\rangle_\alpha^+\langle y,y\rangle_\alpha^+$ and hence

$$|\langle x,y\rangle_\alpha^-| \leq \sqrt{\langle x,x\rangle_\alpha^+}\sqrt{\langle y,y\rangle_\alpha^+}. \tag{10}$$

By (8) and (9), we have

$$0 \leq \langle x,x\rangle_\alpha^+ + 2r\langle x,y\rangle_\alpha^+ + r^2\langle y,y\rangle_\alpha^+, \forall r \in \mathbf{R}.$$

Let $r = -\langle x,y\rangle_\alpha^+/\langle y,y\rangle_\alpha^+$. Then $(\langle x,y\rangle_\alpha^+)^2 \leq \langle x,x\rangle_\alpha^+\langle y,y\rangle_\alpha^+$ and hence

$$|\langle x,y\rangle_\alpha^+| \leq \sqrt{\langle x,x\rangle_\alpha^+}\sqrt{\langle y,y\rangle_\alpha^+}. \tag{11}$$

Now, by (10) and (11), $m'_\alpha \leq \sqrt{\langle x,x\rangle_\alpha^+}\sqrt{\langle y,y\rangle_\alpha^+}$.

Hence, by Definition 2.10 and Lemma 2.8,

$$|\langle x,y\rangle| \leq \|x\|\ \|y\|.$$

If $y=0$, then $\langle x,0\rangle = \langle x,0\rangle + \langle x,0\rangle$ and hence $\langle x,0\rangle_\alpha^- = \langle x,0\rangle_\alpha^- + \langle x,0\rangle_\alpha^-$. Thus, $\langle x,0\rangle_\alpha^- = 0$, for all $\alpha \in (0,1]$. Similarly $\langle x,0\rangle_\alpha^+ = 0$. Consequently, $\langle x,y\rangle = \langle x,0\rangle = \tilde{0}$, which implies that

$$|\langle x,y\rangle| = \tilde{0} \leq \tilde{0} = \|x\|\ \|y\|.$$

□

Theorem 3.3. *The function $\|.\|$ defined in Definition 3.1 is a fuzzy norm.*

Proof. (F1) By Definition 3.1(IP5), $\inf_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$, if $x \neq 0$.

(F2) $\|x\| = \tilde{0}$ if and only if $\langle x,x\rangle = \tilde{0}$ if and only if $x=0$.

(F3) $\|rx\| = \sqrt{\langle rx,rx\rangle} = \sqrt{\tilde{r}^2\langle x,x\rangle} = \sqrt{\tilde{r}^2}\sqrt{\langle x,x\rangle} = |\tilde{r}|\sqrt{\langle x,x\rangle} = |\tilde{r}|\ \|x\|$.

(F4) By Theorem 2.19, it is sufficient to show that $\|x+y\| \leq \|x\| + \|y\|$.We have

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y,x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\
&= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\ \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

and hence

$$\|x+y\| \leq \|x\| + \|y\|.$$

□

A fuzzy Hilbert space is a complete fuzzy inner product space with the fuzzy norm defined by (1).

Theorem 3.4. *Let X be a fuzzy inner product space. For all $x,y \in X$, if $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $\langle x_n,y_n\rangle \longrightarrow \langle x,y\rangle$.*

Proof. Let $[\langle x_n,y_n\rangle]_\alpha = [\langle x_n,y_n\rangle_\alpha^-, \langle x_n,y_n\rangle_\alpha^+]$ and $[\langle x,y\rangle]_\alpha = [\langle x,y\rangle_\alpha^-, \langle x,y\rangle_\alpha^+]$. Consider

$$\begin{aligned}
|\langle x_n,y_n\rangle_\alpha^- - \langle x,y\rangle_\alpha^+| &\leq |\langle x_n,y_n\rangle_\alpha^- - \langle x_n,y\rangle_\alpha^+| + |\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+| \\
&= |\langle x_n,y_n - y\rangle_\alpha^-| + |\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+| \\
&\leq \|x_n\|_\alpha^+\|y_n - y\|_\alpha^+ + |\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+|. \qquad (12)
\end{aligned}$$

We show that $|\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+| \leq \|y\|_\alpha^+\|x_n - x\|_\alpha^+$.

Case 1: If $0 \leq \langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+$, we have $0 \leq \langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+ \leq \langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^-$ and thus

$$|\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^+| \leq |\langle x_n,y\rangle_\alpha^+ - \langle x,y\rangle_\alpha^-| = |\langle x_n - x,y\rangle_\alpha^+| \leq \|x_n - x\|_\alpha^+\|y\|_\alpha^+.$$

Case 2: If $0 \leq \langle x, y\rangle_\alpha^+ - \langle x_n, y\rangle_\alpha^+$, we have $0 \leq \langle x, y\rangle_\alpha^+ - \langle x_n, y\rangle_\alpha^+ \leq \langle x, y\rangle_\alpha^+ - \langle x_n, y\rangle_\alpha^-$ and thus

$$|\langle x_n, y\rangle_\alpha^+ - \langle x, y\rangle_\alpha^+| \leq |\langle x, y\rangle_\alpha^+ - \langle x_n, y\rangle_\alpha^-| = |\langle x - x_n, y\rangle_\alpha^+| \leq \|x - x_n\|_\alpha^+ \|y\|_\alpha^+.$$

Hence

$$|\langle x_n, y\rangle_\alpha^+ - \langle x, y\rangle_\alpha^+| \leq \|y\|_\alpha^+ \|x_n - x\|_\alpha^+. \tag{13}$$

Then, by (12), (13)

$$|\langle x_n, y_n\rangle_\alpha^- - \langle x, y\rangle_\alpha^+| \leq \|x_n\|_\alpha^+ \|y_n - y\|_\alpha^+ + \|y\|_\alpha^+ \|x_n - x\|_\alpha^+.$$

Since $x_n \longrightarrow x$ and $y_n \longrightarrow y$, it follows from Definition 2.20 that

$$\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = \lim_{n\to\infty} \|y_n - y\|_\alpha^+ = 0,$$

and hence

$$\lim_{n\to\infty} |\langle x_n, y_n\rangle_\alpha^- - \langle x, y\rangle_\alpha^+| = 0, \quad \forall \alpha \in (0, 1]. \tag{14}$$

Similarly we have

$$\lim_{n\to\infty} |\langle x_n, y_n\rangle_\alpha^+ - \langle x, y\rangle_\alpha^-| = 0, \forall a \in (0, 1]. \tag{15}$$

Now, we have

$$[\langle x_n, y_n\rangle - \langle x, y\rangle]_\alpha = [\langle x_n, y_n\rangle_\alpha^- - \langle x, y\rangle_\alpha^+, \langle x_n, y_n\rangle_\alpha^+ - \langle x, y\rangle_\alpha^-].$$

Therefore,

$$|\langle x_n, y_n\rangle - \langle x, y\rangle|_\alpha^+ = \max(|\langle x_n, y_n\rangle_\alpha^- - \langle x, y\rangle_\alpha^+|, |\langle x_n, y_n\rangle_\alpha^+ - \langle x, y\rangle_\alpha^-|),$$

and, in view of (14) and (15), $\lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle|_\alpha^+ = 0$ for all $\alpha \in (0, 1]$. Hence, by Definition 2.16, $\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$.

□

Definition 3.5. [4] Two fuzzy normed linear spaces $(X, \|.\|)$ and $(X^*, \|.\|^*)$ are called congruent if there exists an isometry of $(X, \|.\|)$ onto $(X^*, \|.\|^*)$.

Definition 3.6. [4] A complete fuzzy normed linear space $(X^*, \|.\|^*)$ is a completion of a fuzzy normed linear space $(X, \|.\|)$ if
(i) $(X, \|.\|)$ is congruent to a subspace $(X_0, \|.\|^*)$ of $(X^*, \|.\|^*)$, and
(ii) the closure $\overline{X}_0$ of X_0, is all of X^*; i.e., $\overline{X}_0 = X^*$.

Definition 3.7. Let X be a fuzzy inner product space.
i) Let $\{x_n\}$ and $\{y_n\}$ be a Cauchy sequences in X. Then $\{x_n\}$ is said to be equivalent to $\{y_n\}$, written $\{x_n\} \sim \{y_n\}$, if and only if

$$\lim_{n\to\infty} \|x_n - y_n\|_\alpha^+ = 0.$$

Note that, by [4,th.3], the relation $\sim$ is an equivalence relation.
ii) The collection of all equivalence classes of $\sim$ is denoted by X^*. Let $x^*, y^* \in X^*$, we define $x^* + y^*$ as the class represented by $\{x_n + y_n\}$, where $\{x_n\} \in x^*$, $\{y_n\} \in y^*$. Furthermore, if $r \in \mathbf{R}$ and $\{x_n\} \in x^*$, we define rx^* as the class containing $\{rx_n\}$. Note that again in view of the Theorem 3 in [4], the space X^* together with the operations of addition and scalar multiplication defined above is a linear space.

Definition 3.8. Let $x^*, y^* \in X^*$ with representatives $\{x_n\}$ and $\{y_n\}$, respectively. Assume $\alpha \in (0, 1]$ and $\{\alpha_k\}$ in a strictly increasing sequence converging to α. Define

$$[\langle x^*, y^*\rangle]_\alpha = \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^+_{\alpha_k}\right]. \qquad (I)$$

Lemma 3.9. *The function $\langle .,.\rangle$ defined in Definition 3.8, is a fuzzy inner product on X^*.*

Proof. Let X^* be as in Definition 3.8. We show that (I) is well-defined; i.e., the limits on right side of (I) exist and are independent $\{x_n\} \in x^*$, $\{y_n\} \in y^*$ and $\{\alpha_k\}$.

