EXISTENCE AND UNIQUENESS OF THE SOLUTION OF NONLINEAR FUZZY VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper the fixed point theorem of Schauder is used to prove the existence of a continuous solution of the nonlinear fuzzy Volterra integral equations. Then using some conditions the uniqueness of the solution is investigated.

1. Introduction

It is well known that the fixed point theorems in Banach spaces are considered as powerful tools for solving the existence problems associated with integral equations. Most of these fixed point theorems are based on the concept of compact operators acting on some appropriate Banach spaces. Such results can be found in [2, 3, 4, 5, 7, 9, 10, 13, 15, 16, 17, 18].

In [12] nonlinear integral equation of Volterra type are considered as follows:

$$x(t) = Tx(t) = g(t) + \int_a^t k(t,s)f(s,x(s))ds \ , t \in [a,b].$$

first, they introduced the compact and completely continuous operators.

Definition 1.1. Let X be a Banach space and let $T : X \longrightarrow X$ be an operator. T is said to be compact, if it maps bounded sets of X into relatively compact sets. Moreover, T is said to be completely continuous, if it is continuous and compact.

In the special case, where $X = C[a,b]$, they used the following theorem to prove the compactness of T.

Theorem 1.2. *(Arzela-Ascoli's theorem) A necessary and sufficient condition that a family of continuous functions defined on the compact interval* $[a,b]$ *be compact in* $C[a,b]$ *is that this family be uniformly bounded and equicontinuous.*

Then by the following theorem they showed that the integral equation has a solution.

Theorem 1.3. *(Schauder's fixed point theorem) Let K be a closed convex subset of a Banach space X. If* $T : K \longrightarrow K$ *is continuous and* $K = \overline{T(K)}$ *is compact, then T has a fixed point in K.*

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Finally, to prove the uniqueness of the solution of the equation they offered a theorem. In this work, we extend their process for nonlinear fuzzy Volterra integral equations.

2. Preliminaries

Definition 2.1. [11] A fuzzy number is a function $u : \Re \longrightarrow [0,1]$ having the properties:

(i) u is normal,
that is $\exists x_0 \in \Re$ with $u(x_0) = 1$.
(ii) u is fuzzy convex set,
that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}\ \forall x, y \in \Re\, \lambda \in [0,1]$.
(iii) u is upper semi-continuous on $\Re$.
(iv) the support $\overline{\{x \in \Re : u(x) > 0\}}$ is a compact set.

The set of all fuzzy real numbers is denoted by ε^1. For $0 < \alpha \leq 1$, let us define $[u]^\alpha = \{x \in \Re : u(x) \geq \alpha\}$ and $[u]^0 = \overline{\{x \in \Re : u(x) > 0\}}$. Also, we define $u_-^\alpha = \inf [u]^\alpha$ and $u_+^\alpha = \sup [u]^\alpha$.

For $u, v \in \varepsilon^1$ and $\lambda \in \Re$, we have the sum $u + v$ and the product λu defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha\ \forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ is the usual addition of two intervals (as subsets of $\Re$), and $\lambda[u]^\alpha$ is the usual product between a scaler and a subset of $\Re$. We denote by $\sum$ the sum of real numbers and also the sum of fuzzy numbers with respect to $+$ (if the terms are fuzzy numbers).

Also, we use the Hausdorff distance between fuzzy numbers given by $d_\infty : \varepsilon^1 \times \varepsilon^1 \longrightarrow \Re^+ \bigcup \{0\}$ as in [8],

$$d_\infty(u,v) = \sup_{\alpha \in [0,1]} \{d_H([u]^\alpha, [v]^\alpha)\} = \sup_{\alpha \in [0,1]} \max\{|u_-^\alpha - v_-^\alpha|, |u_+^\alpha - v_+^\alpha|\}$$

where $[u]^\alpha = [u_-^\alpha, u_+^\alpha], [v]^\alpha = [v_-^\alpha, v_+^\alpha] \subseteq \Re$ and d_H is the Hausdorff distance between intervals.

In [6] Dubois and Prade introduced the concept of support function of a fuzzy number.

