

## SOME TYPES OF $(\in, \in \vee q)$ -INTERVAL-VALUED FUZZY IDEALS OF BCI ALGEBRAS

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ABSTRACT. In this paper, we introduce the notions of interval-valued and  $(\in, \in \vee q)$ -interval-valued fuzzy  $(p, q$ - and  $a$ -) ideals of BCI algebras and investigate some of their properties. We then derive characterization theorems for these generalized interval-valued fuzzy ideals and discuss their relationship.

### 1. Introduction

In recent years, because of its applicability, the study of  $t$ -norm-based logical systems has become an increasingly important topic in the field of logic. As it is well known, BCK and BCI algebras, introduced by Imai and Iséki [12,15], are two classes of algebras of logic which have been extensively investigated by many researchers [7,8,15-33,35]. Iorgulescu [13,14] showed that pocrimis and BCK algebras with condition (S) are categorically isomorphic. Hence, most of the algebras related to an  $t$ -norm based logic, for example, MTL algebras [10], BL algebras [11], hoop, MV algebras (i.e., lattice implication algebras) and Boolean algebras, are subclasses of BCK algebras. This shows that BCK/BCI algebras are very general structures.

After the introduction of fuzzy sets by Zadeh [34], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroups, called  $(\in, \in \vee q)$ -fuzzy subgroups, were introduced in a paper of Bhakat and Das [2] using the combined notions of “belongingness” and “quasicoincidence” of fuzzy points and fuzzy sets, introduced by Pu and Liu [31]. In fact, the  $(\in, \in \vee q)$ -fuzzy subgroup is an important generalization of Rosenfeld’s fuzzy subgroup. It is now natural to investigate similar type of generalizations of existing fuzzy subsystems with other algebraic structures. Jun [21,22] introduced the concept of  $(\alpha, \beta)$ -fuzzy subalgebras (ideals) of a BCK/BCI algebra and investigated related results. Recently, Davvaz [4] applied this theory to near-rings and obtained some useful results. Davvaz and Corsini [5,6] also redefined fuzzy  $H_v$ -submodules and fuzzy  $H_v$ -ideals. We discuss this topic further in this paper.

In section 2, we recall some basic definitions and results of BCI algebras. In section

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3, we introduce the notions of interval-valued fuzzy ( $p$ -,  $q$ - and  $a$ -) ideals of BCI algebras, which are, respectively, generalizations of fuzzy ( $p$ -,  $q$ - and  $a$ -) ideals, and investigate some of their properties. Section 4 is divided into four subsections. In section 4.1, we briefly review  $(\in, \in \vee q)$ -interval-valued fuzzy ideals of BCI algebras. In section 4.2, we investigate the properties of  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideals of BCI algebras. The notions of  $(\in, \in \vee q)$ -interval-valued fuzzy ( $q$ - and  $a$ -) ideals of BCI algebras and the relationship among these generalized interval-valued fuzzy ideals of BCI algebras are discussed in section 4.3 and 4.4, respectively.

## 2. Preliminaries

By a BCI algebra we mean an algebra  $(X, *, 0)$  of type  $(2,0)$  satisfying the axioms:

- (i)  $((x * y) * (x * z)) * (z * y) = 0$ ;
- (ii)  $(x * (x * y)) * y = 0$ ;
- (iii)  $x * x = 0$ ;
- (iv)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$ .

We can define a partial ordering “ $\leq$ ” by  $x \leq y$  if and only if  $x * y = 0$ . In what follows,  $X$  will denote a BCI algebra unless otherwise specified.

**Proposition 2.1.** [7, 8, 16, 17] *In any BCI algebra  $X$ , we have:*

- (1)  $(x * y) * z = (x * z) * y$ ,
- (2)  $(x * z) * (y * z) \leq x * y$ ,
- (3)  $(x * y) * (x * z) \leq z * y$ ,
- (4)  $x * 0 = x$ ,
- (5)  $0 * (x * y) = (0 * x) * (0 * y)$ ,
- (6)  $x * (x * (x * y)) = x * y$ .