First we show that the limits on the right side (I) exist. Fix k. We show that

$$|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| \le \|x_n - x_m\|^+_{\alpha_k}\|y_m\|^+_{\alpha_k} + \|y_n - y_m\|^+_{\alpha_k}\|x_n\|^+_{\alpha_k}.$$

There are two cases.

Case 1: $|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| = \langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}$. Then

$$\begin{aligned}
|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| &= \langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}\\
&\le \langle x_n, y_n\rangle^+_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}\\
&\le |\langle x_n, y_n\rangle^+_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}|\\
&\le \|x_n - x_m\|^+_{\alpha_k}\|y_m\|^+_{\alpha_k} + \|y_n - y_m\|^+_{\alpha_k}\|x_n\|^+_{\alpha_k}.
\end{aligned}$$

Case 2: $|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| = \langle x_m, y_m\rangle^-_{\alpha_k} - \langle x_n, y_n\rangle^-_{\alpha_k}$. Then

$$\begin{aligned}
|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| &= \langle x_m, y_m\rangle^-_{\alpha_k} - \langle x_n, y_n\rangle^-_{\alpha_k}\\
&\le \langle x_m, y_m\rangle^+_{\alpha_k} - \langle x_n, y_n\rangle^-_{\alpha_k}\\
&\le |\langle x_m, y_m\rangle^+_{\alpha_k} - \langle x_n, y_n\rangle^-_{\alpha_k}|\\
&\le \|x_n - x_m\|^+_{\alpha_k}\|y_m\|^+_{\alpha_k} + \|y_n - y_m\|^+_{\alpha_k}\|x_n\|^+_{\alpha_k}.
\end{aligned}$$

Hence

$$|\langle x_n, y_n\rangle^-_{\alpha_k} - \langle x_m, y_m\rangle^-_{\alpha_k}| \le \|x_n - x_m\|^+_{\alpha_k}\|y_m\|^+_{\alpha_k} + \|y_n - y_m\|^+_{\alpha_k}\|x_n\|^+_{\alpha_k}.$$

Since $\{x_n\}$, $\{y_n\}$ are Cauchy sequences, $\lim_{m,n\to\infty}\|x_n - x_m\| = \lim_{m,n\to\infty}\|y_n - y_m\| = 0$. Hence $\{\langle x_n, y_n\rangle^-_{\alpha_k}\}_{n\ge1}$ is a Cauchy sequence in $\mathbf{R}$. Also since $\mathbf{R}$ is complete, $\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}$ exists. Similarly, $\lim_{n\to\infty}\langle x_n, y_n\rangle^+_{\alpha_k}$ exists.

Since $\alpha_k \le \alpha_{k+1} \le \alpha$, it follows from Lemma 2.9(a) that

$$\langle x_n, y_n\rangle^-_{\alpha_k} \le \langle x_n, y_n\rangle^-_{\alpha_{k+1}} \le \langle x_n, y_n\rangle^-_{\alpha}, \quad for \ \ all \ \ n \ge 1.$$

Hence

$$\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k} \le \lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_{k+1}} \le \lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha}.$$

Since $\{\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}\}_{k\ge1}$ is increasing and bounded, $\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}$ exists.

Also since $\alpha_k \le \alpha_{k+1} \le \alpha$, it follows from Lemma 2.9(a) that

$$\langle x_n, y_n\rangle^+_{\alpha} \le \langle x_n, y_n\rangle^+_{\alpha_{k+1}} \le \langle x_n, y_n\rangle^+_{\alpha_k}, \quad for \ \ all \ \ n \ge 1.$$

Hence

$$\lim_{n\to\infty}\langle x_n,y_n\rangle^+_\alpha \le \lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_{k+1}} \le \lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k}.$$

Since $\{\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k}\}_{k\ge 1}$ is decreasing and bounded, $\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k}$ exists.

Now we show that (I) is independent of $\{x_n\}\in x^*$ and $\{y_n\}\in y^*$. Let $\{x'_n\}\sim\{x_n\}$ and $\{y'_n\}\sim\{y_n\}$. We have

$$|\langle x_n,y_n\rangle^-_{\alpha_k} - \langle x'_n,y'_n\rangle^-_{\alpha_k}| \le \|x_n-x'_n\|^+_{\alpha_k}\|y'_n\|^+_{\alpha_k} + \|y_n-y'_n\|^+_{\alpha_k}\|x_n\|^+_{\alpha_k}.$$

Since $\{x_n\}\sim\{x'_n\}$ and $\{y_n\}\sim\{y'_n\}$, $\lim_{n\to\infty}\|x_n-x'_n\| = \lim_{n\to\infty}\|y_n-y'_n\| = 0$ or, equivalently, $\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha_k} = \lim_{n\to\infty}\langle x'_n,y'_n\rangle^-_{\alpha_k}$ and hence $\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x'_n,y'_n\rangle^-_{\alpha_k}$.

Similarly

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x'_n,y'_n\rangle^+_{\alpha_k}.$$

Now we show that (I) is also independent of the choice of $\{\alpha_k\}$. Let $\alpha_k\nearrow\alpha$ and $\alpha'_k\nearrow\alpha$. Then we can form a sequence $\beta_k\nearrow\alpha$ such that $\{\beta_k\}$ has $\{\alpha'_k\}$ and $\{\alpha_k\}$ as subsequences. Thus

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\beta_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha'_k},$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\beta_k} = \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha'_k}.$$

We show that $\langle .,.\rangle$ defined in Definition 3.8 is a fuzzy real number. Let $x^*,y^*\in X^*$. Assume $\{x_n\}\in x^*$, $\{y_n\}\in y^*$ and let $\{\alpha_k\}$ be a strictly increasing sequence converging to $\alpha\in(0,1]$. Since $\alpha_k\le\alpha$, it follows that

$$\langle x_n,y_n\rangle^-_{\alpha_k} \le \langle x_n,y_n\rangle^-_{\alpha} \le \langle x_n,y_n\rangle^+_{\alpha} \le \langle x_n,y_n\rangle^+_{\alpha_k}$$

and hence

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha_k} \le \lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha} \le \lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha} \le \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k}.$$

So, $[\langle x^*,y^*\rangle]_\alpha\neq\emptyset$, for all $\alpha\in(0,1]$.

We show that $[\langle x^*,y^*\rangle]_\alpha$, $\alpha\in(0,1]$, satisfy the conditions of Lemma 2.9:

(a) Let $0<\beta_1<\beta_2$. Assume that $\alpha_k\nearrow\beta_1$, $\alpha'_k\nearrow\beta_2$ and $\alpha'_k>\beta_1$, for all k. Then, $\alpha_k\le\alpha'_k$, for all k, hence $\langle x_n,y_n\rangle^-_{\alpha_k}\le\langle x_n,y_n\rangle^-_{\alpha'_k}$ and thus

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha_k} \le \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\alpha'_k}.$$

Similarly,

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha'_k} \le \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^+_{\alpha_k}.$$

Hence $[< x^*,y^* >]_{\beta_2}\subseteq[< x^*,y^* >]_{\beta_1}$.

(b) Let $\{\beta_i\}$ be strictly increasing and $\beta_i\nearrow\alpha$. Assume $\beta_{i-1}<\alpha_{ki}\le\beta_i$ and $\alpha_{ki}\nearrow\beta_i$, for all i. We have

$$\langle x^*,y^*\rangle^-_\alpha = \lim_{i\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\beta_i} = \lim_{i\to\infty}\lim_{n\to\infty}\langle x_n,y_n\rangle^-_{\beta_{i-1}}.$$

Since $\alpha_{ki} \leq \beta_i$, it follows that $\langle x_n, y_n\rangle^-_{\alpha_{ki}} \leq \langle x_n, y_n\rangle^-_{\beta_i}$, and hence

$$\lim_{i\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_{ki}} \leq \lim_{i\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\beta_i}.$$

Also since $\beta_{i-1} \leq \alpha_{ki}$, it follows that $\langle x_n, y_n\rangle^-_{\beta_{i-1}} \leq \langle x_n, y_n\rangle^-_{\alpha_{ki}}$, and hence

$$\lim_{i\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\beta_{i-1}} \leq \lim_{i\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_{ki}}.$$

Thus

$$\langle x^*, y^*\rangle^-_\alpha = \lim_{i\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_{ki}} = \lim_{i\to\infty}\langle x^*, y^*\rangle^-_{\beta_i}.$$

Similarly $\langle x^*, y^*\rangle^+_\alpha = \lim\limits_{i\to\infty}\langle x^*, y^*\rangle^+_{\beta_i}$. Hence

$$[\langle x^*, y^*\rangle^-_\alpha, \langle x^*, y^*\rangle^+_\alpha] = [\lim_{i\to\infty}\langle x^*, y^*\rangle^-_{\beta_i}, \lim_{i\to\infty}\langle x^*, y^*\rangle^+_{\beta_i}].$$

(c) Let $\alpha \in (0,1]$ and $\alpha_k \nearrow \alpha$. Since $\alpha_1 \leq \alpha_k$, it follows that $\langle x_n, y_n\rangle^-_{\alpha_1} \leq \langle x_n, y_n\rangle^-_{\alpha_k}$ and thus $\lim\limits_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_1} \leq \lim\limits_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}$. Hence

$$-\infty < \lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_1} \leq \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}.$$

Similarly

$$\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^+_{\alpha_k} \leq \lim_{n\to\infty}\langle x_n, y_n\rangle^+_{\alpha_1} < +\infty.$$

Hence $\langle x^*, y^*\rangle$ is a fuzzy real number i.e. $\langle x^*, y^*\rangle \in F(\mathbf{R})$.

We show that the function $\langle ., .\rangle$ defined in Definition 3.8, is a fuzzy inner product on X^*.

