FUZZY LINEAR PROGRAMMING WITH GRADES OF SATISFACTION IN CONSTRAINTS

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Abstract. We present a new model and a new approach for solving fuzzy linear programming (FLP) problems with various utilities for the satisfaction of the fuzzy constraints. The model, constructed as a multi-objective linear programming problem, provides flexibility for the decision maker (DM), and allows for the assignment of distinct weights to the constraints and the objective function. The desired solution is obtained by solving a crisp problem controlled by a parameter. We establish the validity of the proposed model and study the effect of the control parameter on the solution. We also illustrate the efficiency of the model and present three algorithms for solving the FLP problem, the first of which obtains a desired solution by solving a single crisp problem. The other two algorithms, interact with the decision maker, and compute a solution which achieves a given satisfaction level. Finally, we present an illustrative example showing that the solutions obtained are often even more satisfactory than asked for.

1. Introduction

In the beginning of the 20th century, modeling complex real-word problems by precise mathematical models was a major concern in science and engineering. However, due to imprecision in data, crisp mathematical models may not be appropriate for all problems and hence the concept and techniques of probability theory were used instead. In the 1960s, the effectiveness of probability theory in modeling practical problems, especially in artificial intelligence was questioned. At almost the same time as chaos theory was being developed for handling nonlinear dynamical systems in physics and mathematics, Zadeh [26] introduced the concept of fuzzy sets. Since then, fuzzy set theory has been applied to many fields, including Management Science, Artificial Intelligence and Expert Systems, Control Theory, Mathematics and Statistics. Since Bellman and Zadeh proposed the concept of fuzzy decision making [1, 5, 12, 13, 14 16, 17], fuzzy mathematical programming has been an active area of research. Fuzzy linear programming has been classified into various categories, depending on how the imprecise parameters are modeled. In this paper, we only consider fuzzy linear programming problems with fuzzy constraints. Several methods have been proposed for solving such problems [3, 19, 20, 21, 27]. Reference [3] presents a recent review of various approaches and solution methods. However,

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these methods eventually assign a single value for the utility of satisfaction of constraints. A nonlinear model is presented in [7] to consider different utilities for the constraints by the assigning weights and models which produce different levels of constraint satisfaction as imposed by the solution, are also proposed in [6, 23]. However, the decision maker does not have any control over individual importance levels. Recently, a weighting two-phase scheme has been proposed in [11] to obtain efficient solutions to fuzzy multiple objective linear programming problems allowing arbitrary weights for the objective functions.

In this paper, we present a model in which each constraint is allowed to have a distinct level of utility (or importance), as specified by the decision maker (DM). The model is linear and has the advantage that is allows the DM to assign a weight corresponding to the level of importance for the satisfaction of each constraint and the objective function. Moreover, the solution strategy does not involve a two-phase scheme. We establish several results to demonstrate the validity of our model and the effect of its control parameter on the solution. We also perform numerical experiments to show the effectiveness of the model.

The outline of the remainder of the paper is as follows. Section 2 describes linear programming with fuzzy constraints. Section 3 presents a brief review of multiobjective linear programming (MOLP) and characterizes Pareto optimal solutions. We introduce our model and an approach to its solution in Section 4, where we also prove several analytical results establishing the validity of the model. We also give the outlines of three algorithms: Algorithm 1 produces a solution desired by the DM by solving a single crisp problem. Algorithms 2 and 3, interact with the DM, and produce various trial solutions achieving various levels of satisfaction. An illustrative example is given in Section 5. We see that the solutions obtained are often even more satisfactory than desired. Section 6 concludes the paper.

2. Problem Definition

Many practical optimization problems are characterized by the flexibility in the problem constraints. A suitable model for linear programming problem with such "soft constraints is [3, 4, 9, 19, 20, 21]:

$$\begin{aligned} Max \quad & c^T x \\ s.t. \quad & (Ax)_i \preceq b_i, \quad i = 1, \dots, m \\ & x \geq 0, \end{aligned} \tag{1}$$

where $c, x \in \Re^n, A \in \Re^{m \times n}, b \in \Re^m$ and the symbol $\preceq$ denotes that the constraint is fuzzy. It is also assumed that the decision maker is willing to accept moderate violations in the satisfaction of the constraints according to specification [3]. Suppose that the tolerances p_i for the fuzzy constraints are known. Verdegay [20], assigned a membership function to the ith fuzzy constraint as follows:

$$\mu_i(x) = \begin{cases} 1 & : \sum_j a_{ij} x_j < b_i \\ 1 - (\sum_j a_{ij} x_j - b_i)/p_i & : b_i \leq \sum_j a_{ij} x_j \leq b_i + p_i \\ 0 & : \sum_j a_{ij} x_j > b_i + p_i. \end{cases}$$

The membership functions, being monotonically nonincreasing, imply that the more the resources consumed, the less satisfied the decision maker will be. So, $\mu_i(x)$ can be interpreted as the degree of satisfaction (fulfillment) of the ith constraint at point x. The decision maker would like the constraints to be satisfied 'as much as possible' [4].

It can be shown that if the membership functions are continuous and monotonic, and trade-offs amongst the fuzzy constraints are allowed, then problem (1) is equivalent to the following crisp parametric linear programming problem [20]:

$$\begin{aligned} & Max \quad c^T x \\ & s.t. \quad (Ax)_i \leq b_i + (1-\alpha)p_i, \quad i = 1, \ldots, m \\ & x \geq 0, \quad 0 \leq \alpha \leq 1. \end{aligned} \tag{2}$$

In problem (2), α relates to the DM's level of satisfaction with resources consumed, and thus we assume that α measures the reliability of the solution.