A non-empty subset  $I$  of  $X$  is called an ideal of  $X$  if it satisfies (I1)  $0 \in I$ ; (I2)  $x * y \in I$  and  $y \in I$  imply  $x \in I$ . A non-empty subset  $I$  of  $X$  is called a  $p$ -ideal of  $X$  if it satisfies (I1) and (I3)  $(x * z) * (y * z) \in I$  and  $y \in I$  imply  $x \in I$ . A non-empty subset  $I$  of  $X$  is called a  $q$ -ideal of  $X$  if it satisfies (I1) and (I4)  $x * (y * z) \in I$  and  $y \in I$  imply  $x * z \in I$ . A non-empty subset  $I$  of  $X$  is called an  $a$ -ideal of  $X$  if it satisfies (I1) and (I5)  $(x * z) * (0 * y) \in I$  and  $z \in I$  imply  $y * x \in I$  (see [25,27]).

**Theorem 2.2.** [22] (i) *Every  $p$ - (resp.,  $q$ -,  $a$ -) ideal of a BCI algebra is an ideal, but the converse is not true.*

(ii) *A non-empty subset  $I$  of a BCI algebra  $X$  is an  $a$ -ideal of  $X$  if and only if it is both a  $p$ -ideal and a  $q$ -ideal.*

We now review some fuzzy logic concepts. Recall that the real unit interval  $[0,1]$  with the totally ordered relation “ $\leq$ ” is a complete lattice, with  $\wedge = \min$  and  $\vee = \max$ , 0 and 1 being the least and greatest element respectively.

**Definition 2.3.** [34] (i) A fuzzy set of  $X$  is a function  $\mu : X \rightarrow [0, 1]$ ;

(ii) For a fuzzy set  $\mu$  of  $X$  and  $t \in (0, 1]$ , the crisp set  $\mu_t = \{x \in X \mid \mu(x) \geq t\}$  is called the *level subset* of  $\mu$ .

**Definition 2.4.** [23] A fuzzy set  $\mu$  of  $X$  is called a fuzzy ideal of  $X$  if it satisfies:

- (F1)  $\mu(0) \geq \mu(x), \forall x \in X$ ,

$$(F2) \mu(x) \geq \min\{\mu(x * y), \mu(y)\}, \forall x, y \in X.$$

**Definition 2.5** (23,25,29). (i) A fuzzy set  $\mu$  of  $X$  is called a fuzzy  $p$ -ideal of  $X$  if it satisfies (F1) and (F3)  $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$ , for all  $x, y, z \in X$ .

(ii) A fuzzy set  $\mu$  of  $X$  is called a fuzzy  $q$ -ideal of  $X$  if it satisfies (F1) and

$$(F4) \mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}, \text{ for all } x, y, z \in X.$$

(iii) A fuzzy set  $\mu$  of  $X$  is called a fuzzy  $a$ -ideal of  $X$  if it satisfies (F1) and

$$(F5) \mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}, \text{ for all } x, y, z \in X.$$

**Theorem 2.6.** [29] *A fuzzy set  $\mu$  of  $X$  is a fuzzy  $(p, q-, a-)$ -ideal of  $X$  if and only if, for all  $t \in (0, 1]$ , each non-empty level subset  $\mu_t$  is a  $(p, q-, a-)$ -ideal of  $X$ , respectively.*

The following theorem, which is a consequence of Theorem 2.2 and Theorem 2.6, shows the connection between these four types of fuzzy ideals of BCI algebras.

**Theorem 2.7.** [29] (i) *Every fuzzy  $p$ - (resp.,  $q-, a-$ )-ideal of a BCI algebra is a fuzzy ideal, but the converse is not true.*

(ii) *A fuzzy set  $\mu$  of any BCI algebra  $X$  is a fuzzy  $a$ -ideal of  $X$  if and only if it is both a fuzzy  $p$ -ideal and a fuzzy  $q$ -ideal.*

### 3. Some Types of Interval-valued Fuzzy Ideals

We now review some interval-valued fuzzy logic concepts. Let  $\bar{a} = [a^-, a^+]$  be a closed interval of  $[0, 1]$ , where  $0 \leq a^- \leq a^+ \leq 1$ . We denote by  $D[0, 1]$  the set of all such closed intervals of  $[0, 1]$ . The reader can see [30] for more details on the order relation " $\leq$ ".