(IP1) Let $x^*, y^*, z^* \in X^*$. We have

$$[\langle x^* + y^*, z^*\rangle]_\alpha = \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n + y_n, z_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n + y_n, z_n\rangle^+_{\alpha_k}\right].$$

Since X is a fuzzy inner product space,

$$[\langle x^* + y^*, z^*\rangle]_\alpha =$$

$$\left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, z_n\rangle^-_{\alpha_k} + \lim_{k\to\infty}\lim_{n\to\infty}\langle y_n, z_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, z_n\rangle^+_{\alpha_k} + \lim_{k\to\infty}\lim_{n\to\infty}\langle y_n, z_n\rangle^+_{\alpha_k}\right]$$

$$= [\langle x^*, z^*\rangle]_\alpha + [\langle y^*, z^*\rangle]_\alpha .$$

(IP2) Let $x^*, y^* \in X^*$ and $r \in \mathbf{R}$. We have

$$\begin{aligned}
[\langle rx^*, y^*\rangle]_\alpha &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle rx_n, y_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle rx_n, y_n\rangle^+_{\alpha_k}\right] \\
&= \begin{cases} \left[\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} r\langle x_n, y_n\rangle^-_{\alpha_k}, \lim\limits_{k\to\infty}\lim\limits_{n\to\infty} r\langle x_n, y_n\rangle^+_{\alpha_k}\right] & , \quad r \geq 0 \\ \left[\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} r\langle x_n, y_n\rangle^+_{\alpha_k}, \lim\limits_{k\to\infty}\lim\limits_{n\to\infty} r\langle x_n, y_n\rangle^-_{\alpha_k}\right] & , \quad r < 0 \end{cases} \\
&= \begin{cases} \left[r\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} \langle x_n, y_n\rangle^-_{\alpha_k}, r\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} \langle x_n, y_n\rangle^+_{\alpha_k}\right] & , \quad r \geq 0 \\ \left[r\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} \langle x_n, y_n\rangle^+_{\alpha_k}, r\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} \langle x_n, y_n\rangle^-_{\alpha_k}\right] & , \quad r < 0 \end{cases} \\
&= [\tilde{r}\langle x^*, y^*\rangle]_\alpha .
\end{aligned}$$

Hence $\tilde{r}\langle x^*, y^*\rangle = \langle rx^*, y^*\rangle$.

(IP3) Let $x^*, y^* \in X^*$. We have

$$\begin{aligned}
[\langle x^*, y^*\rangle]_\alpha &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, y_n\rangle^+_{\alpha_k}\right] \\
&= \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle y_n, x_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle y_n, x_n\rangle^+_{\alpha_k}\right] \\
&= [\langle y^*, x^*\rangle]_\alpha .
\end{aligned}$$

Hence $< x^*, y^* >=< y^*, x^* >$.

(IP4) Let $x^* \in X^*$. We have

$$[\langle x^*, x^*\rangle]_\alpha = \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, x_n\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x_n, x_n\rangle^+_{\alpha_k}\right],$$

since $\langle x_n, x_n\rangle \geq \tilde{0}$, for all $n \geq 1$, it follows that $0 \leq \langle x_n, x_n\rangle^-_{\alpha_k} \leq \langle x_n, y_n\rangle^+_{\alpha_k}$ and hence

$$\langle x^*, x^*\rangle \geq \tilde{0}.$$

(IP5) Let $x^* \in X^*$ and $x^* \neq 0^*$. Then by Theorem 3 ([4]), X^* is a fuzzy normed space and

$$[\|x^*\|]_\alpha = \left[\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^+_{\alpha_k}\right]. \tag{II}$$

By part (B) of Definition 1 in [4], there exists $\alpha_0 \in (0, 1]$ such that $\inf\limits_{0<\alpha\leq\alpha_0}\|x^*\|^-_\alpha > 0$. Since $\|x^*\|^-_{\alpha_0} \leq \|x^*\|^-_\alpha$, for all $\alpha_0 < \alpha$, it follows that $\inf\limits_{\alpha\in(0,1]}\|x^*\|^-_\alpha > 0$ and hence

$$0 < \inf_{0<\alpha\leq 1}\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^-_{\beta^\alpha_k} \quad for \quad all \quad \beta^\alpha_k \quad (\beta^\alpha_k \nearrow \alpha).$$

Then,

$$0<\inf_{0<\alpha\leq 1}\lim_{k\to\infty}\lim_{n\to\infty}\sqrt{\langle x_n,x_n\rangle^-_{\beta^\alpha_k}}=\inf_{0<\alpha\leq 1}\sqrt{\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^-_{\beta^\alpha_k}}.$$

Therefore $\inf\limits_{0<\alpha\leq 1}\lim\limits_{k\to\infty}\lim\limits_{n\to\infty}\langle x_n,x_n\rangle^-_{\beta^\alpha_k}>0$. Thus $\inf\limits_{0<\alpha\leq 1}a^\alpha_{\langle x^*,x^*\rangle}>0$.

(IP6) Let $x^*=0^*$. Then

$$[\langle x^*,x^*\rangle]_\alpha=\left[\lim_{k\to\infty}\lim_{n\to\infty}\langle 0,0\rangle^-_{\alpha_k},\lim_{k\to\infty}\lim_{n\to\infty}\langle 0,0\rangle^+_{\alpha_k}\right]=[0,0],$$

and hence $\langle x^*,x^*\rangle=\tilde{0}$.

Conversely, let $\langle x^*,x^*\rangle=\tilde{0}$. Then $[\langle x^*,x^*\rangle]_\alpha=[0,0]$ and hence

$$[\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^-_{\alpha_k},\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^+_{\alpha_k}]=[0,0].$$

By (II),

$$\begin{aligned}[\|x^*\|]_\alpha &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^-_{\alpha_k},\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^+_{\alpha_k}\right]\\ &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\sqrt{\langle x_n,x_n\rangle^-_{\beta^\alpha_k}},\lim_{k\to\infty}\lim_{n\to\infty}\sqrt{\langle x_n,x_n\rangle^+_{\beta^\alpha_k}}\right]\\ &= \left[\sqrt{\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^-_{\beta^\alpha_k}},\sqrt{\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^+_{\beta^\alpha_k}}\right]=[0,0].\end{aligned}$$

Hence $\|x^*\|=\tilde{0}$ and thus $x^*=0^*$.

Therefore X^* is a fuzzy inner product space and by Definition 3.1,

$$\|x^*\|_1=\sqrt{\langle x^*,x^*\rangle},\quad \forall x^*\in X^*. \tag{III}$$

□

Theorem 3.10. *For every fuzzy inner product space X there is a completion. In fact, X^* is a completion of X.*

Proof. By (II) and (III) in the proof of Theorem 3.9 we have

$$\begin{aligned}[\|x^*\|_1]_\alpha &= \left[\sqrt{\langle\widetilde{x},\widetilde{x}\rangle}\right]_\alpha=\left[\sqrt{\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^-_{\alpha_k}},\sqrt{\lim_{k\to\infty}\lim_{n\to\infty}\langle x_n,x_n\rangle^+_{\alpha_k}}\right]\\ &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\sqrt{\langle x_n,x_n\rangle^-_{\alpha_k}},\lim_{k\to\infty}\lim_{n\to\infty}\sqrt{\langle x_n,x_n\rangle^+_{\alpha_k}}\right]\\ &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^-_{\alpha_k},\lim_{k\to\infty}\lim_{n\to\infty}\|x_n\|^+_{\alpha_k}\right]=[\|x^*\|]_\alpha\,.\end{aligned}$$

Hence, by Theorem 3 in [4], X^* is a completion of the fuzzy normed linear space X and

$$\begin{aligned}\varphi: X &\longrightarrow X^*\\ x &\longmapsto \{x\}\end{aligned}$$

is a linear isometry.

Now we show that $\langle \varphi(x), \varphi(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$. We have

$$\begin{aligned}[\langle \varphi(x), \varphi(y)\rangle]_\alpha &= \left[\lim_{k\to\infty}\lim_{n\to\infty}\langle x, y\rangle^-_{\alpha_k}, \lim_{k\to\infty}\lim_{n\to\infty}\langle x, y\rangle^+_{\alpha_k}\right] \\ &= \left[\lim_{k\to\infty}\langle x, y\rangle^-_{\alpha_k}, \lim_{k\to\infty}\langle x, y\rangle^+_{\alpha_k}\right].\end{aligned}$$

Then by Lemma 2.9 (b),

$$[\langle \varphi(x), \varphi(y)\rangle]_\alpha = \left[\langle x, y\rangle^-_\alpha, \langle x, y\rangle^+_\alpha\right].$$

Hence X^* is a completion of X. □

4. **Orthogonal Complements Direct Sums**

In this chapter we show that every subspace of a fuzzy Hilbert space has a complementary subspace.

Definition 4.1. (Orthogonality) An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if $\langle x, y\rangle = 0$. We also say that x and y are orthogonal, and we write $x \perp y$. Similarly, for subset $A, B \subseteq X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.

Lemma 4.2. *(Parallelogram inequality) Let X be a fuzzy inner product space. Then*

$$(\|x+y\|^-_\alpha)^2 + (\|x-y\|^-_\alpha)^2 \le 2((\|x\|^-_\alpha)^2 + (\|y\|^-_\alpha)^2)$$

for all $x, y \in X$ and $\alpha \in (0, 1]$.

