EXISTENCE AND UNIQUENESS OF THE SOLUTION OF FUZZY-VALUED INTEGRAL EQUATIONS OF MIXED TYPE

F. MOKHTARNEJAD AND R. EZZATI

Abstract. In this paper, existence theorems for the fuzzy Volterra-Fredholm integral equations of mixed type (FVFIEMT) involving fuzzy number valued mappings have been investigated. Then, by using Banach's contraction principle, sufficient conditions for the existence of a unique solution of FVFIEMT are given. Finally, illustrative examples are presented to validate the obtained results.

1. Introduction

The study of population dynamics, parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic and various physical and biological models leads to integral equations. Since these real-world problems are too complex to be defined in precise terms, imprecision is often involved. Analyzing such problems requires the use of fuzzy information. Therefore, the fuzzy concept proposed by Zadeh [27, 28] is deemed to be quite useful in many applications. Thus, the need for solving Volterra-Fredholm integral equations of mixed type, with the parameters being completely or partially represented by fuzzy numbers, is legitimated.

The concept of fuzzy measures and fuzzy integrals for single-valued mappings was initiated by Sugeno [23] and then investigated by Dubois and Prade [10], Kaleva [13], Goetschel and Voxman [12], Nanda [16] and others [19, 25]. The Henstock-integral of fuzzy-valued functions is defined in [24]. The fuzzy Riemann integral and its numerical integration was investigated by Wu in [26]. Using the Lebesque type concept for integration, Kaleva [13] defined the integral of fuzzy function.

In [21], the authors proved the existence of the solution of fuzzy functional equations. The authors of [18] defined the concepts of fuzzy random variable and the expectation of a fuzzy random variable. The study of fuzzy integral equations is done by Mordeson and Newman in [15].

Many authors applied the fixed point theorems like the Darbo's theorem and the Banach's fixed point principle as a tool to prove the existence and uniqueness of the solution of fuzzy integral equations ([6, 7, 17]). Seikkala [20] considered the

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existence of solution of fuzzy integral equations using the Banach fixed point principle. In [22], the authors presented some existence theorems for certain Volterra integral equations involving fuzzy set valued mappings.

Various techniques to construct the numerical methods for fuzzy integral equations are proposed by many researchers. In [8], the author presented a numerical method for the solution of nonlinear fuzzy Fredholm integral equations using a fixed point technique. In [11], a new approach based on fuzzy Bernstein polynomials is proposed for solving fuzzy Fredholm integral equation of the second kind. Solving linear fuzzy Fredholm integral equations of the second kind using fuzzy Haar wavelet is done in [29]. In [14], using the parametric form of a fuzzy number, a fuzzy Fredholm integral equations of the second kind is converted to two linear systems of integral equations in the crisp case. For other works, see [1, 3, 4, 9].

In this work, we consider FVFIEMT as

$$x(t,s) = f(t,s) + \int_0^t \int_\Omega g(t,s,\sigma,\tau,x(\sigma,\tau))d\sigma d\tau, (t,s) \in [0,T]\times\Omega, 0 \le T \le 1 \quad (1)$$

involving fuzzy number valued mappings. Through this paper we suppose that $\Omega = [a,b]$. Some theoretical background that are needed in the rest of the paper, are briefly reviewed in Section 2. In Section 3, main section of the paper, by using Banach's contraction principle, we prove a theorem for the existence of unique solution of FVFIEMT defined in (1). Section 4 contains the illustrative examples for the validity of the obtained results. Finally, conclusion is drawn in Section 5.

2. **Preliminaries**

The symbol $F_k(\mathbb{R}^n)$ denotes the family of nonempty, compact, convex subsets of $\mathbb{R}^n$. Addition and scalar multiplication in $F_k(\mathbb{R}^n)$ are defined as usual. $\overline{U}$ denotes the closure of U, where U is contained in $\mathbb{R}^n$. Denote

$$E^n = \{x : \mathbb{R}^n \mapsto [0,1] |\ x\ satifies\ (1)-(4)\},$$

where

(1) x is normal, i.e. $\exists x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
(2) x is fuzzy convex set (i.e. $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\} \forall x, y \in \mathbb{R}, \lambda \in [0,1]$),
(3) x is upper semi-continuous on $\mathbb{R}$,
(4) $\overline{\{x \in \mathbb{R}^n | u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