**Definition 3.1.** [3,32] (i) An interval-valued fuzzy set of  $X$  is  $\bar{F} : X \rightarrow D[0, 1]$ , where, for each  $x \in X$ ,  $\bar{F}(x) = [F^-(x), F^+(x)] \in D[0, 1]$ .

(ii) Let  $\bar{F}$  be an interval-valued fuzzy set of  $X$ . Then, for every  $[0, 0] < \bar{t} \leq [1, 1]$ , the crisp set  $\bar{F}_{\bar{t}} = \{x \in X | \bar{F}(x) \geq \bar{t}\}$  is called the level subset of  $\bar{F}$ .

Note that since every  $a \in [0, 1]$  is in correspondence with the interval  $[a, a] \in D[0, 1]$ , hence a fuzzy set is a particular case of interval-valued fuzzy sets. Also, we can consider an interval-valued fuzzy set  $\bar{F}$  of  $X$  to be a pair of fuzzy sets  $(F^-, F^+)$  of  $X$  such that  $F^-(x) \leq F^+(x)$  for all  $x \in X$ .

We refer the reader to [30] for more details on operations on two interval-valued fuzzy sets of  $X$ .

Define  $\text{rmin}$  by  $\text{rmin}\{\bar{a}_i, \bar{b}_i\} = [\min\{a_i^-, b_i^-\}, \min\{a_i^+, b_i^+\}]$ , where  $\bar{a}_i = [a_i^-, a_i^+]$ ,  $\bar{b}_i = [b_i^-, b_i^+] \in D[0, 1]$ ,  $i \in I$ .

**Definition 3.2** (20). An interval-valued fuzzy set  $\bar{F}$  of  $X$  is called an interval-valued fuzzy ideal of  $X$  if  $\bar{F}$  satisfies:

$$(\text{intF1}) \bar{F}(0) \geq \bar{F}(x), \text{ for all } x \in X,$$

$$(\text{intF2}) \bar{F}(x) \geq \text{rmin}\{\bar{F}(x * y), \bar{F}(y)\}, \text{ for all } x, y \in X.$$

**Remark 3.3.** Every fuzzy ideal of  $X$  is a particular case of an interval-valued fuzzy ideal.

**Theorem 3.4.** [20] *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an interval-valued fuzzy ideal of  $X$  if and only if the set  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is an ideal of  $X$  for all  $[0, 0] < \bar{t} \leq [1, 1]$ .*

**Definition 3.5.** (i) An interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an interval-valued fuzzy  $p$ -ideal of  $X$  if

(intF3)  $\bar{F}(x) \geq \text{rmin}\{\bar{F}((x * z) * (y * z)), \bar{F}(y)\}$ , for all  $x, y, z \in X$ .

(ii) An interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an interval-valued fuzzy  $q$ -ideal of  $X$  if

(intF4)  $\bar{F}(x * z) \geq \text{rmin}\{\bar{F}(x * (y * z)), \bar{F}(y)\}$ , for all  $x, y, z \in X$ .

(iii) An interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an interval-valued fuzzy  $a$ -ideal of  $X$  if

(intF5)  $\bar{F}(y * x) \geq \text{rmin}\{\bar{F}((x * z) * (0 * y)), \bar{F}(z)\}$ , for all  $x, y, z \in X$ .

**Example 3.6.** (i) Let  $X = \{0, 1, 2, 3\}$  be a proper BCI algebra with Cayley table as follows:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.8, 0.9]$ ,  $\bar{F}(1) = \bar{F}(2) = [0.7, 0.8]$ , and  $\bar{F}(3) = [0.2, 0.3]$ . It is easy to verify that  $\bar{F}$  is an interval-valued fuzzy  $p$ -ideal of  $X$ .

(ii) Let  $X = \{0, 1, 2\}$  be a proper BCI algebra with Cayley table as follows:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.8, 0.9]$ , and  $\bar{F}(1) = \bar{F}(2) = [0.2, 0.3]$ . It is easy to verify that  $\bar{F}$  is an interval-valued fuzzy  $q$ -ideal of  $X$ .