Proof. Let $x, y \in X$. We have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle,$$

and

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle.$$

Let $[\langle x, y\rangle] = [\langle x, y\rangle^-_\alpha, \langle x, y\rangle^+_\alpha]$. Then

$$(\|x+y\|^-_\alpha)^2 = (\|x\|^-_\alpha)^2 + 2\langle x, y\rangle^-_\alpha + (\|y\|^-_\alpha)^2,$$

and

$$(\|x-y\|^-_\alpha)^2 = (\|x\|^-_\alpha)^2 - 2\langle x, y\rangle^+_\alpha + (\|y\|^-_\alpha)^2.$$

Thus

$$(\|x+y\|^-_\alpha)^2 + (\|x-y\|^-_\alpha)^2 = 2((\|x\|^-_\alpha)^2 + (\|y\|^-_\alpha)^2) + 2(\langle x, y\rangle^-_\alpha - \langle x, y\rangle^+_\alpha).$$

Since $\langle x, y\rangle^-_\alpha \le \langle x, y\rangle^+_\alpha$,

$$(\|x+y\|^-_\alpha)^2 + (\|x-y\|^-_\alpha)^2 \le 2((\|x\|^-_\alpha)^2 + (\|y\|^-_\alpha)^2).$$

□

Theorem 4.3. *Let X be a fuzzy inner product space and $M \neq \emptyset$ a convex subset of X such that $(M, \|.\|_\alpha^-)$ is complete. Then for every $x \in X$ and $\alpha \in (0,1]$ there exists a unique $y_\alpha \in M$ such that*

$$\delta = \inf_{y \in M} \|x - y\|_\alpha^- = \|x - y_\alpha\|_\alpha^-.$$

Proof. The proof is similar to the proof of Theorem 3.3.1 of [6]. □

Lemma 4.4. *In Theorem 4.3, let M be a subspace of Y. Let $x \in X$ and let $\alpha \in (0,1]$. Then, for all $y \in Y$,*

$$[\langle z_\alpha, y \rangle]_\alpha = \{0\} \quad where \quad z_\alpha = x - y_\alpha.$$

Proof. Assume that there exists $y_1 \in Y$ such that

$$[\langle z_\alpha, y_1 \rangle]_\alpha = [a^\alpha, b^\alpha] \neq \{0\}. \tag{16}$$

Clearly, $y_1 \neq 0$ since otherwise $\langle z_\alpha, y_1 \rangle = \widetilde{0}$. Furthermore, for any scalar t,

$$\|z_\alpha - ty_1\|^2 = \langle z_\alpha - ty_1, z_\alpha - ty_1 \rangle = \langle z_\alpha, z_\alpha \rangle - 2t\langle z_\alpha, y_1 \rangle + t^2 \langle y_1, y_1 \rangle.$$

Hence

$$\begin{cases} (\|z_\alpha - ty_1\|_\alpha^-)^2 = (\|z_\alpha\|_\alpha^-)^2 - 2tb^\alpha + t^2(\|y_1\|_\alpha^-)^2 & t > 0 \\ (\|z_\alpha - ty_1\|_\alpha^-)^2 = (\|z_\alpha\|_\alpha^-)^2 - 2ta^\alpha + t^2(\|y_1\|_\alpha^-)^2 & t < 0. \end{cases} \tag{17}$$

We have the following four cases.

Case 1: Let $b^\alpha > 0$. If we choose $t = b^\alpha / (\|y_1\|_\alpha^-)^2 > 0$. Then from (17) we have

$$(\|z_\alpha - ty_1\|_\alpha^-)^2 = (\|z_\alpha\|_\alpha^-)^2 - (b^\alpha)^2/(\|y_1\|_\alpha^-)^2 < (\|z_\alpha\|_\alpha^-)^2 = \delta^2.$$

But this is impossible because we have $z_\alpha - ty_1 = x - y_2$, where $y_2 = y_\alpha + ty_1 \in Y$, so that by the definition of δ, $\|z_\alpha - ty_1\|_\alpha^- \geq \delta$. Hence $b^\alpha \leq 0$.

Case 2: Let $b^\alpha < 0$. Since $a^\alpha \leq b^\alpha$, it follows that $a^\alpha < 0$. If we choose $t = a^\alpha / (\|y_1\|_\alpha^-)^2 < 0$, then from (17) we have

$$(\|z_\alpha - ty_1\|_\alpha^-)^2 = (\|z_\alpha\|_\alpha^-)^2 - (a^\alpha)^2/(\|y_1\|_\alpha^-)^2 < (\|z_\alpha\|_\alpha^-)^2 = \delta^2.$$

But this is impossible, because we have $z_\alpha - ty_1 = x - y_2$, where $y_2 = y_\alpha + ty_1 \in Y$ and hence by the definition of δ, $\|z_\alpha - ty_1\|_\alpha^- \geq \delta$. Thus $b^\alpha \geq 0$.

Case 3: Let $a^\alpha < 0$. If we choose $t = a^\alpha / (\|y_1\|_\alpha^-)^2 < 0$, then by a similar proof as in the case 2, it can be seen that this is impossible. Hence $a^\alpha \geq 0$.

Case 4: Let $a^\alpha > 0$. Since $a^\alpha \leq b^\alpha$, it follows that $b^\alpha > 0$. If we choose $t = b^\alpha / (\|y_1\|_\alpha^-)^2 > 0$, then by a similar proof as in the case 1, we see that this is impossible. Hence $a^\alpha \leq 0$.

Thus $a^\alpha = b^\alpha = 0$. Hence (16) is impossible and so the Lemma is proved.

□

Definition 4.5. A vector space X is said to be the direct sum of two subspace Y and Z of X, written $X = Y \oplus Z$, if every $x \in X$ has a unique representation $x = y + z$ $(y \in Y, z \in Z)$. Then Z is called an algebraic complement of Y in X and vice versa, and Y, Z is called a complementary pair of subspaces in X.

Theorem 4.6. *Let Y be any subspace of a fuzzy inner product space X such that the normed spaces $(Y, \|.\|_\alpha^-)$ are complete, for all $\alpha \in (0, 1]$. Then*

$$X = Y \oplus Z \quad where \quad Z = Y^\perp.$$

Proof. Since Y is convex and the normed spaces $(Y, \|.\|_\alpha^-)$ are complete for all $\alpha \in (0, 1]$, Theorem 4.3 and Lemma 4.4 imply that for every $x \in X$ and for every $\alpha \in (0, 1]$ there exists a $y_\alpha \in Y$ such that $x = y_\alpha + z_\alpha$ and, for all $y \in Y$, $[\langle z_\alpha, y\rangle]_\alpha = \{0\}$.

We show that $y_\alpha = y_\beta$ for all $\alpha, \beta \in (0, 1]$. Let $\alpha \le \beta$. We have

$$[\langle z_\alpha, y\rangle]_\beta \subseteq [\langle z_\alpha, y\rangle]_\alpha = \{0\}.$$

Hence $[\langle z_\alpha, y\rangle]_\beta = \{0\}$ and so $[\langle z_\alpha - z_\beta, y\rangle]_\beta = \{0\}$, for all $y \in Y$. Now we have $x = y_\alpha + z_\alpha$ and $x = y_\beta + z_\beta$, which implies that $y_\beta - y_\alpha = z_\alpha - z_\beta$ and thus

$$[\langle y_\beta - y_\alpha, y\rangle]_\beta = \{0\}, \quad for \ \ all \ \ y \in Y.$$

Since $y_\beta - y_\alpha \in Y$, it follows that $[\langle y_\beta - y_\alpha, y_\beta - y_\alpha\rangle]_\beta = \{0\}$. Hence $(\|y_\beta - y_\alpha\|_\beta^-)^2 = 0$. Thus $\|y_\beta - y_\alpha\|_\beta^- = 0$, or, equivalently, $y_\beta = y_\alpha$.

Now we have $x = y + z$ such that $y = y_\alpha$ and $z = z_\alpha$, for all $\alpha \in (0, 1]$. Since

$$[\langle z, y'\rangle]_\alpha = [\langle z_\alpha, y'\rangle]_\alpha = \{0\}, \quad for \ \ all \ \ \alpha \in (0, 1],$$

it follows that $\langle z, y'\rangle = \widetilde{0}$, for all $y' \in Y$. Hence $z \in Y^\perp$.

The proof of uniqueness is the same as in Theorem 3.3.4 of [6]. □

Corollary 4.7. *Let Y be any subspace of a fuzzy Hilbert space X such that the normed spaces $(Y, \|.\|_\alpha^-)$ are complete, for all $\alpha \in (0, 1]$. Then*

$$X = Y \oplus Z \quad where \quad Z = Y^\perp.$$

5. **Fuzzy Norm of Linear Operator**

At first, we introduce the notions of fuzzy bounded and fuzzy norm of linear operators.

Definition 5.1. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be fuzzy bounded, if there is a fuzzy real number η such that

$$\|Tx\| \le \eta\|x\|, \quad for \ \ all \ \ x \in X. \tag{18}$$

The set of all fuzzy bounded linear operators $T : X \longrightarrow Y$, is denoted by $B(X, Y)$.

Remark 5.2. The set $B(X, Y)$ is a real vector space.

Definition 5.3. Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces and let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator. We define $\|T\|$ by,

$$[\|T\|]_\alpha = [\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}], \quad for \ \ all \ \ \alpha \in (0,1]. \tag{19}$$

Then $\|T\|$ is called the fuzzy norm of the operator T.

Notation: We write $\|T\|_\alpha^- = \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$ and $\|T\|_\alpha^+ = \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}$, i.e. $[\|T\|]_\alpha = [\|T\|_\alpha^-, \|T\|_\alpha^+]$, for all $\alpha \in (0,1]$.

Theorem 5.4. *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then $\|T\|$ is a fuzzy real number.*

Proof. Let $\alpha \in (0,1]$ and let $\|Tx\| \leq \eta\|x\|$. Suppose that $\beta < \alpha$ and $\|x\|_\beta^- \leq 1$. Then $\|Tx\|_\beta^- \leq \eta_\beta^-$ and $\eta_\beta^- \leq \eta_\alpha^-$, hence $\|Tx\|_\beta^- \leq \eta_\alpha^-$. Therefore,

$$\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \eta_\alpha^-. \tag{20}$$

Since $\eta_\alpha^- \leq \eta_\alpha^+$,

$$\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \eta_\alpha^+,$$

and thus

$$\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}.$$

Hence $[\|T\|]_\alpha = [\|T\|_\alpha^-, \|T\|_\alpha^+]$ is a nonempty interval, for all $\alpha \in (0,1]$.