For $0 < \alpha \le 1$ denote $[x]^\alpha = \{t \in \mathbb{R}^n | x(t) \ge \alpha\}$. Then from (1)-(4) it follows that the α-level set $[x]^\alpha \in F_k(\mathbb{R}^n)$ for all $\alpha \in [0,1]$.
If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then using Zadeh's extension principle we can extend f to $E^n \times E^n \to E^n$ by the equation

$$\tilde{f}(u,v)(z) = \sup_{\{(x,y):z=f(x,y)\}} \min(u(x), v(y)).$$

It is well known that $[\tilde{f}(u,v)]^\alpha = f([u]^\alpha, [v]^\alpha)$, for all $u, v \in E^n, 0 \le \alpha \le 1$ and f continuous [13]. Clearly, from the above equation, we conclude that $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda \odot u]^\alpha = \lambda[u]^\alpha$ for any real number λ.
Let $D : E^n \times E^n \to \mathbb{R}^+ \cup \{0\}$ be defined by

$$D(u,v) = \sup_{0\le\alpha\le1} H([u]^\alpha, [v]^\alpha), \quad (2)$$

where H is the Hausdorff metric defined in $F_k(\mathbb{R}^n)$. It is proved that (E^n, D) is a complete metric space with the following properties [13, 18]:

(1) $D(u \oplus w, v \oplus w) = D(u, v)\ \forall\ u, v, w \in E^n$,
(2) $D(k \odot u, k \odot v) = |k|\, D(u, v)\ \forall\ u, v \in E^n\ \forall k \in \mathbb{R}$,
(3) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)\ \forall\ u, v, w, e \in E^n$.

Definition 2.1. [5] Let T be $[0,1]$ and for each t in T, let $F(t)$ be a non-empty subset of $\mathbb{R}^n$. Let $\mathbb{F}$ be the set of all point valued functions f from T to $\mathbb{R}^n$ such that f is integrable over T and $f(t) \in F(t)$ for all t in T. Then

$$\int_T F(t)dt = \left\{ \int_T f(t)dt : f \in \mathbb{F} \right\}.$$

Definition 2.2. [13] A mapping $F : T \to E^n$ is strongly measurable if for all $r \in [0,1]$ the set-valued map $F_r : T \to \mathbb{F}_k(\mathbb{R}^n)$ defined by $F_r(t) = [F(t)]^r$ is Lebesgue measurable when $F_k(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric H.

Definition 2.3. [18] A mapping $F : T \to E^n$ is said to be integrably bounded if there is an integrable function g such that $||x|| \leq g(t)$ for ever $x \in F(t)$.

Definition 2.4. [2] A fuzzy valued function $f : [a,b] \times [c,d] \to E^n$ is said to be continuous at $x_0 \in [a,b], y_0 \in [c,d]$, if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(x,y), f(x_0,y_0)) < \epsilon$ whenever $x \in [a,b],\ y \in [c,d]$ and $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$. We say that f is fuzzy continuous on $[a,b] \times [c,d]$ if f is continuous at each $x_0 \in [a,b],\ y_0 \in [c,d]$, denote the space of all such functions by $C([a,b] \times [c,d], E^n)$.

Theorem 2.5. [22] *If $F : T \to E^n$ is continuous then it is integrable.*

Theorem 2.6. [22] *Let $F, G : T \to E^n$ be integrable and $\lambda \in \mathbb{R}$. Then*

(1) $\int (F + G) = \int F + \int G$,
(2) $\int (\lambda F) = \lambda \int F$,
(3) $D(F, G)$ *is integrable*,
(4) $D(\int F, \int G) \leq \int D(F, G)$.

3. Existence Theorems

In [22], the authors considered the nonlinear fuzzy Volterra integral equations

$$x(t) = f(t) + \int_0^t g(t, s, x(s))ds,\ t \in [0, T]$$

and presented an existence theorem for this equation. In this section, we extend the results of [22] to FVFIEMT defined in (1), and prove an existence theorem for this equation.