On the other hand, Werners [21, 22] considered a fuzzy objective function for problem (1) and, similarly to [27], proposed a max-min approach for solving the problem, defining z^0 and z^1 as follows:

$$\begin{aligned} z^0 = & Max \quad c^T x \\ & s.t. \quad (Ax)_i \leq b_i, \quad i = 1, \ldots, m \\ & x \geq 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} z^1 = & Max \quad c^T x \\ & s.t. \quad (Ax)_i \leq b_i + p_i, \quad i = 1, \ldots, m \\ & x \geq 0. \end{aligned} \tag{4}$$

He then considered the following membership function μ_0 for the objective function:

$$\mu_0(x) = \begin{cases} 1 & : \ c^T x > z^1 \\ \frac{c^T x - z^0}{z^1 - z^0} & : \ z^0 \leq c^T x \leq z^1 \\ 0 & : \ c^T x < z^0. \end{cases} \tag{5}$$

We consider $\mu_0(x)$ to be the degree of satisfaction (fulfillment) of the objective function at point x. We now use the max-min operator to obtain the optimal decision. Thus, we find α by solving the following crisp problem:

$$\begin{aligned} &Max \quad \alpha \\ &s.t. \quad \mu_i(x) \geq \alpha, \quad i = 0, \ldots, m \\ &\qquad 0 \leq \alpha \leq 1 \\ &\qquad x \geq 0. \end{aligned} \tag{6}$$

The inherent flexibility in problem (1) offers a trade-off between improving the objective function and satisfying the constraints. In fact, improving the objective function generally results in satisfying the constraints at lower degrees. So far, researchers have treated this problem by setting the same value of α-level for all the constraints, and thus giving equal priority to all constraints and regarding α as the grade of satisfaction for every constraint [15]. However, there exist cases in which the DM desires to consider constraints having various levels of importance [8, 24] and appropriate models are needed to handle these situations. For instance, the different levels of importance of constraints i and j can be posed by considering different (possibly nonlinear) membership functions. However, this approach is complicated, both at the modeling phase and in the process of computing the solution. Other approaches have been proposed, having linear membership functions for fuzzy right hand sides of the constraints [6, 7, 23] and fuzzy constraint coefficients have also been considered [10, 15, 25].

In Section 4, we present a new approach to the problem, modeling the grades of the objective function ($c^T x$) and the constraint satisfactions as multiple objectives. However, we first need to describe certain aspects of multiobjective linear programming.

3. Multiobjective Linear Programming (MOLP)

Consider the multiobjective linear programming (MOLP) problem,

$$\begin{aligned} &Max\ z_1(x) = c_1^T x \\ &Max\ z_2(x) = c_2^T x \\ &\quad \cdots \\ &Max\ z_k(x) = c_k^T x \\ &s.t. \\ &\quad Ax \leq b \\ &\quad x \geq 0, \end{aligned} \tag{7}$$

where,

$$c_i = (c_{i1}, \ldots, c_{in})^T, \quad i = 1, \ldots, k, \tag{8}$$

$$x = (x_1, \dots, x_n)^T, \tag{9}$$

$$A = \begin{bmatrix} a_{11}, \dots, a_{1n} \\ \cdots \\ a_{m1}, \dots, a_{mn} \end{bmatrix}, \tag{10}$$

$$b = (b_1, \dots, b_m)^T. \tag{11}$$

Let $X = \{x \in \Re^n | Ax \leq b, x \geq 0\}$. A definition for a complete solution for this problem is as follows [18] :

Definition 3.1. (Complete Optimal Solution)
The point $x^* \in X$ is said to be a complete optimal solution of the MOLP problem (7) if $z_i(x^*) \geq z_i(x), i = 1, \dots, k$ for all $x \in X$.

However, in general, when the objective functions conflict with one another, a complete optimal solution may not exist and hence, a new concept of optimality, called Pareto optimality, is considered.

Definition 3.2. (Pareto Optimal Solution) The point $x^* \in X$ is said to be a Pareto optimal solution if there does not exist $x \in X$ such that $z_i(x) \geq z_i(x^*)$ for all i and $z_j(x) > z_j(x^*)$ for at least one j.

Now we use the weighted max-min method to solve the MOLP problem. The weighted max-min approach seeks Pareto optimal solutions by first solving the following weighted max-min problem [2]:

$$\begin{aligned} &Max \quad Min_{i=1,\dots,k} \ w_i z_i(x) \\ &s.t. \quad x \in X, \end{aligned} \tag{12}$$

or equivalently,

$$\begin{aligned} &Max \quad \nu \\ &s.t. \quad w_i z_i(x) \geq \nu, \quad i = 1, \dots, k \\ &\qquad\qquad x \in X, \end{aligned} \tag{13}$$

where $w = (w_1, \dots, w_k)^T \geq 0$ is a given vector of nonnegative weight coefficients assigned to the objective functions (usually, $w_i > 0$).

Remark 3.3. Without loss of generality, we assume that $z_i(x) \geq 0, \ i = 1, \dots, k$, for all $x \in X$. If for an objective function we have $z_i(x) < 0$ for some $x \in X$ then by computing,

$$z_i^{min} = Min_{x \in X} z_i(x), \tag{14}$$

and replacing z_i by $\hat{z}_i$ as,

$$\hat{z}_i = z_i(x) - z_i^{min}, \tag{15}$$

we have $\hat{z}_i \geq 0$, for all $x \in X$.