We show that $[\|T\|]_\alpha$, $\alpha \in (0,1]$, satisfies the conditions of Lemma 2.9.

(a) Let $0 < \alpha_1 \leq \alpha_2$. Then

$$\sup_{\beta<\alpha_1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{\beta<\alpha_2} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-.$$

Since $0 < \alpha_1 \leq \alpha_2$, it follows from Lemma 2.9 (a) that $\eta_{\alpha_2}^+ \leq \eta_{\alpha_1}^+$, and thus

$$\inf\{\eta_{\alpha_2}^+ : \|Tx\| \leq \eta\|x\|\} \leq \inf\{\eta_{\alpha_1}^+ : \|Tx\| \leq \eta\|x\|\}.$$

Hence $[\|T\|]_{\alpha_2} \subseteq [\|T\|]_{\alpha_1}$

(b) Let $\{\alpha_k\}$ be a increasing sequence in (0,1] converging to α. Then $\alpha_k \leq \alpha_{k+1} \leq \alpha$ and hence

$$\sup_{\beta<\alpha_k} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-.$$

Thus

$$\sup_k \sup_{\beta<\alpha_k} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-. \tag{21}$$

Let $\varepsilon > 0$. Then there exists $\beta_0 < \alpha$ such that

$$\sup_{\beta<\alpha} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta - \varepsilon \le \sup_{\|x\|^-_{\beta_0} \le 1} \|Tx\|^-_{\beta_0}.$$

Since $\alpha_k \nearrow \alpha$, there is a $0 < k_0$ such that $\beta_0 < \alpha_{k_0} \le \alpha$. Hence

$$\sup_{\|x\|^-_{\beta_0} \le 1} \|Tx\|^-_{\beta_0} \le \sup_{\beta<\alpha_{k_0}} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta \le \sup_k \sup_{\beta<\alpha_k} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta,$$

which implies that

$$\sup_{\beta<\alpha} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta - \varepsilon \quad \le \quad \sup_k \sup_{\beta<\alpha_k} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta.$$

As $\varepsilon \to 0$,

$$\sup_{\beta<\alpha} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta \quad \le \quad \sup_k \sup_{\beta<\alpha_k} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta. \tag{22}$$

Hence by (21) and (22),

$$\lim_{k\to\infty} \sup_{\beta<\alpha_k} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta = \sup_k \sup_{\beta<\alpha_k} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta = \sup_{\beta<\alpha} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta.$$

Since $\alpha_k \le \alpha$,

$$\inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\} \quad \le \quad \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\},$$

therefore

$$\inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\} \quad \le \quad \inf_k \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\}. \tag{23}$$

Let $0 < \varepsilon$. Then there exists a fuzzy real number η_0 such that $\eta^+_{0\alpha} \le \inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\} + \varepsilon$. Since $\alpha_k \nearrow \alpha$ it follows from Lemma 2.9 (b) that $\inf_k \eta^+_{0\alpha_k} = \eta^+_{0\alpha}$. This implies that there is a $0 < k_0$ such that $\eta^+_{0\alpha_{k_0}} \le \eta^+_{0\alpha} + \varepsilon$. Therefore

$$\inf_k \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\} \le \inf\{\eta^+_{\alpha_{k_0}} : \|Tx\| \le \eta\|x\|\} \le \eta^+_{0\alpha} + \varepsilon,$$

and thus

$$\inf_k \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\} \le \inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\} + 2\varepsilon.$$

As $\varepsilon \to 0$,

$$\inf_k \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\} \le \inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\}. \tag{24}$$

Then from (23) and (24),

$$\lim_{k\to\infty} \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\} = \inf_k \inf\{\eta^+_{\alpha_k} : \|Tx\| \le \eta\|x\|\} = \inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\}.$$

Hence

$$[\lim_{k\to\infty} \|T\|^-_{\alpha_k}, \lim_{k\to\infty} \|T\|^+_{\alpha_k}] = [\|T\|^-_\alpha, \|T\|^+_\alpha].$$

(c) Since $0 \le \|Tx\|^-_\beta$, for all $x \in X$ and all $\beta \in (0, 1]$, hence

$$-\infty < 0 \le \sup_{\beta<\alpha} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta.$$

Let η be a fuzzy real number which is $\|Tx\| \leq \eta\|x\|$, for all $x \in X$. Lemma 2.9 (c) implies that $\eta_\alpha^+ < +\infty$, for all $\alpha \in (0,1]$. Hence

$$\inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\} < +\infty.$$

Therefore

$$-\infty < \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\} < +\infty, \quad for \ \ all \ \ \alpha \in (0,1].$$

Thus by Lemma 2.9, $\|T\|$ is a fuzzy real number. □

Lemma 5.5. *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then $\|Tx\| \leq \|T\|\|x\|$, for all $x \in X$.*

Proof. Let $\{\beta_k\}$ be an increasing sequence in (0,1] converging to $\alpha \in (0,1]$. We have

$$\|Tx\|_{\beta_k}^- / \|x\|_{\beta_k}^- \leq \sup_{\|x\|_{\beta_k}^- \leq 1} \|Tx\|_{\beta_k}^- \leq \|T\|_\alpha^-.$$

Hence

$$\|Tx\|_{\beta_k}^- \leq \|T\|_\alpha^- \|x\|_{\beta_k}^-.$$

Since $\beta_k \nearrow \alpha$, it follows from Lemma 2.9 (b) that

$$\|Tx\|_\alpha^- = \lim_{k\to\infty} \|Tx\|_{\beta_k}^- \leq \lim_{k\to\infty} \|T\|_\alpha^- \|x\|_{\beta_k}^- = \|T\|_\alpha^- \lim_{k\to\infty} \|x\|_{\beta_k}^- = \|T\|_\alpha^- \|x\|_\alpha^-.$$

Hence

$$\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^-. \tag{25}$$

Moreover, we have $\|Tx\|_\alpha^+ \leq \eta_\alpha^+ \|x\|_\alpha^+$, which implies that

$$\|Tx\|_\alpha^+ \leq (\inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\})\|x\|_\alpha^+.$$

Thus

$$\|Tx\|_\alpha^+ \leq \|T\|_\alpha^+ \|x\|_\alpha^+. \tag{26}$$

The proof follows from (25) and (26). □

Theorem 5.6. *The vector space $B(X, Y)$ equipped with the norm defined in Definition 5.3 is a fuzzy normed linear space.*

Proof. It is sufficient to show that the norm defined by (2) satisfies the conditions of Definition 2.17.

(F1) Let $0 \neq T \in B(X, Y)$. Then there is an element $0 \neq x_0 \in X$ such that $Tx_0 \neq 0$. Assume that $M = \sup_{0<\alpha\leq 1} \|x_0\|_\alpha^-$ (note that $M < \infty$). We have

$$\inf_{0<\beta\leq 1} \|Tx_0\|_\beta^- \leq \|Tx_0\|_\alpha^- \leq \|T\|_\alpha^- \|x_0\|_\alpha^- \leq \|T\|_\alpha^- M, \quad for \ \ all \ \ \alpha \in (0,1].$$

Hence

$$\inf_{0<\beta\leq 1} \|Tx_0\|_\beta^- \leq M \inf_{0<\alpha\leq 1} \|T\|_\alpha^-.$$

Since $Tx_0 \neq 0$, it follows that $0 < \inf_{0<\alpha\leq 1} \|Tx_0\|_\alpha^-$. Thus $0 < \inf_{0<\alpha\leq 1} \|T\|_\alpha^-$.

(F2) Let $T \in B(X, Y)$. It is clear that, when $T = 0$.
Conversely, let $\|T\| = 0$. Since $\|Tx\| \leq 0\|x\|$, it follows that $\|Tx\| = 0$, for all $x \in X$. Thus by Definition 2.17, (F2) $Tx = 0$, for all $x \in X$, therefore $T = 0$.

(F3) Let $r \in \mathbf{R}$ and $T \in B(X, Y)$. For all $\alpha \in (0, 1]$, we have

$$
\begin{aligned}
[\|rT\|]_\alpha &= [\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|rTx\|_\beta^-, \inf\{\eta_\alpha^+ : \|rTx\| \leq \eta\|x\|\}] \\
&= [r \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, \inf\{\eta_\alpha^+ : \|Tx\| \leq \frac{1}{r}\eta\|x\|\}] \\
&= [r \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, \inf\{r\gamma_\alpha^+ : \|Tx\| \leq \gamma\|x\|\}] \\
&= [r \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, r\inf\{\gamma_\alpha^+ : \|Tx\| \leq \gamma\|x\|\}] \\
&= [r\|T\|]_\alpha.
\end{aligned}
$$

Hence $\|rT\| = r\|T\|$.

(F4) Consider

$$
\begin{aligned}
\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|(T+S)(x)\|_\beta^- &\leq \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} (\|T(x)\|_\beta^- + \|S(x)\|_\beta^-) \\
&\leq \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|S(x)\|_\beta^- + \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|T(x)\|_\beta^-.
\end{aligned}
$$

Hence

$$\|T + S\|_\alpha^- \leq \|T\|_\alpha^- + \|S\|_\alpha^-. \tag{27}$$

Now we have

$$\|(T + S)(x)\| \leq \|Tx\| + \|Sx\| \leq \|T\|\|x\| + \|S\|\|x\| = (\|T\| + \|S\|)\|x\|.$$

Therefore

$$\inf\{\eta_\alpha^+ : \|(T + S)(x)\| \leq \eta\|x\|\} \leq \|T\|_\alpha^+ + \|S\|_\alpha^+.$$

Hence

$$\|T + S\|_\alpha^+ \leq \|T\|_\alpha^+ + \|S\|_\alpha^+. \tag{28}$$

Thus by (27), (28), and Theorem 2.19, (F4) holds. □

Corollary 5.7. *Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. The vector space $B(X, Y)$ is a fuzzy normed linear space.*

Now we show that all linear operators on finite dimensional spaces are fuzzy bounded. We need the following lemmas.