(iii) Consider the BCI algebra  $X$  as in Example 3.6 (i). Define an interval-valued fuzzy set  $\bar{F}$  in  $X$  by  $\bar{F}(0) = [0.8, 0.9]$ ,  $\bar{F}(1) = [0.7, 0.8]$ , and  $\bar{F}(2) = \bar{F}(3) = [0.2, 0.3]$ . It is easy to verify that  $\bar{F}$  is an interval-valued fuzzy  $a$ -ideal of  $X$ .

**Remark 3.7.** Every fuzzy  $(p, q-, a-)$  ideal of  $X$  is a particular case of interval-valued fuzzy  $(p, q-, a-)$  ideal, respectively.

Now we characterize the interval-valued fuzzy  $(p, q-, a-)$  ideals by their level subsets.

**Theorem 3.8.** *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an interval-valued fuzzy  $(p, q-, a-)$  ideal of  $X$  if and only if the set  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is respectively a  $(p, q-, a-)$  ideal of  $X$  for all  $[0, 0] < \bar{t} \leq [1, 1]$ .*

*Proof.* Similar to the proof of Theorem 2.6. □

The following theorem, a consequence of Theorem 2.6 and Theorem 3.8, shows the connections between these four types of interval-valued fuzzy ideals of BCI algebras.

**Theorem 3.9.** (i) Every interval-valued fuzzy  $p$ - (resp.,  $q$ -,  $a$ -)ideal of  $X$  is an interval-valued fuzzy ideal, but the converse is not true;

(ii) An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an interval-valued fuzzy  $a$ -ideal if and only if it is both an interval-valued fuzzy  $p$ -ideal and an interval-valued fuzzy  $q$ -ideal.

#### 4. Some Types of $(\in, \in \vee q)$ -interval-valued Fuzzy Ideals

For any  $\bar{F}(x) = [F^-(x), F^+(x)]$  and  $\bar{t} = [t^-, t^+]$ , we define  $\bar{F}(x) + \bar{t} = [F^-(x) + t^-, F^+(x) + t^+]$ , for all  $x \in X$ . In particular, if  $t^- + F^-(x) > 1$ , we write  $\bar{F}(x) + \bar{t} > [1, 1]$ .

Let  $x \in X$  and  $\bar{t} \in D[0, 1]$ . An interval-valued fuzzy set  $\bar{G}$  of a BCI algebra  $X$  is said to be an interval-valued fuzzy point  $x_{\bar{t}}$ , with support  $x$  and interval value  $\bar{t}$ , if

$$\bar{G}(y) = \begin{cases} \bar{t}(\neq [0, 0]) & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x, \end{cases}$$

for all  $y \in X$ . We say  $x_{\bar{t}}$  belongs to (resp., is quasi-coincident with) an interval-valued fuzzy set  $\bar{F}$ , written by  $x_{\bar{t}} \in \bar{F}$  (resp.,  $x_{\bar{t}} q\bar{F}$ ) if  $\bar{F}(x) \geq \bar{t}$  (resp.  $\bar{F}(x) + \bar{t} > [1, 1]$ ). If  $x_{\bar{t}} \in \bar{F}$  or  $x_{\bar{t}} q\bar{F}$ , then we write  $x_{\bar{t}} \in \vee q \bar{F}$ ; if  $\bar{F}(x) < \bar{t}$  (resp.,  $\bar{F}(x) + \bar{t} \leq [1, 1]$ ), then we say that  $x_{\bar{t}} \bar{\in} \bar{F}$  (resp.,  $x_{\bar{t}} \bar{q}\bar{F}$ ). The symbol  $\bar{\in} \vee q$  means that  $\in \vee q$  does not hold.

An interval-valued fuzzy set  $\bar{F}(x) = [F^-(x), F^+(x)]$  of  $X$  is said to satisfy the condition (E) if the following holds:

(E)  $\bar{F}(x) \leq [0.5, 0.5]$  or  $[0.5, 0.5] < \bar{F}(x)$ , for all  $x \in X$ .