If the optimal solution x^* for the weighted max-min problem ((12) or (13)) is not unique, then it is necessary to perform the Pareto optimality test. i.e. we must solve the following linear programming problem:

$$\begin{aligned} Max \quad & \sum_{i=1}^{k} \epsilon_i \\ s.t. \quad & z_i(x) - \epsilon_i = z_i(x^*), \quad i = 1, \ldots, k \\ & x \in X, \ \ \epsilon \geq 0. \end{aligned} \tag{16}$$

For the optimal solution $(\bar{x}, \bar{\epsilon})$ of the linear programming problem (16), the following theorem holds [16].

Theorem 3.4. *Let x^* and $(\bar{x}, \bar{\epsilon})$ be optimal solutions of problems (13) and (16), respectivly. We have: (1) If $\bar{\epsilon_i} = 0, i = 1, \ldots, k$, then both $\bar{x}$ and x^* are Pareto optimal solutions. (2)If at least one i exists such that $\bar{\epsilon_i} > 0$, then $\bar{x}$ is (but x^* is not) a Pareto optimal solution.*

4. **A New Approach**

LP problems with fuzzy constraints have been considered by many researchers [9, p. 79] and, in particular, several approaches have been proposed to solve problem (1). In the two-phase approach of [6], first the crisp linear programming problem,

$$\begin{aligned} Max \quad & \nu \\ s.t. \quad & 1 \geq \mu_i(x) \geq \nu \geq 0, \ i = 0, \ldots, m \\ & x \geq 0, \end{aligned} \tag{17}$$

is solved in the first phase. If x' is an optimal solution of (17), the phase II problem is as follows:

$$\begin{aligned} Max \quad & \sum_{i=0}^{m} \alpha_i \\ s.t. \quad & 1 \geq \mu_i(x) \geq \alpha_i \geq \mu_i(x'), \ i = 0, \ldots, m \\ & x \geq 0. \end{aligned} \tag{18}$$

Now we denote the optimal solution of the two-phase problem by (ν^*, α^*, x^*), where (ν^*, x') is an optimal solution of (17) and (α^*, x^*) is an optimal solution of (18). The advantage of the model (18) in the two-phase approach over the traditional max-min operator (17) is in its capability to improve upon the solution having higher membership values, if possible (in fact, [6] shows an example illustrating the improvement). We note that at the optimal solution of (18) we will have $\mu_i(x^*) = \alpha_i^*, \ i = 0, \ldots m$. Thus, problem (18) may equivalently be posed as:

$$
\begin{aligned}
Max \quad & \sum_{i=0}^{m} \mu_i(x) \\
s.t. \quad & 1 \geq \mu_i(x) \geq \mu_i(x'),\ i = 0, \dots, m \\
& x \geq 0.
\end{aligned}
\tag{18'}
$$

The two-phase approach of [6] for problem (1) considers the DM's levels of satisfaction for the objective function and the constraints as various objectives having equal weights in an MOLP model. In phase I, using the max-min principle, model (17) is solved for an optimal solution. Phase II considers the model (18) to improve upon the solution of (17), if possible.

Here, we present a new approach that considers both optimizing the objective function as well as satisfying the constraints at higher achievement levels, if possible. We propose the multi-objective model below:

$$
\begin{aligned}
Max \quad & (w_0\alpha_0, w_1\alpha_1, \dots, w_m\alpha_m)^T \\
s.t. \quad & (Ax)_i \leq b_i + (1 - \alpha_i)p_i,\ i = 1, \dots, m \\
& \alpha_0 = \frac{c^T x - z^0}{z^1 - z^0} \\
& 0 \leq \alpha_i \leq 1, i = 1, \dots, m \\
& x \geq 0,
\end{aligned}
\tag{19}
$$

where each $w_i > 0$ is a given weight associated with α_i .

Using the max-min approach for (19), we solve the following problem:

$$
\begin{aligned}
Max \quad & \nu \\
s.t. \quad & w_0\alpha_0 \geq \nu \\
& w_1\alpha_1 \geq \nu \\
& \vdots \\
& w_m\alpha_m \geq \nu \\
& \alpha_0 = \frac{c^T x - z^0}{z^1 - z^0} \\
& (Ax)_i \leq b_i + (1 - \alpha_i)p_i,\ i = 1, \dots, m \\
& 0 \leq \alpha_i \leq 1,\ \ i = 1, \dots, m \\
& x \geq 0,
\end{aligned}
\tag{20}
$$

where z^0 and z^1 are optimal objective values corresponding to problems (3) and (4), respectively. Let $X = \{(\nu, \alpha, x) \mid (\nu, \alpha, x) \text{ is feasible solution of (20)}\}$.

Remark 4.1. (1) If (20) does not have a feasible solution then, for at least one i, there does not exist x such that $(Ax)_i \leq b_i + p_i$.
(2) The feasible space of (18) is a subspace of the feasible space of (20) with $w_i = 1,\ i = 0, \dots, m$.
(3) Having z^0 and z^1 defined by (3) and (4), it is obvious that for all feasible x we will have $0 \leq \alpha_0 = \frac{c^T x - z^0}{z^1 - z^0} \leq 1$.

Next, we propose a new model and an approach for solving problem (19). The proposed model has the advantage of producing a DM's desired solution by solving a single crisp problem using an appropriate parameter, set initially to enforce a desired solution. Analytical results establish the validity of the model. We propose solving the following problem:

$$\begin{aligned} Max \quad & \nu + \frac{1}{M}\sum_{i=0}^{m} \alpha_i \\ s.t. \quad & w_0\alpha_0 \geq \nu \\ & w_1\alpha_1 \geq \nu \\ & \vdots \\ & w_m\alpha_m \geq \nu \\ & \alpha_0 = \frac{c^T x - z^0}{z^1 - z^0} \\ & (Ax)_i \leq b_i + (1-\alpha_i)p_i, \ i = 1, \ldots, m \\ & 0 \leq \alpha_i \leq 1, \ \ i = 1, \ldots, m \\ & x \geq 0, \end{aligned} \tag{21}$$

where M is a (big enough) positive number set to achieve a DM's desired solution. Let $\alpha = (\alpha_0, \ldots, \alpha_m)^T$. The following results provide some characterizations of the solution of (21).