Lemma 5.8. *Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Furthermore, Let $T : X \longrightarrow Y$ be a linear operator which is $\sup_{\|x\|_\alpha^- \leq 1} \|Tx\|_\alpha^+ < \infty$, for all $\alpha \in (0, 1]$, and $\sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- < \infty$. Then T is a fuzzy bounded linear operator.*

Proof. Let $N_\alpha = \sup_{\|x\|_\alpha^- \le 1} \|Tx\|_\alpha^+$, for all $\alpha \in (0,1]$. If $\alpha \le \gamma$, then $\|x\|_\alpha^- \le \|x\|_\gamma^-$ and $\|Tx\|_\gamma^+ \le \|Tx\|_\alpha^+$. This implies that $N_\gamma \le N_\alpha$. Assume that $M_\alpha = \inf_{\beta<\alpha} N_\beta$. If $\alpha < \gamma$, then

$$M_\gamma \le M_\alpha. \tag{29}$$

Let $\{\alpha_k\}$ be an increasing sequence in $(0,1]$ converging to α. Since $\alpha_k \le \alpha$ it follows that $M_\alpha \le M_{\alpha_k}$ and hence

$$M_\alpha \le \inf_k M_{\alpha_k}. \tag{30}$$

Let $0 < \varepsilon$. Then there is a $\beta_0 < \alpha$ such that $N_{\beta_0} \le M_\alpha + \varepsilon$. Since $\alpha_k \nearrow \alpha$, there exists $0 < k_0$ such that $\beta_0 < \alpha_{k_0}$. Then $\inf_k M_{\alpha_k} \le M_{\alpha_{k_0}} \le N_{\beta_0}$ and hence $\inf_k M_{\alpha_k} \le M_\alpha + \varepsilon$. As $\varepsilon \to 0$, we have

$$\inf_k M_{\alpha_k} \le M_\alpha. \tag{31}$$

Hence by (30), (31), $M_\alpha = \inf_k M_{\alpha_k}$ and thus

$$M_\alpha = \inf_k M_{\alpha_k} = \lim_{k\to\infty} M_{\alpha_k}. \tag{32}$$

Now we have $N_\alpha \le N_\beta$, for all $\beta < \alpha$, thus $N_\alpha \le \inf_{\beta<\alpha} N_\beta = M_\alpha$. Then

$$\|Tx\|_\alpha^+ \le N_\alpha \|x\|_\alpha^- \le N_\alpha \|x\|_\alpha^+ \le M_\alpha \|x\|_\alpha^+. \tag{33}$$

Let $M = \sup_{0<\beta\le 1} \sup_{\|x\|_\beta^- \le 1} \|Tx\|_\beta^-$. Then,

$$\|Tx\|_\alpha^- \le M\|x\|_\alpha^-, \quad for \quad all \quad x \in X \quad and \quad \alpha \in (0,1]. \tag{34}$$

We define $[\eta]_\alpha = [M, M + M_\alpha]$, for all $\alpha \in (0,1]$. By (29), (32), and Lemma 2.9, the family $[\eta]_\alpha$, $\alpha \in (0,1]$, represents the $\alpha-$level sets of a fuzzy real number η. By (33) and (34), we obtain that $\|Tx\| \le \eta\|x\|$. Hence T is a fuzzy bounded linear operator. □

Lemma 5.9. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces with $\dim X < \infty$. Furthermore, let $T : X \to Y$ be a linear operator. Then $\sup_{\|x\|_\alpha^- \le 1} \|Tx\|_\alpha^+ < \infty$, for all $\alpha \in (0,1]$, and $\sup_{0<\beta\le 1} \sup_{\|x\|_\beta^- \le 1} \|Tx\|_\beta^- < \infty$.*

Proof. Let $\{e_1, ..., e_n\}$ be a basis for X. For $x = \sum a_i e_i$, since T is linear,

$$\|Tx\| = \|\sum a_i T e_i\| \le \sum |a_i| \|Te_i\|.$$

Let $\gamma = \max_{1\le i\le n} \|Te_i\|$. Then

$$\|Tx\| \le \gamma(\sum |a_i|). \tag{35}$$

By Proposition 3.3 [3], there is a fuzzy real number $0 < \eta$ such that

$$\eta(\sum |a_i|) \le \|x\|, \quad for \quad all \quad x \in X. \tag{36}$$

Let $\alpha \in (0,1]$. Then from (35) and (36), we have

$$\|Tx\|_\alpha^+ \leq \gamma_\alpha^+(\sum |a_i|) \leq (\gamma_\alpha^+/\eta_\alpha^-)\|x\|_\alpha^-,$$

and hence

$$\sup_{\|x\|_\alpha^- \leq 1} \|Tx\|_\alpha^+ \leq \gamma_\alpha^+/\eta_\alpha^- < \infty.$$

Now we have

$$\|x\| = \|\sum a_i e_i\| \leq \sum |a_i|\|e_i\| \leq (\max_{1\leq i\leq n} \|e_i\|)\sum |a_i|. \tag{37}$$

We define a norm $\|.\|_0$ on X by $[\|x\|_0]_\alpha = [\sum |a_i|, \sum |a_i|]$, for all $\alpha \in (0,1]$. By (36) and (37), the norm $\|.\|$ is equivalent to the norm $\|.\|_0$. This implies that there is a fuzzy real number μ such that $\|x\|_0 \leq \mu\|x\|$, for all $x \in X$. Now we obtain

$$\|Tx\|_\beta^- \leq \gamma_\beta^-(\sum |a_i|) \leq \gamma_\beta^- \mu_\beta^- \|x\|_\beta^-.$$

Hence

$$\sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \gamma_\beta^- \mu_\beta^-,$$

and thus

$$\sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{0<\beta\leq 1} \gamma_\beta^- \mu_\beta^- \leq \gamma_1^- \mu_1^- < \infty.$$

□

Theorem 5.10. *Let $(X, \|.\|)$ be a finite dimensional fuzzy normed linear space. Then every linear operator on X is fuzzy bounded.*

Proof. By Lemmas 5.8 and 5.9, every linear operator on X is fuzzy bounded. □

6. Strongly Continuous Linear Operator

In this section we investigate the relationship between (strong) continuity and fuzzy boundedness.

Theorem 6.1. *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then T is continuous.*

Proof. By Lemma 5.5, $\|Tx\| \leq \|T\|\|x\|$, for all $x \in X$. Hence $\|Tx\|_\alpha^+ \leq \|T\|_\alpha^+\|x\|_\alpha^+$, for all $\alpha \in (0,1]$. Let $\{x_n\}$ be a sequence in X converging to $x \in X$. Then $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$. Hence

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \|Tx_n - Tx\|_\alpha^+ &= \lim_{n\to\infty} \|T(x_n - x)\|_\alpha^+ \\ &\leq \lim_{n\to\infty} (\|T\|_\alpha^+ \|x_n - x\|_\alpha^+) \\ &= \|T\|_\alpha^+ \lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0. \end{aligned}$$

Thus $\lim_{n\to\infty} \|Tx_n - Tx\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$. It follows that T is continuous. □

Note: Fuzzy boundedness is not equivalent to continuity. In the next example we show that continuity does not imply fuzzy boundedness.

Example 6.2. Let X be a vector space and $\mathbf{B}=\{e_i\}_{i=1}^{\infty}$ a basis for X ($\dim X = \infty$). Define fuzzy norms $\|.\|$ and $\|.\|_0$ on X by

$$[\|x\|]_\alpha = [\sum_{i=1}^{n} |a_i|, \sum_{i=1}^{n} (1/\alpha)^i |a_i|] \ \textit{and} \ [\|x\|_0]_\alpha = [\sum_{i=1}^{n} |a_i|, \sum_{i=1}^{n} |a_i|], \ \textit{where} \ x = \sum_{i=1}^{n} a_i e_i,$$

for all $\alpha \in (0,1]$. It is clear that $\|.\|$ and $\|.\|_0$ are fuzzy norms on X.

We define $T : (X, \|.\|) \to (X, \|.\|_0)$ by $Te_n = ne_n$ for all $e_n \in \mathbf{B}$. Let $\{x_n\} \subseteq (X, \|.\|)$ be a sequence which is $x_n \to 0$. Assume that $x_n = \sum_{i=1}^{k_n} a_{ni} e_i$. Then $Tx_n = \sum_{i=1}^{k_n} a_{ni} i e_i$, hence $\|Tx_n\|_{0\alpha}^{+} = \sum_{i=1}^{k_n} |a_{ni} i|$, for all $\alpha \in (0,1]$. Let $\beta < 1$. We have

$$\|Tx_n\|_{0\alpha}^{+} = \sum_{i=1}^{k_n} |a_{ni} i| = \sum_{i=1}^{k_n} i|a_{ni}| \leq \sum_{i=1}^{k_n} (1/\beta)^i |a_{ni}| = \|x_n\|_\beta^{+}.$$

Since $x_n \to 0$, it follows that $\|x_n\|_\beta^{+} \to 0$, for all $\beta \in (0,1]$, hence $\|Tx_n\|_{0\alpha}^{+} \to 0$, for all $\alpha \in (0,1]$. Thus T is continuous.