Let $u \in \varepsilon^1$, define $S_u : [0,1] \times \{\pm 1\} \rightarrow \Re$ by

$$S_u(\alpha, p) = S(p, [u]^\alpha) = \sup\{< p, a > ;\quad a \in [u]^\alpha\}$$

for $(\alpha, p) \in [0,1] \times \{\pm 1\}$, where $S(., [u]^\alpha)$ is the support function of $[u]^\alpha$. Call S_u the support function of the fuzzy number u. Note that the supremum is actually attained, since the set $[u]^\alpha$ is compact, and so can be replaced by the maximum.

Also, for each $\alpha \in [0,1]$

$$\begin{aligned} d_H([u]^\alpha, [v]^\alpha) &= \sup\{\ |S_u(\alpha,p) - S_v(\alpha,p)|\ ;\quad p \in \{\pm 1\}\ \} \\ &= \sup\{\ |S(p,[u]^\alpha) - S(p,[v]^\alpha)|\ ;\quad p \in \{\pm 1\}\ \}. \end{aligned}$$

We define $||.||_F = d_\infty(., \tilde{o})$, where $\tilde{o} = \chi_{\{0\}}$. Then we have the following theorem and it is known.

Theorem 2.2. , [1].

(i) The function $||.||_F : \varepsilon^1 \to \Re^+ \cup \{0\}$ *has the usual properties of the norm, that is,*

$$||u||_F = o \quad iff \quad u = \tilde{o},$$
$$||\lambda u||_F = |\lambda| ||u||_F,$$
$$||u + v||_F \leq ||u||_F + ||v||_F.$$

(ii) $|\ ||u||_F - ||v||_F\ | \leq d_\infty(u, v)$ *and* $d_\infty(u, v) \leq ||u||_F + ||v||_F$ *for any* $u, v \in \varepsilon^1$.

The following theorem will be helpful in what follows:

Theorem 2.3. [19].

(i) $(\varepsilon^1, d_\infty)$ *is a complete metric space.*
(ii) $d_\infty(u + v, v + w) = d_\infty(u, w) \quad \forall u, v, w \in \varepsilon^1$.
(iii) $d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v) \quad \forall u, v \in \varepsilon^1, \quad \forall \lambda \in \Re$.
(iv) $d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e) \quad \forall u, v, w, e \in \varepsilon^1$.

The usual definition of continuity of functions between metric spaces will be used.

Definition 2.4. [6]. A fuzzy-number-valued function $f : [a, b] \to \varepsilon^1$ is continuous at $t_0 \in [a, b]$ if for every $\epsilon > 0$ there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$d_\infty(f(t), f(t_0)) < \epsilon$$

for all $t \in [a, b]$ with $|t - t_0| < \delta$. We denote

$$C[a, b] = \{f;\ f : [a, b] \to \varepsilon^1 \text{ is continuous}\}.$$

In [19] Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral for a fuzzy-number-valued function.

Let $f : [a, b] \longrightarrow \varepsilon^1$. For $\Delta_n : a = x_0 < x_1 < ... < x_n = b$ a partition of the interval $[a, b]$, let us consider the intermadiate points $\zeta_i \in [x_{i-1}, x_i], i = 1, ..., n$, and $\delta : [a, b] \longrightarrow \Re^+$. The division $P = \{([x_{i-1}, x_i]; \zeta_i); i = 1, ..., n\}$ denoted shortly by $P = (\Delta_n, \zeta)$ is said to be $\delta - fine$ if:

$$[x_{i-1}, x_i] \subseteq (\zeta_i - \delta(\zeta_i), \zeta_i + \delta(\zeta_i)).$$

The function f is called Henstock integrable to $I \in \varepsilon^1$ if for every $\epsilon > 0$ there is a function $\delta : [a, b] \longrightarrow \Re^+$ such that for any $\delta - fine$ division P we have:

$$d_\infty(\Sigma_{i=1}^n (x_i - x_{i-1}) f(\zeta_i), I) < \epsilon.$$

I is called the Henstock integral of f and is denoted by:

$$(FH) \int_a^b f(t)dt.$$

If the above $\delta : [a, b] \longrightarrow \Re^+$ is a constant function, then one recaptures the concept of Riemann integral introduced by Goestchel and Voxman [11].