Theorem 4.2. *If $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is an optimal solution for problem (21), then $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is a Pareto optimal solution for problem (19).*

Proof. Suppose, on the contrary, that $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is not a Pareto optimal solution for (19). Then, by Definition 3.2, there exists $(\nu^{**}, \alpha^{**}, x^{**}) \in X$ (the solution space of problem (19)) such that:

$$\begin{aligned} & \alpha_i^{**} \geq \bar{\alpha}_i, \ i = 0, \ldots, m \\ & \exists j: \ \alpha_j^{**} > \bar{\alpha}_j \ . \end{aligned} \tag{22}$$

Note that $\alpha_0^{**} \geq \bar{\alpha}_0$ if and only if $c^T x^{**} \geq c^T \bar{x}$. From (22), we have:

$$w_i\alpha_i^{**} \geq w_i\bar{\alpha}_i, \ i = 0, \ldots, m.$$

Since $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is a feasible solution for problem (21), hence $w_i\bar{\alpha}_i \geq \bar{\nu}$, and

$$w_i\alpha_i^{**} \geq \bar{\nu}. \tag{23}$$

Since $(\nu^{**}, \alpha^{**}, x^{**}) \in X$, hence:

$$(Ax^{**})_i \leq b_i + (1 - \alpha_i^{**})\, p_i \ . \tag{24}$$

From (23) and (24), we see that $(\bar{\nu}, \alpha^{**}, x^{**})$ is a feasible solution for problem (21), and from (22) we have:

$$\sum_{i=0}^{m} \alpha_i^{**} > \sum_{i=0}^{m} \bar{\alpha}_i \Rightarrow \bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \alpha_i^{**} > \bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i \ . \tag{25}$$

This implies that $(\bar{\nu}, \alpha^{**}, x^{**})$ is a better solution than the optimal solution $(\bar{\nu}, \bar{\alpha}, \bar{x})$ for problem (21), which is a contradiction. □

Lemma 4.3. *If (ν^*, α^*, x^*) is an optimal solution for the two-phase problems (17) and (18) and $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is an optimal solution for problem (21) then we have:*
(i) $v^* = min_{i=0,\ldots,m}\ \{\alpha_i^*\}$.
(ii) $\bar{v} = min_{i=0,\ldots,m}\ \{w_i\bar{\alpha}_i\}$.

Proof. (i) From the constraints in (17) and (18) respectively, we have $\mu_i(x') \geq \nu^*$ and $\alpha_i^* \geq \mu_i(x')$, where x' is the optimal solution of (17). Thus, $\alpha_i^* \geq \nu^*$ and hence,

$$\nu^* \leq Min_{i=0,\ldots,m}\{\alpha_i^*\}. \tag{a}$$

By (18), $\mu_i(x^*) \geq Min\ _{i=0,\ldots,m}\{\alpha_i^*\}$ and hence it is easily seen that the point $(Min\ _{i=0,\ldots,m}\{\alpha_i^*\}, x^*)$ is a feasible solution for problem (17). Since ν^* is an optimal value for problem (17) hence

$$\nu^* \geq Min_{i=0,\ldots,m}\{\alpha_i^*\}. \tag{b}$$

We conclude that

$$\nu^* = Min_{i=0,\ldots,m}\{\alpha_i^*\}.$$

(ii) From the constraints in (21) we have $w_i\bar{\alpha}_i \geq \bar{\nu}$ and hence,

$$\bar{\nu} \leq Min_{i=0,\ldots,m}\{w_i\bar{\alpha}_i\}. \tag{c}$$

It is easily seen that the point $(Min\ _{i=0,\ldots,m}\{w_i\bar{\alpha}_i\}, \bar{\alpha}, \bar{x})$ is a feasible solution for problem (21). Since $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is optimal for problem (21), we must have

$$\bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i \geq Min\{w_i\bar{\alpha}_i\} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i \ ,$$

which implies:

$$\bar{\nu} \geq Min_{i=0,\ldots,m}\{w_i\bar{\alpha}_i\}. \tag{d}$$

Hence:

$$\bar{\nu} = Min_{i=0,\ldots,m}\{w_i\bar{\alpha}_i\}.$$

□

Lemma 4.4. *If (ν^*, x^*) is an optimal solution for (17) and $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is an optimal solution for (21) with $w_i = 1,\ i = 0, \ldots, m$, then $\nu^* \geq \bar{\nu}$.*

Proof. The constraints of (21) imply:

$$\bar{\alpha}_i \geq \bar{\nu},\ i = 0, \ldots, m, \tag{26}$$

and

$$(A\bar{x})_i \leq b_i + (1 - \bar{\alpha}_i)p_i,\ i = 1, \ldots, m. \tag{27}$$

From (27) we deduce that,

$$(1 - \bar{\alpha}_i)p_i \geq (A\bar{x})_i - b_i \Rightarrow \bar{\alpha}_i \leq 1 - \frac{(A\bar{x})_i - b_i}{p_i}. \tag{28}$$

Now, by (28) and the definition of $\mu_i(x)$, we have

$$\bar{\alpha}_i \leq \mu_i(\bar{x}),\ i = 0, \ldots, m. \tag{29}$$

By Lemma 4.3, $\bar{\nu} = Min_{i=0,\ldots,m}\{\bar{\alpha}_i\}$, and thus $\bar{v} \geq 0$. Now, by (26) and (29) and the fact that $\mu_i(x) \leq 1$, we can write,

$$0 \leq \bar{\nu} \leq \mu_i(\bar{x}) \leq 1,\ i = 0, \ldots, m. \tag{30}$$

Hence $(\bar{\nu}, \bar{x})$ is a feasible point for (17). Since (ν^*, x^*) is an optimal solution for (17), then we must have $v^* \geq \bar{\nu}$. □