We now show that T is not bounded. If T is fuzzy bounded, then there exists a fuzzy real number η such that $\|Tx\|_0 \leq \eta \|x\|$, for all $x \in X$, and hence $\|Tx\|_{01}^{+} \leq \eta_1^{+} \|x\|_1^{+}$, for all $x \in X$. However,

$$\|Te_n\|_{01}^{+} = \|ne_n\|_{01}^{+} = |n| = n \leq \eta_1^{+} \|e_n\|_1^{+} = \eta_1^{+},$$

hence $n \leq \eta_1^{+}$, for all $n \in \mathbf{N}$, and thus $\eta_1^{+} = +\infty$, which is a contradiction.

Definition 6.3. Let $T : X \longrightarrow Y$ be a linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then T is called strongly continuous if for any real number $0 < \varepsilon$, we can find a fuzzy real number $0 < \delta$ such that if $\|x\|_\alpha^{+} < \delta_\alpha^{-}$, $\|Tx\|_\alpha^{+} < \varepsilon$ and if $\|x\|_\alpha^{-} < \delta_\alpha^{+}$, $\|Tx\|_\alpha^{-} < \varepsilon$ $(\alpha \in (0,1], x \in X)$.

Now we show that the notions of strong continuity and fuzzy boundedness are equivalent.

Lemma 6.4. *Let $T : X \longrightarrow Y$ be a linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be a fuzzy normed linear spaces. Assume $M_\alpha = \sup_{\alpha \leq \beta} \sup_{\|x\|_\beta^{+} \leq 1} \|Tx\|_\beta^{+} < \infty$, for all $\alpha \in (0,1]$, and $N = \sup_{0 < \beta \leq 1} \sup_{\|x\|_\beta^{-} \leq 1} \|Tx\|_\beta^{-} < \infty$. Then T is a fuzzy bounded linear operator.*

Proof. Let $\gamma < \alpha$. Then $M_\alpha \leq M_\gamma$. Assume that $F_\alpha = \inf_{\beta < \alpha} M_\beta$, for all $\alpha \in (0,1]$. If $\alpha < \gamma$, then

$$F_\gamma \leq F_\alpha. \tag{38}$$

Let $\{\alpha_k\}$ be an increasing sequence in $(0,1]$ converging to α. Since $\alpha_k \leq \alpha$, it follows that $F\alpha \leq F\alpha_k$ and hence

$$F\alpha \leq \inf_k F\alpha_k. \tag{39}$$

Let $0 < \varepsilon$. Then there is a $\beta_0 < \alpha$ such that $M_{\beta_0} \leq F_\alpha + \varepsilon$. Since $\alpha_k \nearrow \alpha$, there exists a $0 < k_0$ such that $\beta_0 < \alpha_{k_0}$. Hence $\inf_k F_{\alpha_k} \leq F_{\alpha_{k_0}} \leq M_{\beta_0}$ and thus $\inf_k F_{\alpha_k} \leq F_\alpha + \varepsilon$. Since $\varepsilon \to 0$,

$$\inf_k F_{\alpha_k} \leq F_\alpha. \tag{40}$$

By (38), (39),

$$F_\alpha = \inf_k F_{\alpha_k} = \lim_{k\to\infty} F_{\alpha_k}. \tag{41}$$

We define $[\eta]_\alpha = [N, N + F_\alpha]$ for all $\alpha \in (0,1]$. By (41) and Lemma 2.9, the family $[\eta]_\alpha$, $\alpha \in (0,1]$, represents the $\alpha-$level sets of the fuzzy real number η.

Now we have $M_\alpha \leq M_\beta$, for all $\beta < \alpha$. So $M_\alpha \leq F_\alpha$, and hence

$$\|Tx\|_\alpha^+ \leq M_\alpha \|x\|_\alpha^+ \leq F_\alpha \|x\|_\alpha^+ \leq (F_\alpha + N)\|x\|_\alpha^+. \tag{42}$$

Since $N = \sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$,

$$\|Tx\|_\alpha^- \leq N\|x\|_\alpha^-, \quad for \;\; all \;\; \alpha \in (0,1] \;\; and \;\; x \in X. \tag{43}$$

Then from (42), (43), $\|Tx\| \leq \eta\|x\|$, for all $x \in X$. Consequently, T is a fuzzy bounded linear operator. □

Theorem 6.5. *Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ a linear operator. Then T is fuzzy bounded if and only if T is strongly continuous.*

Proof. Let T be a fuzzy bounded linear operator and let $0 < \varepsilon$. Suppose that $\delta = \varepsilon/\|T\|$. Assume $\|x\|_\alpha^- < \delta_\alpha^+$. Then

$$\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^- < \|T\|_\alpha^- \delta_\alpha^+ = \|T\|_\alpha^- (\varepsilon/\|T\|_\alpha^-) = \varepsilon.$$

Hence, $\|Tx\|_\alpha^- \leq \varepsilon$.
Next assume $\|x\|_\alpha^+ < \delta_\alpha^-$. Then

$$\|Tx\|_\alpha^+ \leq \|T\|_\alpha^+ \|x\|_\alpha^+ < \|T\|_\alpha^+ \delta_\alpha^- = \|T\|_\alpha^+ (\varepsilon/\|T\|_\alpha^+) = \varepsilon.$$

Thus $\|Tx\|_\alpha^+ \leq \varepsilon$. Consequently, T is strongly continuous.

Conversely, let T be strongly continuous. Assume $N = \sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$ and $0 < \varepsilon$. Since T is strongly continuous, there exists $0 < \delta \in F(\mathbf{R})$ such that $\|Tx\|_\alpha^+ < \varepsilon$ if $\|x\|_\alpha^+ < \delta_\alpha^-$ and $\|Tx\|_\alpha^- < \varepsilon$ if $\|x\|_\alpha^- < \delta_\alpha^+$, for all $\alpha \in (0,1]$ and all $x \in X$.
Let $\|x\|_\beta^- \leq 1$. We have $\|\delta_\beta^+ x\|_\beta^- = \delta_\beta^+ \|x\|_\beta^- < \delta_\beta^+$. Hence $\|T(\delta_\beta^+ x)\|_\beta^- < \varepsilon$ and thus $\|Tx\|_\beta^- < \varepsilon/\delta_\beta^+$. It follows that $\sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- < \varepsilon/\delta_\beta^+$, which implies

$$\sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{0<\beta\leq 1} \varepsilon/\delta_\beta^+.$$

Since $\delta_1^+ < \delta_\beta^+$, for all $\beta \in (0,1]$, hence $\varepsilon/\delta_\beta^+ < \varepsilon/\delta_1^+$, for all $\beta \in (0,1]$, and therefore $\sup_{0<\beta\leq 1} \varepsilon/\delta_\beta^+ \leq \varepsilon/\delta_1^+$. Thus

$$N = \sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \varepsilon/\delta_1^+ < \infty. \tag{44}$$

Let $M_\alpha = \sup_{\alpha\leq\beta} \sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+$, for all $\alpha \in (0,1]$. Suppose that $\|x\|_\beta^+ \leq 1$. Then $\|\delta_\beta^- x\|_\beta^+ = \delta_\beta^- \|x\|_\beta^+ < \delta_\beta^-$. Hence $\|T(\delta_\beta^- x)\|_\beta^+ < \varepsilon$ and thus $\|Tx\|_\beta^+ < \varepsilon/\delta_\beta^-$. So

$$\sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+ < \varepsilon/\delta_\beta^-.$$

Hence

$$\sup_{\alpha\leq\beta} \sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+ \leq \sup_{\alpha\leq\beta} \varepsilon/\delta_\beta^-.$$

Since $\delta_\alpha^- < \delta_\beta^-$, for all $\alpha \leq \beta$, it follows that $\varepsilon/\delta_\beta^- < \varepsilon/\delta_\alpha^-$, for all $\alpha \leq \beta$, and hence $\sup_{\alpha\leq\beta} \varepsilon/\delta_\beta^- \leq \varepsilon/\delta_\alpha^-$. Thus

$$M_\alpha = \sup_{\alpha\leq\beta} \sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+ \leq \varepsilon/\delta_\alpha^- < \infty, \quad for \ \ all \ \ \alpha \in (0,1]. \tag{45}$$

Consequently, by (44), (45) and Lemma 6.4, T is fuzzy bounded. □

Proposition 6.6. *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Assume $M_\alpha = \sup_{\alpha\leq\beta} \sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+$, for all $\alpha \in (0,1]$, and $N = \sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$. Then $N < \infty$ and $M_\alpha < \infty$, for all $\alpha \in (0,1]$.*

Proof. Since T is a fuzzy bounded linear operator, $\|Tx\| \leq \|T\| \|x\|$, for all $x \in X$. Hence $\|Tx\|_\beta^- \leq \|T\|_\beta^- \|x\|_\beta^-$, for all $\beta \in (0,1]$, and thus for all $\beta \in (0,1]$, $\sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \|T\|_\beta^-$. This implies that

$$\sup_{0<\beta\leq 1} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \leq \sup_{0<\beta\leq 1} \|T\|_\beta^-.$$

Since $\|T\|_\beta^- \leq \|T\|_1^-$, for all $\beta \in (0,1]$, it follows that $\sup_{0<\beta\leq 1} \|T\|_\beta^- \leq \|T\|_1^-$. Hence $N \leq \|T\|_1^- < \infty$.

Let $\alpha \in (0,1]$. Since $\|Tx\|_\beta^+ \leq \|T\|_\beta^+ \|x\|_\beta^+$, for all $\beta \in (0,1]$, $\sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+ \leq \|T\|_\beta^+$ and hence

$$\sup_{\alpha\leq\beta} \sup_{\|x\|_\beta^+ \leq 1} \|Tx\|_\beta^+ \leq \sup_{\alpha\leq\beta} \|T\|_\beta^+.$$

Since $\|T\|_\beta^+ \leq \|T\|_\alpha^+$, for all $\alpha \leq \beta$, it follows that $\sup_{\alpha\leq\beta} \|T\|_\beta^+ \leq \|T\|_\alpha^+$. Hence

$$M_\alpha \leq \|T\|_\alpha^+ < \infty \quad for \ \ all \ \ \alpha \in (0,1].$$

□

7. **Some Properties of** $B(X,Y)$

In this section we show that $B(X,Y)$ is complete.