In this case $I \in \varepsilon^1$ will be called the Riemann integral of f on $[a, b]$ and will be denoted by:

$$(FR) \int_a^b f(t)dt.$$

Theorem 2.5. [19].

(i) If f and g are Henstock integrable mapping and if $d_\infty(f(t), g(t))$ is Lebesgue integrable, then:

$$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ (FH)\int_a^b g(t)dt\ ) \leq (L)\int_a^b d_\infty(f(t), g(t))dt.$$

(ii) Let $f : [a, b] \longrightarrow \varepsilon^1$ be a Henstock integrable bounded mapping. Then for any fixed $u \in [a, b]$, the function $\varphi_u : [a, b] \longrightarrow \Re$ defined by $\varphi_u(t) = d_\infty(f(u), f(t))$ is Lebesgue integrable on $[a, b]$.

3. Existence and Uniqueness of the Solution of Nonlinear Fuzzy Volterra Integral Equations

Before proposing the existence and uniqueness theorems, we are going to propose the following theorem which helps us to prove the mentioned theorems:

Theorem 3.1. *Consider the following functions:*

$$f : [a, b] \times \varepsilon^1 \longrightarrow \varepsilon^1, \quad k : [a, b] \times [a, b] \longrightarrow \Re$$

then:

$$d_\infty(\ k(s,t)f(t,x(t))\ ,\ k(s,t)f(t,y(t))\ ) = |k(s,t)|\ d_\infty(\ f(t,x(t))\ ,\ f(t,y(t))\ ) \quad \textit{and}$$

$$||k(s,t)f(t,x(t))||_F = |k(s,t)|\ ||f(t,x(t))||_F.$$

Proof. Using definition of the Hausdorff distance we have:

$d_\infty(\ k(s,t)f(t,x(t))\ ,\ k(s,t)f(t,y(t))\ ) =$

$\sup_{\alpha\in[0,1]}\{\ d_H(\ [k(s,t)f(t,x(t))]^\alpha\ ,\ [k(s,t)f(t,y(t))]^\alpha\ )\ \} =$

$\sup_{\alpha\in[0,1]}\{\ d_H(\ k(s,t)[f(t,x(t))]^\alpha\ ,\ k(s,t)[f(t,y(t))]^\alpha\ )\ \} =$

$\sup_{\alpha\in[0,1]}\{\ |k(s,t)|\ d_H(\ [f(t,x(t))]^\alpha\ ,\ [f(t,y(t))]^\alpha\ )\ \} =$

$|k(s,t)|\ \sup_{\alpha\in[0,1]}\{\ d_H(\ [f(t,x(t))]^\alpha\ ,\ [f(t,y(t))]^\alpha\ )\ \} =$

$|k(s,t)|\ d_\infty(\ f(t,x(t))\ ,\ f(t,y(t))\ ).$

□

So for the second property we have:

$$\begin{aligned} ||k(s,t)f(t,x(t))||_F &= d_\infty(\ k(s,t)f(t,x(t))\ ,\ \tilde{o}\ ) \\ &= |k(s,t)|\ d_\infty(\ f(t,x(t))\ ,\ \tilde{o}\ ) \\ &= |k(s,t)|\ ||f(t,x(t))||_F. \end{aligned}$$

Our result is given by the following theorem:

Theorem 3.2. *Consider the following nonlinear fuzzy Volterra integral equation:*

$$x(t) = Tx(t) = g(t) + (FH)\int_a^t k(t,s)f(s,x(s))ds \ , \qquad t \in [a,b] \tag{1}$$

Suppose that $g : [a,b] \to \varepsilon^1$ is a continuous function on the interval $[a,b]$. Also, suppose that the kernel k satisfies the following conditions,

(i) $\forall s \in [a,b] \quad \forall t \leq t_0 \quad k(t,s) \geq k(t_0,s)$.
(ii) the function $t \mapsto \int_a^t k(t,s)ds$ is continuous on $[a,b]$.
(iii) $\forall t_0 \in [a,b], \qquad s \mapsto k(t_0,s) \in L^1[a,b]$.