Lemma 4.5. *If (ν^*, α^*, x^*) is an optimal solution for the two-phase problems (17) and (18) then (ν^*, α^*, x^*) is feasible for (21) with $w_i = 1,\ i = 0, \ldots, m$.*

Proof. By the constraints of (18) we have,

$$\alpha_i^* \geq \mu_i(x')\ ,\ i = 0, \ldots, m,$$

where x' is optimal for (17). By the constraints of (17) we have,

$$\mu_i(x') \geq \nu^*\ .$$

Hence,

$$\alpha_i^* \geq \nu^*,\ i = 0, \ldots, m. \tag{31}$$

Also, it follows from $\alpha_i^* \leq \mu_i(x^*),\ i = 0, \ldots, m$, in (18) that

$$\begin{aligned} \alpha_i^* \leq \mu_i(x^*) \Rightarrow \alpha_i^* \leq 1 - \frac{(Ax^*)_i - b_i}{p_i},\ 0 \leq \alpha_i^* \leq 1,\ i = 1, \ldots, m \\ \Rightarrow (Ax^*)_i \leq b_i + (1 - \alpha_i^*)p_i. \end{aligned} \tag{32}$$

From (31) and (32) we conclude that (ν^*, α^*, x^*) is feasible for (21) with $w_i = 1,\ i = 0, \ldots, m$. □

Theorem 4.6. *If (ν^*, α^*, x^*) is an optimal solution for the two-phase problems and $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is an optimal solution for (21) with $w_i = 1,\ i = 0, \ldots, m$, then (i) $\sum_{i=0}^m \bar{\alpha}_i \geq \sum_{i=0}^m \alpha_i^*$. (ii) If M is large enough then $\nu^* = \bar{\nu}$.*

Proof. (i) By Lemma 4.5, (ν^*, α^*, x^*) is feasible for (21) and hence

$$\alpha_i^* \geq \nu^*,\ i = 0, \ldots, m, \tag{33}$$

and

$$(Ax^*)_i \leq b_i + (1 - \alpha_i^*)p_i,\ i = 1, \ldots, m. \tag{34}$$

By Lemma 4.4, $\nu^* \geq \bar{\nu}$. Using this fact and (33), we can write,

$$\alpha_i^* \geq \bar{\nu},\ i = 0, \ldots, m. \tag{35}$$

From (34) and (35) we conclude that $(\bar{\nu}, \alpha^*, x^*)$ is feasible for (21). But, since $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is optimal, hence,

$$\bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i \geq \bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \alpha_i^* \Rightarrow \sum_{i=0}^{m} \bar{\alpha}_i \geq \sum_{i=0}^{m} \alpha_i^*. \tag{36}$$

(ii) Assume, on the contrary, that $\nu^* \neq \bar{\nu}$. By Lemma 4.4, we have $\nu^* \geq \bar{\nu}$. Let

$$\nu^* - \bar{\nu} = p. \tag{37}$$

Now, for large enough M , we have,

$$\frac{\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^*}{M} < p = \nu^* - \bar{\nu}, \tag{38}$$

implying that,

$$\nu^* + \frac{1}{M}\sum_{i=0}^{m} \alpha_i^* > \bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i\ . \tag{39}$$

By Lemma 4.5, (ν^*, α^*, x^*) is feasible for (21). Thus, (ν^*, α^*, x^*) is a better solution for (21) than $(\bar{\nu}, \bar{\alpha}, \bar{x})$, which is a contradiction. □

Remark 4.7. Corresponding results can also be stated for the optimal solutions $(\nu^{**}, \alpha^{**}, x^{**})$ and $(\bar{\nu}, \bar{\alpha}, \bar{x})$ of (20) and (21), respectively, with a common set of weights.

(i) $\sum_{i=0}^{m} \bar{\alpha}_i \geq \sum_{i=0}^{m} \alpha_i^{**}$.

(ii) If M is large enough then $\nu^{**} = \bar{\nu}$.

We have seen in Theorem 4.2 that for any value of M, an optimal solution for (21) is a Pareto optimal solution for (19). Also, by Theorem 4.6, we know that if $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is optimal for (21) with $w_i = 1,\ i = 0, \ldots, m$, and (ν^*, α^*, x^*) is optimal for the two-phase problem, then $\sum_{i=0}^{m} \bar{\alpha}_i \geq \sum_{i=0}^{m} \alpha_i^*$. Moreover, for large enough M, the difference between ν^* and $\bar{\nu}$ goes to zero. We now establish a result relating the value of M to the gap between ν^* and $\bar{\nu}$.