In what follows, unless otherwise specified, we all interval-valued fuzzy sets of  $X$  must satisfy the comparability conditions and condition (E).

**4.1.  $(\in, \in \vee q)$ -interval-valued Fuzzy Ideals.** In this subsection, we review  $(\in, \in \vee q)$ -interval-valued fuzzy ideals of  $X$  and state some of their properties [30].

**Definition 4.1.1.** [30] An interval-valued fuzzy set  $\bar{F}$  of  $X$  is said to be an  $(\in, \in \vee q)$ -interval-valued fuzzy ideal of  $X$  if, for all  $[0, 0] < \bar{t} \leq [1, 1]$ ,  $[0, 0] < \bar{r} \leq [1, 1]$  and for all  $x, y \in X$ ,

(F1')  $x_{\bar{t}} \in \bar{F}$  implies  $0_{\bar{t}} \in \vee q \bar{F}$ ,

(F2')  $(x * y)_{\bar{t}} \in \bar{F}$  and  $y_{\bar{r}} \in \bar{F}$  imply  $x_{\min\{\bar{t}, \bar{r}\}} \in \vee q \bar{F}$ .

**Definition 4.1.2.** [30] An interval-valued fuzzy set  $\bar{F}$  of  $X$  is said to be an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy ideal of  $X$  if, for all  $[0, 0] < \bar{t} \leq [1, 1]$ ,  $[0, 0] < \bar{r} \leq [1, 1]$  and all  $x, y \in X$ ,

(F3')  $0_{\bar{t}} \bar{\in} \bar{F}$  implies  $x_{\bar{t}} \bar{\in} \vee \bar{q} \bar{F}$ ,

(F4')  $x_{\min\{\bar{t}, \bar{r}\}} \bar{\in} \bar{F}$  implies  $(x * y)_{\bar{t}} \bar{\in} \vee \bar{q} \bar{F}$  or  $y_{\bar{r}} \bar{\in} \vee \bar{q} \bar{F}$ .

**Theorem 4.1.3.** [30] *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy ideal (resp.,  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy ideal) of  $X$  if and only if  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is an ideal of  $X$  for all  $[0, 0] < \bar{t} \leq [0.5, 0.5]$  (resp.,  $[0.5, 0.5] < \bar{t} \leq [1, 1]$ ).*

**4.2.  $(\in, \in \vee q)$ -interval-valued Fuzzy  $p$ -ideals.** In this subsection, we introduce the concept of  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideals of  $X$  and derive their properties.

**Definition 4.2.1.** An  $(\in, \in \vee q)$ -interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal of  $X$  if

$$(F5') \quad \bar{F}(x) \geq \text{rmin}\{\bar{F}((x * z) * (y * z)), \bar{F}(y), [0.5, 0.5]\}, \text{ for all } x, y, z \in X.$$

**Example 4.2.2.** Consider the BCI algebra  $X$  of Example 3.6(i) and define an interval-valued fuzzy set  $\bar{F}$  in  $X$  by  $\bar{F}(0) = [0.7, 0.8]$ ,  $\bar{F}(1) = \bar{F}(2) = [0.8, 0.9]$ , and  $\bar{F}(3) = [0.2, 0.3]$ . Then  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $p$ -ideal.

**Proposition 4.2.3.** *Every interval-valued fuzzy  $p$ -ideal of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal, but the converse is not true in general.*

*Proof.* Similar to the proof of Theorem 5.2.3 in [30]. □

**Proposition 4.2.4.** *Let  $\bar{F}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy ideal of  $X$ . Then the following are equivalent:*

- (i)  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal.
- (ii)  $\forall x \in X, \quad \bar{F}(x) \geq \text{rmin}\{\bar{F}(0 * (0 * x)), [0.5, 0.5]\}.$

*Proof.* The proposition follows from Proposition 2.3 and Theorem 2.10 in [23]. □

*Next, we characterize the  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideals by their level subsets.*

**Theorem 4.2.5.** *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal of  $X$  if and only if  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is a  $p$ -ideal of  $X$  for all  $[0, 0] < \bar{t} \leq [0.5, 0.5]$ .*