Moreover, suppose that the function $f : [a,b]\times\varepsilon^1 \to \varepsilon^1$ is continuous and satisfies the following condition,

$$\forall u \in \varepsilon^1 \quad ||f(s,u)||_F \leq c_1||u||_F + c_2 \tag{2}$$

where c_1, c_2 are two positive constant. Finally, suppose that the kernel k satisfies the following condition

$$\sup_{a\leq t\leq b}\int_a^t |k(t,s)| \ ds < \frac{1}{c_1}. \tag{3}$$

Then the equation (1) has a solution $x \in C[a,b]$.

Proof. First, we prove that the operator T in (1) is completely continuous from $C[a,b]$ into $C[a,b]$. To prove that $T(C[a,b]) \subset C[a,b]$, we proceed as follows.

Let $x \in C[a,b]$ and $t, t_0 \in [a,b]$, with $t < t_0$. Then, by using (iv) of the theorem 2.3 we have

$d_\infty(Tx(t), Tx(t_0)) =$

$d_\infty(\ g(t) + (FH)\int_a^t k(t,s)f(s,x(s))ds \ , \ g(t_0) + (FH)\int_a^{t_0} k(t_0,s)f(s,x(s))ds \ ) \leq$

$d_\infty(\ g(t) \ , \ g(t_0) \ ) + d_\infty(\ (FH)\int_a^t k(t,s)f(s,x(s))ds \ , \ (FH)\int_a^{t_0} k(t_0,s)f(s,x(s))ds \ )$.

On the other hand, using theorem 2.5 and the definition of fuzzy norm we have

$d_\infty(\ (FH)\int_a^t k(t,s)f(s,x(s))ds \ , \ (FH)\int_a^{t_0} k(t_0,s)f(s,x(s))ds \ ) =$

$d_\infty(\ (FH)\int_a^t k(t,s)f(s,x(s))ds \ ,$

$(FH)\int_a^t k(t_0,s)f(s,x(s))ds \ + \ (FH)\int_t^{t_0} k(t_0,s)f(s,x(s))ds \ ) \leq$

$d_\infty(\ (FH)\int_a^t k(t,s)f(s,x(s))ds \ , \ (FH)\int_a^t k(t_0,s)f(s,x(s))ds \ ) +$

$d_\infty(\ (FH)\int_t^{t_0} k(t_0,s)f(s,x(s))ds \ , \ \tilde{o} \ ) \leq$

$(L)\int_a^t \ d_\infty(\ k(t,s)f(s,x(s)), k(t_0,s)f(s,x(s)) \ )ds \ +$

$||(FH)\int_t^{t_0} k(t_0,s)f(s,x(s))ds||_F \le$

$(L)\int_a^t(\ k(t,s)-k(t_0,s)\ )||f(s,x(s))||_F\ ds +$

$(L)\int_t^{t_0} |k(t_0,s)|\ ||f(s,x(s))||_F\ ds,$

where, to get the last inequality we use (i) as follow

$d_\infty(\ k(t,s)f(s,x(s)), k(t_0,s)f(s,x(s))\ ) =$

$\sup_\alpha\{\ d_H(\ [k(t,s)f(s,x(s))]^\alpha, [k(t_0,s)f(s,x(s))]^\alpha\ )\ \} =$

$\sup_\alpha\{\ \sup_p\{\ |\ S(p,[k(t,s)f(s,x(s))]^\alpha) - S(p,[k(t_0,s)f(s,x(s))]^\alpha)\ |\ \}\ \} =$

$\sup_\alpha\{\ \sup_p\{\ |\ S(p,k(t,s)[f(s,x(s))]^\alpha) - S(p,k(t_0,s)[f(s,x(s))]^\alpha)\ |\ \}\ \} =$

$\sup_\alpha\{\ \sup_p\{\ |\ k(t,s)S(p,[f(s,x(s))]^\alpha) - k(t_0,s)S(p,[f(s,x(s))]^\alpha)\ |\ \}\ \} =$

$\sup_\alpha\{\ \sup_p\{\ |\ (\ k(t,s)-k(t_0,s)\ )S(p,[f(s,x(s))]^\alpha)\ |\ \}\ \} =$