Theorem 3.1. *Let β and L be positive numbers. Suppose:*

(1) $f : [0,T] \times \Omega \to E^n$ *iscontinuous*,
(2) $g : U \to E^n$ *iscontinuous where*
$U = \{(t, s, \sigma, \tau, x) | 0 \leq \tau \leq t \leq T, x \in E^n \text{ and } D(x(\sigma,\tau), f(t,s)) \leq \beta\}$,

(3) *g satisfies Lipschitz condition with respect to x on U, i.e.* $D(g(t,s,\sigma,\tau,x), g(t,s,\sigma,\tau,y)) \le LD(x,y)$ *if* $(t,s,\sigma,\tau,x),(t,s,\sigma,\tau,y) \in U$.

If $\beta L < M$, *where* $M = \sup_U D(g(t,s,\sigma,\tau,x),\tilde{0})$, $N = \int_\Omega d\sigma$, *and also* $T \le \frac{\beta}{MN}$ *then there is a unique solution of FVFIEMT defined in (1).*

Proof. As the notations in [22], let φ be the space of continuous functions from $[0,T]\times\Omega$ into (E^n, D) with $H_1(\psi, f) \le \beta$, i.e. $\varphi = \{\psi | \psi : [0,T]\times\Omega \to E^n$ continuous and $H_1(\psi,f) \le \beta\}$, where $H_1(\psi, f) = \sup_{0\le t\le T, s\in\Omega} D(\psi(t,s), f(t,s))$ in which D is defined in [5]. We define the operator $A : \varphi \to \varphi$ as follows:

$$A\psi(t,s) = f(t,s) + \int_0^t \int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau.$$

Now, we prove that $A : \varphi \to \varphi$. To this end, we have to prove that $A\psi$ is continuous whenever $\psi \in \varphi$ and also $H_1(A\psi, f) \le \beta$. From (1) and by using the properties of D, we have

$$\begin{aligned}
&D(A\psi(t+h,s+k), A\psi(t,s)) = \\
&D\bigg(f(t+h,s+k) + \int_0^{t+h}\int_\Omega g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau, f(t,s) \\
&+ \int_0^t\int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau\bigg) \le D(f(t+h,s+k), f(t,s)) + \\
&D\bigg(\int_0^{t+h}\int_\Omega g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau), \int_0^t\int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau\bigg) \\
&\le D(f(t+h,s+k), f(t,s)) + \int_t^{t+h}\int_\Omega D\Big(g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau)),\tilde{0}\Big)d\sigma d\tau \\
&+ \int_0^t D\bigg(\int_\Omega g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau))d\sigma, \int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma\bigg)d\tau \\
&\le \frac{\epsilon}{2} + \int_t^{t+h}\int_\Omega D(g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau)),\tilde{0})d\sigma d\tau \\
&+ \int_0^t\int_\Omega D(g(t+h,s+k,\sigma,\tau,\psi(\sigma,\tau)), g(t,s,\sigma,\tau,\psi(\sigma,\tau)))d\sigma d\tau. \qquad (3)
\end{aligned}$$

Since f is continuous function and g is integrable, so the right-hand side of (3) tends to zero. Therefore, $A\psi$ is a continuous. Now we prove that $A\psi \in \varphi$. We have

$$\begin{aligned}
H_1(A\psi, f) &= \sup_{0\le t\le T, s\in\Omega} D(A\psi(t,s), f(t,s)) \\
&= \sup_{0\le t\le T, s\in\Omega} D\left(f(t,s) + \int_0^t\int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau, f(t,s)\right) \\
&= \sup_{0\le t\le T, s\in\Omega} D(\int_0^t\int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau, \tilde{0}) \\
&\le \sup_{0\le t\le T, s\in\Omega} \int_0^t\int_\Omega D(g(t,s,\sigma,\tau,\psi(\sigma,\tau),\tilde{0})d\sigma d\tau \le MNT \le \beta.
\end{aligned}$$

So $A\psi \in \varphi$. Clearly, operator A maps φ into itself. In [13], the authors show that $C([0,T], E^n)$ is a complete metric space with the metric

$$H_2(\psi, f) = \sup_{0\le t\le T} D(\psi(t), f(t)).$$

So, a standard proof applies to show that $C([0,T]\times\Omega,E^n)$ is a complete metric space with the metric H_1. Now, we show that φ is a closed subset of $C([0,T]\times\Omega,E^n)$. Let (ψ_n) be a sequence in φ converging to ψ in $C([0,T]\times\Omega,E^n)$. Using the properties of D, we conclude that for sufficiently large n and all positive sufficiently small ϵ, we have