*Proof.* Similar to the proof of Theorem 4.1.3. □

*We have a corresponding result when  $\bar{F}_{\bar{t}}$  is a  $p$ -ideal of  $X$ , for all  $[0.5, 0.5] < \bar{t} \leq [1, 1]$ .*

**Definition 4.2.6.** An  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $p$ -ideal of  $X$  if

$$(F6') \quad \text{rmax}\{\bar{F}(x), [0.5, 0.5]\} \geq \text{rmin}\{\bar{F}((x * z) * (y * z)), \bar{F}(y)\}, \text{ for all } x, y, z \in X.$$

**Example 4.2.7.** Consider the BCI algebra  $X$  of Example 3.6(i) and define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.8, 0.9]$ ,  $\bar{F}(1) = [0.6, 0.7]$ ,  $\bar{F}(2) = [0.2, 0.3]$  and  $\bar{F}(3) = [0.3, 0.4]$ . Then  $\bar{F}$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $p$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $p$ -ideal of  $X$ .

**Theorem 4.2.8.** *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $p$ -ideal of  $X$  if and only if  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is a  $p$ -ideal of  $X$  for all  $[0.5, 0.5] < \bar{t} \leq [1, 1]$ .*

*Proof.* Similar to the proof of Theorem 4.1.3. □

**Remark 4.2.9.** *Let  $\bar{F}$  be an interval-valued fuzzy set of  $X$  and  $J = \{\bar{t} \mid \bar{a} < \bar{t} \leq \bar{b} \text{ and } \bar{F}_{\bar{t}} \text{ is an empty set or a } p\text{-ideal of } X\}$ . In particular, if  $\bar{a} = [0, 0]$  and  $\bar{b} = [1, 1]$ , then  $\bar{F}$  is an ordinary interval-valued fuzzy  $p$ -ideal of  $X$  (Theorem 3.8); if  $\bar{a} = [0, 0]$  and  $\bar{b} = [0.5, 0.5]$ , then  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal of  $X$  (Theorem 4.2.5); if  $\bar{a} = [0.5, 0.5]$  and  $\bar{b} = [1, 1]$ , then  $\bar{F}$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $p$ -ideal of  $X$  (Theorem 4.2.8).*

**4.3.  $(\in, \in \vee q)$ -interval-valued Fuzzy  $q$ -ideals.** *In this subsection, we introduce the concept of  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideals of  $X$  and obtain some related properties.*

**Definition 4.3.1.** *An  $(\in, \in \vee q)$ -interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal of  $X$  if*

$$(F7') \quad \bar{F}(x * z) \geq \text{rmin}\{\bar{F}(x * (y * z)), \bar{F}(y), [0.5, 0.5]\}, \text{ for all } x, y, z \in X.$$

**Example 4.3.2.** *Consider the BCI algebra  $X$  of Example 3.6(ii) and define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.7, 0.8]$ ,  $\bar{F}(1) = [0.8, 0.9]$  and  $\bar{F}(2) = [0.2, 0.3]$ . Then  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $q$ -ideal.*

**Proposition 4.3.3.** *Every interval-valued fuzzy  $q$ -ideal of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal, but the converse is not true in general.*

*Proof.* Similar to the proof of Proposition 4.2.3. □

**Proposition 4.3.4.** *Let  $\bar{F}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy ideal of  $X$ . Then the following are equivalent:*

- (i)  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal.
- (ii)  $\bar{F}(x * y) \geq \text{rmin}\{\bar{F}(x * (0 * y)), [0.5, 0.5]\}, \forall x, y \in X$ .