$(\ k(t,s)-k(t_0,s)\ )\sup_\alpha\{\ \sup_p\{\ |\ S(p,[f(s,x(s))]^\alpha)\ |\ \}\ \} =$

$(\ k(t,s)-k(t_0,s)\ )\sup_\alpha\{\ \sup_p\{\ |\ S(p,[f(s,x(s))]^\alpha) - S(p,[\tilde{o}]^\alpha)\ |\ \}\ \} =$

$(\ k(t,s)-k(t_0,s)\ )\sup_\alpha\{\ d_H(\ [f(s,x(s))]^\alpha\ ,\ [\tilde{o}]^\alpha\ )\ \} =$

$(\ k(t,s)-k(t_0,s)\ )d_\infty(\ f(s,x(s))\ ,\ \tilde{o}\ ) =$

$(\ k(t,s)-k(t_0,s)\ )||\ f(s,x(s))\ ||_F\ .$

Therefore,

$$\begin{aligned} d_\infty(Tx(t),Tx(t_0)) &\le d_\infty(\ g(t)\ ,\ g(t_0)\ ) \\ &+ (L)\int_a^t(\ k(t,s)-k(t_0,s)\ )||f(s,x(s))||_F\ ds \\ &+ (L)\int_t^{t_0}|k(t_0,s)|\ ||f(s,x(s))||_F\ ds. \end{aligned} \tag{4}$$

By using (i), we have

$$\begin{aligned} t<t_0 &\Rightarrow k(t,s)\ge k(t_0,s) \\ &\Rightarrow k(t,s)-k(t_0,s)\ge 0 \\ &\Rightarrow (L)\int_a^t(\ k(t,s)-k(t_0,s)\ )\ ds\ge 0. \end{aligned}$$

Also,

$$\begin{aligned} 0 &\leq (L)\int_a^t (\ k(t,s) - k(t_0,s)\ )\ ds \\ &= |\ (L)\int_a^t (\ k(t,s) - k(t_0,s)\ )\ ds\ | \\ &= |\ (L)\int_a^t k(t,s)\ ds - (L)\int_a^t k(t_0,s)\ ds\ | \\ &= |\ (L)\int_a^t k(t,s)\ ds - (L)\int_a^{t_0} k(t_0,s)\ ds + (L)\int_t^{t_0} k(t_0,s)\ ds\ | \\ &\leq |\ (L)\int_a^t k(t,s)\ ds - (L)\int_a^{t_0} k(t_0,s)\ ds\ | + |\ (L)\int_t^{t_0} k(t_0,s)\ ds\ | \\ &\leq |\ (L)\int_a^t k(t,s)\ ds - (L)\int_a^{t_0} k(t_0,s)\ ds\ | + (L)\int_t^{t_0} |k(t_0,s)|\ ds. \end{aligned} \tag{5}$$

By using (ii), (iii) and (5), we deduce that

$$\lim_{t\to t_0}\ (L)\int_a^t (k(t,s) - k(t_0,s))\ ds = \lim_{t\to t_0}\ (L)\int_t^{t_0} |k(t_0,s)|\ ds = 0. \tag{6}$$

Since x is bounded on $[a,b]$ and the function f satisfies (2), there exists a constant $M > 0$, such that

$$\forall s \in [a,b], \qquad ||f(s,x(s))||_F \leq M. \tag{7}$$

Combining (4) and (7), we have

$d_\infty(Tx(t), Tx(t_0)) \leq$

$d_\infty(\ g(t), g(t_0)\ ) +$

$(L)\int_a^t(\ k(t,s) - k(t_0,s)\ )||f(s,x(s))||_F\ ds + (L)\int_t^{t_0} |k(t_0,s)|\ ||f(s,x(s))||_F\ ds \leq$

$d_\infty(\ g(t), g(t_0)\ ) + (L)\int_a^t(\ k(t,s) - k(t_0,s)\ ).\ M\ ds + (L)\int_t^{t_0} |k(t_0,s)|.\ M\ ds =$

$d_\infty(\ g(t), g(t_0)\ ) + M.\ (\ (L)\int_a^t(\ k(t,s) - k(t_0,s)\ )\ ds + (L)\int_t^{t_0} |k(t_0,s)|\ ds\ ).$

Therefore, using (6) and the fact that g is continuous, we have

$$t \to t_0 \qquad \Rightarrow \qquad d_\infty(Tx(t), Tx(t_0)) \to 0$$

. This shows that

$$Tx \in C[a,b].$$

Also, from (ii) and continuity of the function f on $[a,b] \times \varepsilon^1$ we deduce that T is continuous on $C[a,b]$.