Theorem 7.1. *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then $\|T\| \le \eta$ whenever $\|Tx\| \le \eta\|x\|$ $(\eta \in F(\mathbf{R}))$.*

Proof. Let $\|Tx\| \le \eta\|x\|$. By (20) in proof of Theorem 5.4, $\|T\|^-_\alpha \le \eta^-_\alpha$, for all $\alpha \in (0,1]$.

We have $\|T\|^+_\alpha = \inf\{\eta^+_\alpha : \|Tx\| \le \eta\|x\|\} \le \eta^+_\alpha$, then $\|T\|^+_\alpha \le \eta^+_\alpha$, for all $\alpha \in (0,1]$. Consequently, $\|T\| \le \eta$. □

Theorem 7.2. *Let $(Y, \|.\|)$ be a complete fuzzy normed linear space and $(X, \|.\|)$ be a fuzzy normed linear space. Then $B(X,Y)$ is complete fuzzy normed linear space.*

Proof. By Corollary 5.7, $B(X,Y)$ is a fuzzy normed linear space.

We consider an arbitrary Cauchy sequence $\{T_n\}$ in $B(X,Y)$ and show that $\{T_n\}$ converges to an operator $T \in B(X,Y)$. Since $\{T_n\}$ is Cauchy,

$$\lim_{m,n\to\infty} \|T_n - T_m\|^+_\alpha = 0, \quad for \ \ all \ \ \alpha \in (0,1].$$

We have $\|(T_n - T_m)(x)\| \le \|T_n - T_m\|\|x\|$. Then

$$\lim_{m,n\to\infty} \|(T_n - T_m)(x)\|^+_\alpha = 0, for \ \ all \ \ \alpha \in (0,1].$$

Hence $\{T_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{T_n(x)\}$ converges, say $T_n(x) \to y$, this defines an operator $T : X \to Y$, where $y = Tx$. Also since

$$\lim_{n\to\infty} T_n(\alpha x + \beta t) = \lim_{n\to\infty}(\alpha T_n x + \beta T_n t) = \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n t = \alpha Tx + \beta Tt,$$

$for \ \ all \ \ x, t \in X$,T$islinear$.

We prove that T is a fuzzy bounded. Let $N = \sup_{0<\beta\le 1} \sup_{\|x\|^-_\beta \le 1} \|Tx\|^-_\beta$ and $M_\alpha = \sup_{\alpha\le\beta} \sup_{\|x\|^+_\beta \le 1} \|Tx\|^+_\beta$, for all $\alpha \in (0,1]$. Suppose that $0 < \varepsilon$. Since $Tx = \lim_{n\to\infty} T_n x$, for all $\alpha \in (0,1]$, there exists $0 < M_{\alpha,x}$ such that

$$\|Tx - T_n x\|^+_\alpha < \varepsilon, \quad for \ \ all \ \ M_{\alpha,x} \le n.$$

Since $\{T_n\}$ is Cauchy, for all $\alpha \in (0,1]$ there exists $0 < N_\alpha$ such that

$$\|T_n - T_m\|^+_\alpha < \varepsilon, \quad for \ \ all \ \ N_\alpha \le m, n.$$

Let $\|x\|^+_\beta \le 1$ and $\alpha \le \beta$. Assume $N_\alpha, M_{\alpha,x} \le k$. Then

$$\begin{aligned}
\|Tx\|^+_\beta &\le \|(T - T_{N_\alpha})(x)\|^+_\beta + \|T_{N_\alpha} x\|^+_\beta \\
&\le \|(T - T_{N_\alpha})(x)\|^+_\beta + \|T_{N_\alpha}\|^+_\beta \|x\|^+_\beta \\
&\le \|(T - T_{N_\alpha})(x)\|^+_\beta + \|T_{N_\alpha}\|^+_\alpha \\
&\le \|(T - T_k)(x)\|^+_\beta + \|(T_k - T_{N_\alpha})(x)\|^+_\beta + \|T_{N_\alpha}\|^+_\alpha \\
&\le \|(T - T_k)(x)\|^+_\alpha + \|T_k - T_{N_\alpha}\|^+_\beta \|x\|^+_\beta + \|T_{N_\alpha}\|^+_\alpha \\
&\le \varepsilon + \varepsilon + \|T_{N_\alpha}\|^+_\alpha.
\end{aligned}$$

Hence we obtain that

$$M_\alpha = \sup_{\alpha\le\beta} \sup_{\|x\|_\beta^+\le 1} \|Tx\|_\beta^+ \le 2\varepsilon + \|T_{N_\alpha}\|_\alpha^+ < \infty. \tag{46}$$

Thus $M_\alpha < \infty$, for all $\alpha \in (0,1]$.

Let $\|x\|_\beta^- \le 1$ and $N_1, M_{1,x} \le k$. Then

$$\begin{aligned}
\|Tx\|_\beta^- &\le \|(T-T_{N_1})(x)\|_\beta^- + \|T_{N_1}x\|_\beta^- \\
&\le \|(T-T_{N_1})(x)\|_\beta^- + \|T_{N_1}\|_\beta^- \|x\|_\beta^- \\
&\le \|(T-T_{N_1})(x)\|_\beta^- + \|T_{N_1}\|_1^- \\
&\le \|(T-T_k)(x)\|_\beta^- + \|(T_k-T_{N_1})(x)\|_\beta^- + \|T_{N_1}\|_1^- \\
&\le \|(T-T_k)(x)\|_1^+ + \|T_k-T_{N_1}\|_\beta^- \|x\|_\beta^- + \|T_{N_1}\|_1^- \\
&\le \|(T-T_k)(x)\|_1^+ + \|T_k-T_{N_1}\|_1^+ + \|T_{N_1}\|_1^- \\
&\le \varepsilon + \varepsilon + \|T_{N_1}\|_1^-.
\end{aligned}$$

Therefore

$$N = \sup_{0<\beta\le 1} \sup_{\|x\|_\beta^-\le 1} \|Tx\|_\beta^- \le 2\varepsilon + \|T_{N_1}\|_1^- < \infty. \tag{47}$$

Hence $N < \infty$. By (46),(47) and Lemma 6.4, T is a fuzzy bounded linear operator.

Now we show that $\|T_n - T\| \to 0$. Let $0 < \varepsilon$. Suppose that $N_1 \le k$ and $\|x\|_\beta^- \le 1$. Assume $N_1, M_{1,x} \le n$. Then

$$\begin{aligned}
\|(T-T_k)(x)\|_\beta^- &\le \|(T-T_n)(x)\|_\beta^- + \|(T_k-T_n)(x)\|_\beta^- \\
&\le \|(T-T_n)(x)\|_1^+ + \|T_k-T_n\|_\beta^- \|x\|_\beta^- \\
&\le \|(T-T_n)(x)\|_1^+ + \|T_k-T_n\|_1^+ \\
&\le \varepsilon + \varepsilon.
\end{aligned}$$

Therefore

$$\sup_{0<\beta\le 1} \sup_{\|x\|_\beta^-\le 1} \|(T-T_k)(x)\|_\beta^- \le 2\varepsilon. \tag{48}$$

Let $\alpha \in (0,1]$ and $N_\alpha \le k$. Assume $\alpha \le \beta$ and $\|x\|_\beta^+ \le 1$. Suppose that $N_\alpha, M_{\alpha,x} \le n$,

$$\begin{aligned}
\|(T-T_k)(x)\|_\beta^+ &\le \|(T-T_n)(x)\|_\beta^+ + \|(T_k-T_n)(x)\|_\beta^+ \\
&\le \|(T-T_n)(x)\|_\alpha^+ + \|T_k-T_n\|_\beta^+ \|x\|_\beta^+ \\
&\le \|(T-T_n)(x)\|_\alpha^+ + \|T_k-T_n\|_\alpha^+ \\
&\le \varepsilon + \varepsilon.
\end{aligned}$$

Therefore

$$\sup_{\alpha\le\beta} \sup_{\|x\|_\beta^+\le 1} \|T - T_k(x)\|_\beta^+ \le 2\varepsilon. \tag{49}$$

Let $N^k = \sup_{0<\beta\le 1}\sup_{\|x\|_\beta^-\le 1} \|(T-T_k)(x)\|_\beta^-$ and $M_\alpha^k = \sup_{\alpha\le\beta}\sup_{\|x\|_\beta^+\le 1} \|(T-T_k)(x)\|_\beta^+$, for all $\alpha \in (0,1]$. By the proof of Lemma 6.4, $\|(T-T_k)(x)\| \le \eta_k\|x\|$ where $[\eta_k]_\alpha = [N^k, M_\alpha^k + N^k]$, for all $\alpha \in (0,1]$. Then from (48), (49), $\lim_{k\to\infty}\eta_k = 0$. Since by Theorem 7.1, $\|T - T_k\| \le \eta_k$ it follows that $\lim_{k\to\infty}\|T-T_k\| = 0$.

Consequently, $B(X,Y)$ is a complete fuzzy normed linear space. □

Theorem 7.3. *Let* $(X,\|.\|)$, $(Y,\|.\|)$ *and* $(Z,\|.\|)$ *be fuzzy normed linear spaces. Furthermore, let* $T \in B(X,Y)$ *and* $S \in B(Y,Z)$. *Then* $ST \in B(X,Z)$ *and* $\|ST\| \leq \|S\|\|T\|$.

Proof. We have $\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$. Hence $ST \in B(X,Z)$ and by Theorem 7.1, $\|ST\| \leq \|S\|\|T\|$. □

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