*Proof.* the proof follows from Proposition 3.16 in [23]. □

*Next, we characterize the  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideals by their level subsets.*

**Theorem 4.3.5.** *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal of  $X$  if and only if  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is a  $q$ -ideal of  $X$  for all  $[0, 0] < \bar{t} \leq [0.5, 0.5]$ .*

*Proof.* Similar to the proof of Theorem 4.1.3. □

*We have a corresponding result when  $\bar{F}_{\bar{t}}$  is a  $q$ -ideal of  $X$ , for all  $[0.5, 0.5] < \bar{t} \leq [1, 1]$ .*

**Definition 4.3.6.** An  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $q$ -ideal of  $X$  if

$$(F8') \text{rmax}\{\bar{F}(x * z), [0.5, 0.5]\} \geq \text{rmin}\{\bar{F}(x * (y * z)), \bar{F}(y)\}, \text{ for all } x, y, z \in X.$$

**Example 4.3.7.** Consider the BCI algebra  $X$  of Example 3.6(ii) and define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.8, 0.9]$ ,  $\bar{F}(1) = [0.2, 0.3]$  and  $\bar{F}(2) = [0.5, 0.5]$ . Then  $\bar{F}$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $q$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $q$ -ideal of  $X$ .

**Theorem 4.3.8.** An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $q$ -ideal of  $X$  if and only if  $\bar{F}_{\bar{t}}(\neq \emptyset)$  is a  $q$ -ideal of  $X$  for all  $[0.5, 0.5] < \bar{t} \leq [1, 1]$ .

*Proof.* Similar to the proof of Theorem 4.1.3. □

**Remark 4.3.9.** Let  $\bar{F}$  be an interval-valued fuzzy set of  $X$  and  $J = \{\bar{t} \mid \bar{a} < \bar{t} \leq \bar{b} \text{ and } \bar{F}_{\bar{t}} \text{ is an empty set or a } q\text{-ideal of } X\}$ . In particular, if  $\bar{a} = [0, 0]$  and  $\bar{b} = [1, 1]$ , then  $\bar{F}$  is an ordinary interval-valued fuzzy  $q$ -ideal of  $X$  (Theorem 3.8). If  $\bar{a} = [0, 0]$  and  $\bar{b} = [0.5, 0.5]$ , then  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal of  $X$  (Theorem 4.3.5). If  $\bar{a} = [0.5, 0.5]$  and  $\bar{b} = [1, 1]$ , then  $\bar{F}$  is an  $(\bar{\in}, \bar{\in} \vee \bar{q})$ -interval-valued fuzzy  $q$ -ideal of  $X$  (Theorem 4.3.8).

**4.4.  $(\in, \in \vee q)$ -interval-valued Fuzzy  $a$ -ideals.** In this subsection, we introduce the concept of  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideals of  $X$  and obtain some related properties.

**Definition 4.4.1.** An  $(\in, \in \vee q)$ -interval-valued fuzzy ideal  $\bar{F}$  of  $X$  is called an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  if

$$(F9') \bar{F}(y * x) \geq \text{rmin}\{\bar{F}((x * z) * (0 * y)), \bar{F}(z), [0.5, 0.5]\}, \text{ for all } x, y, z \in X.$$

**Example 4.4.2.** Let  $X = \{0, 1, 2\}$  be a proper BCI algebra with Cayley table as follows:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Now define an interval-valued fuzzy set  $\bar{F}$  of  $X$  by  $\bar{F}(0) = [0.7, 0.8]$  and  $\bar{F}(1) = \bar{F}(2) = [0.8, 0.9]$ . Then  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $a$ -ideal.

**Proposition 4.4.3.** Every interval-valued fuzzy  $a$ -ideal of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal, but the converse is not true in general.

*Proof.* Similar to the proof of Proposition 4.2.3. □

**Proposition 4.4.4.** Let  $\bar{F}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy ideal of  $X$ . Then the following are equivalent:

- (i)  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal.
- (ii)  $\bar{F}(y * (x * z)) \geq \text{rmin}\{\bar{F}((x * z) * (0 * y)), [0.5, 0.5]\}, \forall x, y, z \in X.$



(iii)  $\overline{F}(y * x) \geq \text{rmin}\{\overline{F}(x * (0 * y)), [0.5, 0.5]\}, \forall x, y \in X$ .

*Proof.* The theorem follows from Theorem 3.5 in [29].  $\square$

Next, we characterize the  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideals by their level subsets.