Let $x_1(s), x_2(s) \in \varepsilon^1$

. Then $d_\infty(Tx_1(t), Tx_2(t)) =$

$d_\infty(\ g(t) + (FH)\int_a^t k(t,s)f(s,x_1(s))ds\ ,\ g(t) + (FH)\int_a^t k(t,s)f(s,x_2(s))ds\ ) =$

$d_\infty(\ (FH)\int_a^t k(t,s)f(s,x_1(s))ds\ ,\ (FH)\int_a^t k(t,s)f(s,x_2(s))ds\ ) \leq$

$(L)\int_a^t\ d_\infty(\ k(t,s)f(s,x_1(s))\ ,\ k(t,s)f(s,x_2(s))\ )\ ds =$

$(L)\int_a^t\ |k(t,s)|\ d_\infty(\ f(s,x_1(s))\ ,\ f(s,x_2(s))\ )\ ds.$

Next, to prove the compactness of T it suffices to show that T satisfies the condition of Arzela-Ascoli's theorem. Suppose $S = \{x_n\ ,\ n \in \mathbf{N}\}$ is a bounded set of $C[a,b]$ with a constant bound c_s.

Then $\forall n \in \mathbf{N}\quad ||x_n(t)||_F \leq c_s$ and consequently,

$$\begin{aligned} ||Tx_n(t)||_F &= ||g(t) + (FH)\int_a^t k(t,s)f(s,x_n(s))\ ds||_F \\ &\leq ||g(t)||_F\ +\ ||(FH)\int_a^t k(t,s)f(s,x_n(s))\ ds||_F \\ &\leq ||g(t)||_F\ + (L)\int_a^t |k(t,s)|\ ||f(s,x_n(s))||_F\ ds \\ &\leq ||g(t)||_F\ +\ (L)\int_a^t |k(t,s)|(\ c_1||x_n(s)||_F + c_2)\ ds \\ &\leq ||g(t)||_F\ +\ (L)\int_a^t |k(t,s)|(\ c_1.c_s + c_2)\ ds \\ &< c_g\ +\ (\ c_1.c_s + c_2)\frac{1}{c_1}\ . \end{aligned}$$

Therefore, $T(S)$ is uniformly bounded.

Moreover, we know that, a family $T(S)$ of functions $Tx_n : [a,b] \longrightarrow \varepsilon^1$ on $[a,b]$ is equicontinuous if and only if:

$$\forall\epsilon > 0\ \ \exists\delta > 0\ ;\ \forall t_1,t_2 \in [a,b]\ ,\ Tx_n \in T(S)$$
$$|t_1 - t_2| < \delta \longrightarrow\ d_\infty(Tx_n(t_1), Tx_n(t_2)) < \epsilon.$$

Now, by substituting x by x_n in (4) and using (6) and (7) we can show that $T(S)$ is equicontinuous. Finally, using Arzela-Ascoli's theorem, we deduce that $T : C[a,b] \longrightarrow C[a,b]$ is completely continuous.

Next, let $R > 0$ be a real number. Consider the closed, convex ball B_R of $C[a,b]$, given by :

$$B_R = \{\ x \in C[a,b]\ ;\ ||x||_F \leq R\ \}.$$

By using (2) we get

$$||Tx||_F^* \leq ||g||_F^* + (c_1R + c_2).\sup_{a\leq t\leq b}\int_a^t |k(t,s)|\ ds. \tag{8}$$

where $||Tx||_F^* = \sup_{a\leq t\leq b} ||Tx(t)||_F$. Whenever

$$R \geq \frac{||g||_F^* + c_2\sup\int_a^t |k(t,s)|ds}{1 - c_1\sup\int_a^t |k(t,s)|ds} = R_0,$$

we have

$$R - Rc_1 sup_{a \leq t \leq b} \int_a^t |k(t,s)| \ ds \geq ||g||_F^* + c_2 sup_{a \leq t \leq b} \int_a^t |k(t,s)| \ ds.$$