Theorem 4.8. *Suppose $(\bar{\nu}, \bar{\alpha}, \bar{x})$ is an optimal solution for problem (21) with $w_i = 1,\ i = 0, \ldots, m$, and (ν^*, α^*, x^*) is an optimal solution obtained from the two-phase method. If $M \geq \frac{m-\epsilon}{\epsilon}$, then $\nu^* - \bar{\nu} < \epsilon$.*

Proof. Suppose, on the contrary, that $\nu^* - \bar{\nu} \geq \epsilon$. By Lemma 4.4, $\nu^* \geq \bar{\nu}$). The difference of the objective function of (21) for $(\bar{\nu}, \bar{\alpha})$ and (ν^*, α^*) is:

$$D = [\bar{\nu} + \frac{1}{M}\sum_{i=0}^{m} \bar{\alpha}_i] - [\nu^* + \frac{1}{M}\sum_{i=0}^{m} \alpha_i^*] = \bar{\nu} - \nu^* + \frac{1}{M}[\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^*]. \quad (40)$$

Letting $\alpha_j^* = Min_{i=0,\ldots,m}\{\alpha_i^*\}$ and $\bar{\alpha}_k = Min_{i=0,\ldots,m}\{\bar{\alpha}_i\}$, we have

$$D = \bar{\nu} - \nu^* + \frac{1}{M}[(\bar{\alpha}_k - \alpha_j^*) + \left(\sum_{i=0, i\neq k}^{m} \bar{\alpha}_i - \sum_{i=0, i\neq j}^{m} \alpha_i^*\right)] \quad (41)$$

Hence by Lemma 4.3,

$$D = \bar{\nu} - \nu^* + \frac{1}{M}[(\bar{\nu} - \nu^*) + \left(\sum_{i=0, i\neq k}^{m} \bar{\alpha}_i - \sum_{i=0, i\neq j}^{m} \alpha_i^*\right)] \quad (42)$$

Now, $0 \leq \bar{\alpha}_i \leq 1,\ 0 \leq \alpha_i^*,\ i = 0, 1, \ldots, m$, and $\nu^* - \bar{\nu} \geq \epsilon$. Therefore,

$$D \leq -\epsilon + \frac{1}{M}[-\epsilon + m]. \quad (43)$$

Since, according to the assumption, $M \geq \frac{m-\epsilon}{\epsilon}$, hence

$$D \leq -\epsilon + \frac{1}{M}[-\epsilon + m] < -\epsilon + \frac{1}{\frac{m-\epsilon}{\epsilon}}[-\epsilon + m] = 0.$$

By Lemma 4.5, (ν^*, α^*, x^*) is feasible for (21) and hence, from $D < 0$, we conclude that (ν^*, α^*, x^*) is a better solution for (21) than the optimal solution $(\bar{\nu}, \bar{\alpha}, \bar{x})$, which is a contradiction. □

Remark 4.9. To obtain (43), we used $-\epsilon = (\bar{\nu} - \nu^*)$ and considered the maximal value of $\sum_{i=0,i\neq k}^{m} \bar{\alpha}_i - \sum_{i=0,i\neq j}^{m} \alpha_i^*$ as m. Since $Min\ \alpha_i^* = \nu^*,\ i = 0, \ldots, m$, hence we can use $m(1 - \nu^*)$ as the maximal value of $\sum_{i=0,i\neq k}^{m} \bar{\alpha}_i - \sum_{i=0,i\neq j}^{m} \alpha_i^*$ and the condition for M in Theorem 4.8 can also be stated more tightly as:

$$M > \frac{m - \epsilon - m\nu^*}{\epsilon}. \quad (44)$$

Theorem 4.10. *If $(\nu^{**}, \alpha^{**}, x^{**})$ and $(\bar{\nu}, \bar{\alpha}, \bar{x})$ are respectively optimal solutions of (20) and (21) with a common set of weights and $M > \frac{m+1}{\epsilon}$, then $\nu^{**} - \bar{\nu} < \epsilon$.*

Proof. The two optimal solutions are feasible for both the problems (20) and (21), since they have the same feasible space. Thus, $\nu^{**} \geq \bar{\nu}$. If, contrary to the statement of the theorem, we have $\nu^{**} - \bar{\nu} \geq \epsilon$, then,

$$D = \{ \left[\bar{\nu} + \frac{1}{M} \sum_{i=0}^{m} \bar{\alpha}_i \right] - \left[\nu^{**} + \frac{1}{M} \sum_{i=0}^{m} \alpha_i^{**} \right] \} \leq -\epsilon + \frac{1}{M} Max[\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^{**}]. \tag{45}$$

Since $\bar{\alpha}_i \leq 1,\ 0 \leq \alpha_i^{**},\ i = 0, \ldots, m$, hence we can use $m+1$ as the maximal value of $\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^{**}$. Thus, we have,

$$D \leq -\epsilon + \frac{1}{M}[m+1]. \tag{46}$$

Since, according to the assumption, $M > \frac{m+1}{\epsilon}$, hence we have $D < 0$ and this contradicts the optimality of $(\bar{\nu}, \bar{\alpha}, \bar{x})$ for problem (21). □

Remark 4.11. To obtain (46), we considered the maximal value of $\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^{**}$ as $m+1$. Since $\alpha_i^{**} \geq \frac{\nu^{**}}{w_i}, i = 0, \ldots, m$, hence we can use $m + 1 - \nu^{**} \sum_{i=0}^{m} \frac{1}{w_i}$ as the maximal value of $\sum_{i=0}^{m} \bar{\alpha}_i - \sum_{i=0}^{m} \alpha_i^{**}$ and the condition for M in Theorem 4.10 can also be stated more tightly as:

$$M > \frac{m + 1 - \nu^{**} \sum_{i=0}^{m} \frac{1}{w_i}}{\epsilon}. \tag{47}$$

Theorem 4.10 assures the achievement of a near optimal solution $(\bar{\nu}, \bar{\alpha}, \bar{x})$ for (20) instead of the exact optimal solution $(\nu^{**}, \alpha^{**}, x^{**})$, if we solve problem (21) using $M > \frac{m+1}{\epsilon}$ or $M > \frac{m+1-\nu^{**} \sum_{i=0}^{m} \frac{1}{w_i}}{\epsilon}$ according to the difference ϵ , between ν^{**} and $\bar{\nu}$ acceptable to the DM.

Thus, the following simple algorithm is proposed.