$$H_1(\psi,f)\leq H_1(\psi,\psi_n)+H_1(\psi_n,f)\leq \epsilon+\beta.$$

So $\psi\in\varphi$. This implies that φ is a closed subset of $C([0,T]\times\Omega,E^n)$. Therefore φ is a complete metric space. Now we show that A is a contraction mapping. For $\phi,\psi\in\varphi$,

$$\begin{aligned}
&H_1(A\phi,A\psi)\\
&=\sup_{0\leq t\leq T,s\in\Omega} D(\int_0^t\int_\Omega g(t,s,\sigma,\tau,\phi(\sigma,\tau))d\sigma d\tau,\int_0^t\int_\Omega g(t,s,\sigma,\tau,\psi(\sigma,\tau))d\sigma d\tau)\\
&\leq \sup_{0\leq t\leq T,s\in\Omega}\int_0^t\int_\Omega D(g(t,s,\sigma,\tau,\phi(\sigma,\tau)),g(t,s,\sigma,\tau,\psi(\sigma,\tau)))d\sigma d\tau\\
&\leq \sup_{0\leq t\leq T,s\in\Omega}\int_0^t\int_\Omega LD(\phi,\psi)d\sigma d\tau\\
&\leq TNLH_1(\phi,\psi)
\end{aligned}$$

where $TNL\in(0,1)$. So $A:\varphi\to\varphi$ is a contraction map. According to this fact φ is a complete metric space and A is a contracting self-map on φ, so it has a unique fixed point $x\in\varphi$. This fixed point is the required unique solution to (1). □

Now, we consider the linear FVFIEMT as

$$x(t,s)=\phi(t,s)+\lambda\int_0^t\int_a^b K(t,s,\tau,\sigma)x(\sigma,\tau)d\sigma d\tau, \tag{4}$$

where $x(t,s)$ is an unknown fuzzy-valued function, $\phi(t,s)$ and $K(t,s,\sigma,\tau)$ are known functions. We show that (4) has a unique fuzzy-valued solution.

Theorem 3.2. *Suppose $K(t,s,\tau,\sigma):[0,T]\times[a,b]\times[0,T]\times[a,b]\to\mathbb{R}$ and $\phi:[0,T]\times[a,b]\to E^n$ are given continuous functions. Suppose that $\lambda\in\mathbb{R}$ is such that $\frac{|\lambda|^nM^n(b-a)^n}{n!}<1$. Let M be such that $|K(t,s,\tau,\sigma)|\leq M$ for all $t,\tau\in[0,T]$, and $s,\sigma\in[a,b]$. Then (4) has a unique fuzzy-valued solution.*

Proof. Define G on $C([0,T]\times[a,b],E^n)$ by

$$Gx(t,s)=\lambda\int_0^t\int_a^b K(t,s,\sigma,\tau)x(\sigma,\tau)d\sigma d\tau+\phi(t,s).$$

It is obvious that $Gx\in C([0,T]\times[a,b],E^n)$ wherever $x\in C([0,T]\times[a,b],E^n)$. Let $x_1,x_2\in C([0,T]\times[a,b],E^n)$. Clearly, we have

$$\begin{aligned}
&D(Gx_1(t,s),Gx_2(t,s))\\
&=|\lambda|D(\int_0^t\int_a^b K(t,s,\sigma,\tau)x_1(\sigma,\tau)d\sigma d\tau,\int_0^t\int_a^b K(t,s,\sigma,\tau)x_2(\sigma,\tau)d\sigma d\tau)\\
&\leq|\lambda|\int_0^t\int_a^b D(K(t,s,\sigma,\tau)x_1(\sigma,\tau),K(t,s,\sigma,\tau)x_2(\sigma,\tau))d\sigma d\tau\\
&\leq|\lambda|M\int_0^t\int_a^b D(x_1(\sigma,\tau),x_2(\sigma,\tau))d\sigma d\tau\\
&\leq|\lambda|Mt(b-a)H_1(x_1,x_2),
\end{aligned}$$