Therefore, by using (8) we have

$$R \geq ||g||_F^* + (c_1 R + c_2) sup_{a \leq t \leq b} \int_a^t |k(t,s)| \ ds \geq ||Tx||_F^*$$

and it shoes that

$$T(B_R) \subset B_R$$

By using Schauder's fixed point theorem, we deduce that the integral equation (1) has a continuous solution on $[a,b]$. □

Remark 3.3. Since $t \longrightarrow \int_a^t k(t,s)ds$ is continuous, then supposition (3) is satisfied for small enough $b-a$. Hence, the previous theorem always ensures the existence of a solution of integral equation in a neighborhood of 'a' point.

The uniqueness of the solution of integral equation under a nonlinear condition, is given by the following theorem:

Theorem 3.4. *Suppose that the function f given by the previous theorem, satisfies the following condition:*

$$\forall u, v \in \varepsilon^1, d_\infty(\ f(s,u), f(s,v) \ ) \leq L(d_\infty(u,v))^r$$

for some constants $L > 0$ and $0 < r \leq 1$.

Then, under the suppositions of the previous theorem, integral equation has a unique continuous solution on $[a,b]$.

Proof. Let M_K be the positive constant given by $M_K = \sup_{a \leq t,s \leq b} |k(t,s)|$. By contradiction, suppose that the integral equation has two different solutions $x, y \in C[a,b]$. Then, there exists $0 \leq \epsilon < 1$ such that,

$$d_\infty^*(x,y) \geq \epsilon,$$

where, $d_\infty^*(x,y) = \sup_{a \leq t \leq b} d_\infty(x(t), y(t))$ denotes the uniform distance between fuzzy-number-valued functions.

Moreover, it is clear that for all $t \in [a,b]$ we have:

$d_\infty(Tx(t), Ty(t)) =$

$d_\infty(g(t) + (FH) \int_a^t k(t,s) f(s,x(s))ds, g(t) + (FH) \int_a^t k(t,s) f(s,y(s))ds) =$

$d_\infty(\ (FH) \int_a^t k(t,s) f(s,x(s))ds \ , \ (FH) \int_a^t k(t,s) f(s,y(s))ds \ ) \leq$

$(L) \int_a^t \ d_\infty(\ k(t,s) f(s,x(s)) \ , \ k(t,s) f(s,y(s)) \ ) \ ds =$

$(L)\int_a^t\ |k(t,s)|\ d_\infty(\ f(s,x(s))\ ,\ f(s,y(s))\ )\ ds \leq$

$(L)\int_a^t\ |k(t,s)|\ L\ (\ d_\infty(x(s),y(s))\ )^r\ ds \leq$

$LM_K(\ d_\infty^*(x,y)\ )^r(t-a).$

By using the previous inequality, we have:

$d_\infty(T^2x(t),T^2y(t)) =$

$d_\infty(g(t)+(FH)\int_a^t k(t,s)f(s,Tx(s))ds, g(t)+(FH)\int_a^t k(t,s)f(s,Ty(s))ds) =$

$d_\infty(\ (FH)\int_a^t k(t,s)f(s,Tx(s))ds\ ,\ (FH)\int_a^t k(t,s)f(s,Ty(s))ds\ ) \leq$

$(L)\int_a^t\ d_\infty(\ k(t,s)f(s,Tx(s))\ ,\ k(t,s)f(s,Ty(s))\ )\ ds =$

$(L)\int_a^t\ |k(t,s)|\ d_\infty(\ f(s,Tx(s))\ ,\ f(s,Ty(s))\ )\ ds \leq$

$(L)\int_a^t\ |k(t,s)|\ L(d_\infty(Tx(s),Ty(s)))^r\ ds \leq$

$(L)\int_a^t\ |k(t,s)|\ L(LM_K(d_\infty^*(x,y))^r(s-a))^r\ ds \leq$

$L^{r+1}M_K^{r+1}(d_\infty^*(x,y))^{r^2}\frac{(t-a)^{r+1}}{r+1}.$

Similarly, we get:

$$d_\infty(T^3x(t),T^3y(t)) \leq (d_\infty^*(x,y))^{r^3}(LM_K)^{r^2+r+1}\frac{(t-a)^{r^2+r+1}}{(r+1)(r^2+r+1)}.$$