Algorithm 1: Solution of problem (21) using an initial tolerance.
Step 0: Choose a value for ϵ.
Step 1: Let

$$M = \frac{m+1}{\epsilon} + 1.$$

Step 2: Solve problem (21).

Algorithm 1 can be generalized to interact with the DM and obtain various approximate solutions of (1) with various values of ϵ .

Algorithm 2: Interactive algorithm for solution of (21).
Step 0: Get an initial value ϵ from the DM.
Step 1: Solve problem (20) and obtain ν^{**}.
Step 2: Solve problem (21) and obtain $\bar{\nu}$.

Step 3: If ($\bar{\nu} = \nu^{**}$ or DM accepts $\bar{\nu}$) then stop.
Step 4: Let

$$\epsilon = \nu^{**} - \bar{\nu}.$$

Step 5: If DM does not accept ϵ then get a new ϵ .
Step 6: Let

$$M = \frac{m+1}{\epsilon} + 1 \quad (\text{or } M = \frac{m + 1 - \nu^{**} \sum_{i=0}^{m} \frac{1}{w_i}}{\epsilon} + 1).$$

Go to Step 2.

By Remark 4.9, we can modify Algorithm 2 to construct an interactive algorithm for solving problem (21) with weights equal to 1 using an M value specified by (44). The corresponding algorithm is as follows.

Algorithm 3: Interactive algorithm for solution of (21) with weights equal to 1.
Step 0: Get an initial value ϵ from the DM.
Step 1: Solve problem (17) and obtain ν^*.
Step 2: Solve problem (21) and obtain $\bar{\nu}$.
Step 3: If ($\bar{\nu} = \nu^*$ or DM accepts $\bar{\nu}$) then stop.
Step 4: Let

$$\epsilon = \nu^* - \bar{\nu}.$$

Step 5: If DM does not accept ϵ then get a new ϵ .
Step 6: Let

$$M = \frac{m - \epsilon}{\epsilon} + 1 \quad (\text{or } M = \frac{m - \epsilon - m\nu^*}{\epsilon} + 1).$$

Go to Step 2.

5. A Numerical Example

Consider the following fuzzy linear programming problem with imprecise resources as given in [6]:

$$\begin{aligned} Max \quad & z(x) = 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ s.t. \quad & g_1(x) = x_1 + x_2 + x_3 + x_4 \preceq 15 \\ & g_2(x) = 7x_1 + 5x_2 + 3x_3 + 2x_4 \preceq 80 \\ & g_3(x) = 3x_1 + 4.4x_2 + 10x_3 + 15x_4 \preceq 100 \\ & x_1, x_2, x_3.x_4 \geq 0. \end{aligned} \tag{48}$$

As in [6], assume that tolerances are $p_1 = 5,\ p_2 = 40,\ p_3 = 30$.

If we set the corresponding problems (3) and (4) for this problem, we obtain $z^0 = 99.28571$ and $z^1 = 130$. The results obtained for this problem with the two-phase approach are: $x^* = (4.048,\ 5.65,\ 7.79,\ 0)^T$,$\alpha_0^* = \alpha_1^* = \alpha_3^* = 0.5$, $\alpha_2^* = 1$ and so we have $\nu^* = 0.5$ as reported in [6].

Our model is:

$$\begin{aligned} Max \quad & \nu + \frac{1}{M}\sum_{i=0}^{3}\alpha_i \\ s.t. \quad & w_0\alpha_0 \geq \nu \\ & w_1\alpha_1 \geq \nu \\ & w_2\alpha_2 \geq \nu \\ & w_3\alpha_3 \geq \nu \\ & \alpha_0 = \frac{4x_1 + 5x_2 + 9x_3 + 11x_4 - 99.28571}{30.7143} \\ & x_1 + x_2 + x_3 + x_4 + 5\alpha_1 \leq 20 \\ & 7x_1 + 5x_2 + 3x_3 + 2x_4 + 40\alpha_2 \leq 120 \\ & 3x_1 + 4.4x_2 + 10x_3 + 15x_4 + 30\alpha_3 \leq 130 \\ & x_1, x_2, x_3.x_4, \nu \geq 0 \\ & 0 \leq \alpha_i \leq 1,\ i = 0, 1, 2, 3. \end{aligned}$$

Set $\epsilon = 0.1$. Then the value of M should be bigger than $\frac{3-0.1}{0.1} = 29$. Using weights equal to 1 and $M = 30$ in Algorithm 1, the optimal solution, obtained by use of Matlab v 7.2.0.232 on a Pentium II PC with a 512 mb RAM, is: $\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_3 = 0.5$, $\bar{\alpha}ha_2 = 1$, $\bar{\nu} = 0.5$, $\bar{x} = (4.1,\ 5.58,\ 7.8,\ 0)^T$ and $z(\bar{x}) = 114.5$ We realize that the satisfaction levels (also optimal solution of (48)) obtained by our method are the same as the ones obtained by the two-phase method reported in [6]. We tried other weights and obtained the following solutions:

$$w = (\frac{1}{5}, \frac{1}{2}, \frac{1}{2}, 1),\ \bar{x} = (2.55,\ 7.3,\ 8.53,\ 0)^T$$
$$\bar{\alpha}_0 = 0.8,\ \bar{\alpha}_1 = 0.32,\ \bar{\alpha}_2 = 1,\ \bar{\alpha}_3 = 0.16,\ z(\bar{x}) = 123.5$$

$$w = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}),\ \bar{x} = (0,\ 11.98,\ 6.69,\ 0)^T$$
$$\bar{\alpha}_0 = 0.69,\ \bar{\alpha}_1 = 0.26,\ \bar{\alpha}_2 = 1,\ \bar{\alpha}_3 = 0.34,\ z(\bar{x}) = 120.1$$

$$w = (\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}),\ \bar{x} = (0,\ 12.17,\ 6.39,\ 0)^T$$
$$\bar{\alpha}_0 = 0.63,\ \bar{\alpha}_1 = 0.29,\ \bar{\alpha}_2 = 1,\ \bar{\alpha}_3 = 0.42,\ z(\bar{x}) = 118.36.$$

The above results show that higher/lower objective values can be achieved by admitting lower/higher levels of preference for the constraints. This feature can be used effectively to the DM's satisfaction.