and

$$\begin{aligned}
&D(G^2x_1(t,s),G^2x_2(t,s)) = \\
&= |\lambda| D(\int_0^t \int_a^b K(t,s,\sigma,\tau)Gx_1(\sigma,\tau)d\sigma d\tau, \int_0^t \int_a^b K(t,s,\sigma,\tau)Gx_2(\sigma,\tau)d\sigma d\tau) \\
&\leq |\lambda| \int_0^t \int_a^b D(K(t,s,\sigma,\tau)Gx_1(\sigma,\tau), K(t,s,\sigma,\tau)Gx_2(\sigma,\tau))d\sigma d\tau \\
&\leq |\lambda| M \int_0^t \int_a^b D(Gx_1(\sigma,\tau), Gx_2(\sigma,\tau))d\sigma d\tau \\
&\leq |\lambda|^2 M^2 N H_1(x_1,x_2) \int_0^t \int_a^b t d\sigma d\tau \leq \frac{|\lambda|^2 M^2 (b-a)^2 H_1(x_1,x_2)}{2!}.
\end{aligned}$$

So, by induction, it follows that

$$D(G^n x_1(t,s), G^n x_2(t,s)) \leq \frac{|\lambda|^n M^n (b-a)^n H_1(x_1,x_2)}{n!}$$

$$\Rightarrow H_1(G^n x_1, G^n x_2) \leq \frac{|\lambda|^n M^n (b-a)^n H_1(x_1,x_2)}{n!}.$$

Clearly, for any given λ, we can always choose $n \in \mathbb{N}$ large enough such that $\frac{|\lambda|^n M^n (b-a)^n}{n!} < 1$. So, G^n is a contraction mapping whenever n is sufficiently large. Applying Banach's fixed point principle, we conclude that the linear FVFIEMT (4) has a unique fuzzy-valued solution. □

4. **Numerical Examples**

In this section, we define $\|.\|$ as the following formula:

$$\|K(t,s,\tau,\sigma)\| = \sup_{t\in[0,T],\ s\in[a,b],\ \tau\in[0,T],\ \sigma\in[a,b]} |K(t,s,\tau,\sigma)|.$$

Example 4.1. Consider the FVFIEMT

$$x(t,s) = (st - st^4)k + 12 \int_0^t \int_0^1 st\sigma^2 \tau x(\tau,\sigma) d\sigma d\tau, \tag{5}$$

where $[k]^\alpha = [1+2\alpha, 4-\alpha^2]$. According to (4), we have $a = 0,\ b = 1,\ \lambda = 12$, $K(t,s,\tau,\sigma) = st\sigma^2\tau$ and $\|K(t,s,\tau,\sigma)\| \leq 1$. Clearly, the functions K and ϕ are continuous functions. So, FVFIEMT defined in (5) satisfies the assumptions of Theorem 3.2. Hence this equation has a unique solution. It is clear that the exact solution of this equation is $x(t,s) = stk$.

Example 4.2. Consider the FVFIEMT

$$x(t,s) = (s^2 + t^2 - s^2t^3 - s^2t^5)k + \int_0^t \int_0^1 8s^2 t\sigma\tau x(\tau,\sigma) d\sigma d\tau, \tag{6}$$

where $[k]^\alpha = [\alpha^2+\alpha, 4-\alpha^3-\alpha]$. It is obvious that FVFIEMT defined in (6) satisfies the assumptions of Theorem 3.2, hence this equation has a unique solution. It is clear that the exact solution of this equation is $x(t,s) = (s^2+t^2)k$.

Example 4.3.

$$x(t,s) = \log(k + kst) - 9s^2t^4k - 4s^2t^5k + \int_0^t \int_0^1 36s^2t^2\sigma\tau e^{x(\tau,\sigma)} d\sigma d\tau, \qquad (7)$$

where $[k]^\alpha = [\alpha^2 + \alpha, 4 - \alpha^3 - \alpha]$. Clearly, FVFIEMT defined in (7) satisfies the assumptions of Theorem 3.1. Hence this equation has a unique solution. It is clear that the exact solution of this equation is $x(t,s) = \log(k + kst)$.

5. **Conclusion**

In this work we considered the existence and uniqueness of the solution of FV-FIEMT defined in (1) involving fuzzy number valued mappings. To do this, we proved 2 theorems (Theorems 3.1-3.2) by using Banach's contraction principle. In these theorems, we presented sufficient conditions for the unique solution of FV-FIEMT in linear and nonlinear cases. We also showed the application of these theorems by some examples.

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R. Ezzati*, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

E-mail address: ezati@kiau.ac.ir

F. Mokhtarnejad, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

E-mail address: fa_mokhtar@yahoo.com

*Corresponding author