More generally, for any positive integer n, we have:

$$d_\infty(T^nx(t),T^ny(t)) \leq \frac{(d_\infty^*(x,y))^{r^n}\ (LM_K)^{r^{n-1}+\ldots+r+1}\ (t-a)^{r^{n-1}+\ldots+r+1}}{(r+1)(r^2+r+1)\ldots(r^{n-1}+\ldots+r+1)}.$$

Therefore,

$d_\infty(T^nx(t),T^ny(t)) \leq$

$(d_\infty^*(x,y))^{r^n}(LM_K)^{r^{n-1}+\ldots+r+1}\frac{(t-a)^{r^{n-1}+\ldots+r+1}}{(r+1)(r^2+r+1)\ldots(r^{n-1}+\ldots+r+1)} \leq$

$(d_\infty^*(x,y))^{r^n}(LM_K)^{r^{n-1}+\ldots+r+1}\frac{(b-a)^{r^{n-1}+\ldots+r+1}}{(r+1)(r^2+r+1)\ldots(r^{n-1}+\ldots+r+1)} =$

$(d_\infty^*(x,y))^{r^n}\frac{(LM_K(b-a))^{r^{n-1}+\ldots+r+1}}{(r+1)(r^2+r+1)\ldots(r^{n-1}+\ldots+r+1)} =$

$d_\infty^*(x,y)\ \frac{(d_\infty^*(x,y))^{r^n-1}(LM_K(b-a))^{r^{n-1}+\ldots+r+1}}{(r+1)(r^2+r+1)\ldots(r^{n-1}+\ldots+r+1)}\ .$

Hence,

$$d^*_\infty(T^nx, T^ny) \leq d^*_\infty(x, y)\ \frac{(d^*_\infty(x, y))^{r^n-1}(LM_K(b-a))^{r^{n-1}+...+r+1}}{(r+1)(r^2+r+1)...(r^{n-1}+...+r+1)}\ .$$

Since $0 < r \leq 1$ and $d^*_\infty(x, y) \geq \epsilon$, for some $0 < \epsilon < 1$ the previous inequality implies:

$$d^*_\infty(T^nx, T^ny) \leq d^*_\infty(x, y)\ \frac{\max(1, (LM_K(b-a))^{\frac{1}{1-r}})}{\epsilon(r+1)(r^2+r+1)...(r^{n-1}+...+r+1)}\ .$$

Since $\epsilon \prod_{i=1}^{n}(r^i + ... + r + 1) \longrightarrow \infty$ as $n \longrightarrow \infty$, there exists $n_0 \in \mathbf{N}$, such that:

$$\frac{\max(1, (LM_K(b-a))^{\frac{1}{1-r}})}{\epsilon(r+1)(r^2+r+1)...(r^{n-1}+...+r+1)} < 1 \qquad \forall n \geq n_0$$

Consequently, we have:

$$d^*_\infty(T^nx, T^ny) < d^*_\infty(x, y) \quad \forall n \geq n_0$$

On the other hand, since x, y are the solutions of integral equation, they are fixed points of T and more generally, they are fixed points of T^n, $\forall n \in \mathbf{N}$.
Consequently, x, y have to satisfy the equality, $d^*_\infty(T^nx, T^ny) = d^*_\infty(x, y)$ which contradicts $d^*_\infty(T^nx, T^ny) < d^*_\infty(x, y)$. Hence integral equation has a unique solution. □

4. Conclusion

In this article, we investigated existence and uniqueness of the solution of nonlinear fuzzy Volterra integral equations. To this end, we implied some conditions on the integral equation and found that under these conditions, and using Arzela-Ascoli's theorem and Schauder's fixed point theorem we can prove existence and uniqueness of the solution for this kind of equations. Some of these conditions which we can point out are continuity and a condition like Lipschitz's condition for nonlinear statement f of integral equation and also the continuity of known function g.

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