We also tried other values of ϵ and weights for problem (48). The set of weights considered are listed in Table 1. Table 2 shows the M values using the formula $M = \frac{m+1}{\epsilon} + 1$, for various values of ϵ , and the M values using the formula

$M = \frac{m+1-\nu^{**}\sum_{i=0}^{m}\frac{1}{w_i}}{\epsilon} + 1$ for each ϵ , are given in Table 3. In Table 3, the row number i corresponds to the weight, as listed in Table 1, and the value on top of each column corresponds to the value of ϵ .

	$(w_0,\ w_1,\ w_2,\ w_3)$
1	(1, 1, 1, 1)
2	(1, 0.5, 0.5, 0.5)
3	(1, 0.25, 0.5, 0.25)
4	(0.5, 0.25, 0.25, 1)
5	(0.5, 0.25, 0.25, 0.5)
6	(0.75, 0.75, 0.25, 0.5)
7	(0.75, 0.5, 0.25, 0.5)
8	(0.25, 0.25, 0.25, 0.5)
9	(0.75, 0.5, 0.75, 0.5)
10	(0.5, 0.75, 0.75, 0.75)

Table 1: Weights for problem (48).

ϵ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
M	5	5.44	6	6.71	7.66	9	11	14.33	21	41

Table 2: The M values using the formula $M = \frac{m+1}{\epsilon} + 1$ corresponding to various ϵ.

$i\backslash\epsilon$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1	3	3.22	3.5	3.86	4.33	5	6	7.67	11	21
2	2.67	2.85	3.08	3.38	3.77	4.33	5.17	6.55	9.33	17.67
3	2.8	3	3.25	3.57	4	4.6	5.5	7	10	19
4	2.25	2.39	2.56	2.78	3.08	3.5	4.12	5.17	7.25	13.5
5	2.39	2.55	2.74	2.99	3.32	3.79	4.48	5.65	7.97	14.94
6	2.83	3.04	3.29	3.62	4.05	4.67	5.58	7.11	10.17	19.33
7	2.67	2.85	3.08	3.38	3.77	4.33	5.17	6.55	9.33	17.67
8	2.88	3.09	3.35	3.69	4.13	4.76	5.7	7.27	10.4	19.8
9	3	3.22	3.5	3.86	4.33	5	6	7.67	11	21
10	3.2	3.44	3.75	4.14	4.67	5.4	6.5	8.33	12	23

Table 3: The M values using the formula $M = \frac{m+1-\nu^{**}\sum_{i=0}^{m}\frac{1}{w_i}}{\epsilon} + 1$ corresponding to various ϵ and weights.

Table 4 shows the difference $\nu^{**} - \bar{\nu}$ for various values of ϵ . Each row i corresponds to the weight given in the ith row of Table 1. The first line of the ith row corresponds to M values as given in Table 2, and the second row corresponds to the M values given in Table 3. The results in Table 4 show that, regardless of how we choose M, the computed solution is always better than desired, that is, the computed value of $\nu^{**} - \bar{\nu}$ is much less than ϵ , the specified desired value by the DM. Moreover, when ϵ is taken small enough, then $\nu^{**} - \bar{\nu} = 0$, i.e. we obtain the exact solution. We realize that in this case, the solution obtained $(\bar{x}, \bar{\alpha})$ is a Pareto optimal solution (according to Theorem 4.2), while if model (20) corresponding to problem (48) were to be solved, there would be no guarantee for (x^{**}, α^{**}) to be Pareto optimal.

$i\backslash\epsilon$	1	.9	.8	.7	.6	.5	.4	.3	.2	.1
1	0	0	0	0	0	0	0	0	0	0
	0.089	0.089	0.089	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
	0.042	0.042	0.042	0.042	0.042	0.042	0	0	0	0
3	0.016	0.016	0.016	0.016	0	0	0	0	0	0
	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0	0
4	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
5	0.012	0.012	0.012	0.012	0	0	0	0	0	0
	0.012	0.012	0.012	0.012	0.012	0.012	0.012	0.012	0	0
6	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0
8	0.022	0.022	0.022	0.022	0.022	0.022	0.022	0	0	0
	0.022	0.022	0.022	0.022	0.022	0.022	0.022	0.022	0.022	0
9	0.044	0.044	0	0	0	0	0	0	0	0
	0.044	0.044	0.044	0.044	0.044	0.044	0	0	0	0
10	0.062	0.062	0.062	0	0	0	0	0	0	0
	0.062	0.062	0.062	0.062	0.062	0.062	0	0	0	0

Table 4: The value obtained for $\nu^{**} - \bar{\nu}$ corresponding to various ϵ and weights.

6. **Conclusions**

We have presented a new model and a new approach for solving a class of linear programming problems having fuzzy constraints. The new model, constructed as a multi-objective linear programming problem, provides flexibility for the decision maker (DM) by allowing desired distinct weight assignments to the constraints and the objective function. We presented a crisp problem, controlled by a parameter, to achieve a DM's desired solution. We proved several results to establish the validity of the proposed model. Computational results demonstrated that the computed solutions were often more satisfactory than desired by the DM.

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