**Theorem 4.4.5.** *An interval-valued fuzzy set  $\overline{F}$  of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  if and only if  $\overline{F}_{\overline{t}}(\neq \emptyset)$  is an  $a$ -ideal of  $X$  for all  $[0, 0] < \overline{t} \leq [0.5, 0.5]$ .*

*Proof.* Similar to the proof of Theorem 4.1.3.  $\square$

We have a corresponding result when  $\overline{F}_{\overline{t}}$  is an  $a$ -ideal of  $X$ , for all  $[0.5, 0.5] < \overline{t} \leq [1, 1]$ .

**Definition 4.4.6.** *An  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -interval-valued fuzzy ideal  $\overline{F}$  of  $X$  is called an  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -interval-valued fuzzy  $a$ -ideal of  $X$  if*

*(F10')  $\text{rmax}\{\overline{F}(y * x), [0.5, 0.5]\} \geq \text{rmin}\{\overline{F}((x * z) * (0 * y)), \overline{F}(z)\}$ , for all  $x, y, z \in X$ .*

**Example 4.4.7.** *Consider the BCI algebra  $X$  of Example 3.6(i) and define an interval-valued fuzzy set  $\overline{F}$  of  $X$  by  $\overline{F}(0) = \overline{F}(1) = [0.8, 0.9]$ ,  $\overline{F}(2) = [0.5, 0.5]$  and  $\overline{F}(3) = [0.2, 0.3]$ . Then  $\overline{F}$  is an  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -interval-valued fuzzy  $a$ -ideal of  $X$ , but it is not an interval-valued fuzzy  $a$ -ideal of  $X$ .*

**Theorem 4.4.8.** *An interval-valued fuzzy set  $\overline{F}$  of  $X$  is an  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -interval-valued fuzzy  $a$ -ideal of  $X$  if and only if  $\overline{F}_{\overline{t}}(\neq \emptyset)$  is an  $a$ -ideal of  $X$  for all  $[0.5, 0.5] < \overline{t} \leq [1, 1]$ .*

*Proof.* Similar to the proof of Theorem 4.1.3.  $\square$

**Remark 4.4.9.** *Let  $\overline{F}$  be an interval-valued fuzzy set of  $X$  and  $J = \{\overline{t} \mid \overline{a} < \overline{t} \leq \overline{b} \text{ and } \overline{F}_{\overline{t}} \text{ is an empty set or an } a\text{-ideal of } X\}$ . In particular, if  $\overline{a} = [0, 0]$  and  $\overline{b} = [1, 1]$ , then  $\overline{F}$  is an ordinary interval-valued fuzzy  $a$ -ideal of  $X$  (Theorem 3.8). If  $\overline{a} = [0, 0]$  and  $\overline{b} = [0.5, 0.5]$ , then  $\overline{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  (Theorem 4.4.5). If  $\overline{a} = [0.5, 0.5]$  and  $\overline{b} = [1, 1]$ , then  $\overline{F}$  is an  $(\overline{\in}, \overline{\in} \vee \overline{q})$ -interval-valued fuzzy  $a$ -ideal of  $X$  (Theorem 4.4.8).*

**Theorem 4.4.10.** *Every  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal, but the converse may not be true.*

*Proof.* The theorem is an immediate consequence of Theorem 2.2(ii), 4.3.5 and 4.4.5. The last part is shown in Example 3.6 (i), where  $\overline{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal of  $X$ , but not an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$ .  $\square$

**Theorem 4.4.11.** *Every  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal, but the converse may not be true.*

*Proof.* The theorem is an immediate consequence of Theorem 2.2(ii), 4.2.5 and 5.4.5. The last part is shown in Example 3.6 (ii), where  $\bar{F}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal of  $X$ , but not an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$ .  $\square$

*The following theorem shows the relationship between these generalized interval-valued fuzzy ideals of BCI algebras.*

**Theorem 4.4.12.** *An interval-valued fuzzy set  $\bar{F}$  of  $X$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy  $a$ -ideal of  $X$  if and only if it is both an  $(\in, \in \vee q)$ -interval-valued fuzzy  $p$ -ideal and an  $(\in, \in \vee q)$ -interval-valued fuzzy  $q$ -ideal.*

*Proof.* Necessity: Theorem 4.4.10 and 4.4.11.

Sufficiency: Proposition 4.3.4, 4.2.4 and 4.4.4.  $\square$

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