EMBEDDING OF THE LATTICE OF IDEALS OF A RING INTO ITS LATTICE OF FUZZY IDEALS

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Abstract. We show that the lattice of all ideals of a ring R can be embedded in the lattice of all its fuzzy ideals in uncountably many ways. For this purpose, we introduce the concept of the generalized characteristic function $\chi_s^r(A)$ of a subset A of a ring R for fixed $r, s \in [0, 1]$ and show that A is an ideal of R if, and only if, its generalized characteristic function $\chi_s^r(A)$ is a fuzzy ideal of R. We also show that the set of all generalized characteristic functions $C_s^r(I(R))$ of the members of $I(R)$ for fixed $r, s \in [0, 1]$ is a complete sublattice of the lattice of all fuzzy ideals of R and establish that this latter lattice is generated by the union of all its complete sublattices $C_s^r(I(R))$.

1. Introduction

The embedding of one type of algebraic structures into another is an important topic in both classical algebra and lattice theory.The problem of embedding a lattice of algebraic substructures of an algebra in the lattice of the corresponding fuzzy algebraic substructures has also been discussed. [1, 2, 6] found in the pioneering paper of Rosenfeld [5] wherein he tried to embed the lattice of ordinary subgroups of a group in its fuzzy subgroup lattice. This embedding can be easily demonstrated via the characteristic function. On the other hand, Tom Head, in his excellent paper [2], demonstrated such an embedding through the representation function Rep. The ingenious construction of the function Rep allows the embedding of the lattice of algebraic substructures into the corresponding lattices of fuzzy algebraic substructures.

In this paper, we introduce the notion of a generalized characteristic function and show that the set of all generalized characteristic functions of ideals of a ring R constitutes a complete sublattice of the lattice of fuzzy ideals $\vartheta(R)$ of the ring R. Moreover, we define a mapping from $I(R)$ to $\vartheta(R)$ which maps an ideal to its generalized characteristic function. This mapping commutes with the arbitrary suprema and infima and hence is a morphism of complete lattices and we use it to demonstrate that the lattice $I(R)$ can be embedded into the lattice $\vartheta(R)$ in uncountably many ways.

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2. **Preliminaries**

In this section we recall some definitions and results. For details we refer the reader to [1, 4].

A fuzzy set is a function from a non empty set X to the closed unit interval.We say that the fuzzy set θ is contained in the fuzzy set η if $\theta(x) \leq \eta(x)$ for every $x \in R$ and write $\theta \subseteq \eta$. It is well known that the fuzzy power set $F(X)$, which is the set of all fuzzy sets in X, constitutes a complete lattice under the ordering of fuzzy set inclusion ($\subseteq$). The supremum $\cup\, \theta_i$ and infimum $\cap\, \theta_i$ of a family of fuzzy sets $\{\theta_i\}$ in X are respectively defined as $\cup\theta_i(x) = \sup\limits_{x\in X} \theta_i(x)$ and $\cap\theta_i(x) = \inf\limits_{x\in X}\{\theta_i(x)\}$. The level subset θ_t for $t \in [0, 1]$ is defined as $\theta_t = \{x \in R : \theta(x) \geq t\}$.

Definition 2.1. Let θ be a fuzzy set in a ring R. Then θ is said to be a fuzzy subring of R if, for all $x, y \in R$,

(1) $\theta(x - y) \geq \min\{\theta(x), \theta(y)\}$,
(2) $\theta(xy) \geq \min\{\theta(x), \theta(y)\}$.

It can be easily verified that if θ is a fuzzy subring of R then $\theta(x) \leq \theta(0)$ and $\theta(x) = \theta(-x)$ for all $x \in R$.

Definition 2.2. Let θ be a fuzzy subring of a ring R. Then θ is called a fuzzy (left, right) ideal of R if $(\theta(xy) \geq \theta(y), \theta(xy) \geq \theta(x))$, $\theta(xy) \geq \max\{\theta(x), \theta(y)\}$ for $x, y \in R$.

Proposition 2.3. *The intersection $\cap\theta_i$ of a family $\{\theta_i\}$ of fuzzy ideals of a ring R is a fuzzy ideal of R.*

Definition 2.4. Let θ be a fuzzy set in a ring R. Then the fuzzy ideal generated by θ is defined as the least fuzzy ideal of R which contains θ

$$\langle\theta\rangle = \bigcap_{\theta\subseteq\theta_i} \{\theta_i : \theta_i \in \vartheta(R)\}.$$

We note that the existence of such a fuzzy ideal is ensured by Proposition 2.1.

Proposition 2.5. *The set of all fuzzy ideals $\vartheta(R)$ of a ring R is a complete lattice under fuzzy set inclusion, where the infimum and supremum of a family of fuzzy ideals of R are defined respectively as the intersection of the family and the ideal generated by their union.*

Let X be a non empty set.We denote by $P(X)$ the set of all subsets of X, by $F(X)$ the set of all fuzzy sets in X and by "$\subseteq$" the ordinary set inclusion as well as the fuzzy set inclusion.

3. **Embedding**

We start with the definition of a generalized characteristic function.

Definition 3.1. Any two valued mapping from a nonempty set X to the closed unit interval $[0, 1]$ is called a generalized characteristic function. For $r, s \in [0, 1]$ such that $r > s$, let $C_s^r(X)$ denote the set of all functions from a non empty set X

to the set $\{r, s\}$; i.e. $C_s^r(X) = \{r, s\}^X$. We define a map χ_s^r from $P(X)$ to $C_s^r(X)$ by

$$\chi_s^r : A \to \chi_s^r(A) \quad \text{for all} \quad A \in P(X),$$

Where, for each A in $P(X), \chi_s^r(A)$ is defined by

$$\chi_s^r(A)(x) = \begin{cases} r & \text{if} \quad x \in A \\ s & \text{if} \quad x \notin A. \end{cases}$$

We call $\chi_s^r(A)$ *the generalized characteristic function of* A.

Since $\{r, s\}$ is a subset of $[0, 1]$, generalized characteristic functions are special instances of fuzzy sets in X. Hence $C_s^r(X) \subseteq F(X)$. Next, we observe that $C_s^r(X)$ is a complete sublattice of the lattice $F(X)$ under fuzzy set inclusion "$\subseteq$". Also $P(X)$ is a complete lattice under set inclusion "$\subseteq$", the function $\chi_s^r : P(X) \to C_s^r(X)$ is an isomorphism of complete lattices and the function support^r: $C_s^r(X) \to P(X)$ defined by support^r $(f) = \{x :\in X : f(x) = r\}$ is the inverse map of χ_s^r.

Proposition 3.2. *Let θ be a fuzzy set in a ring R. Then every non empty level subset θ_t is an ideal of R if and only if θ is a fuzzy ideal of R.*

Proposition 3.3. *Let A be a non empty subset of R. Then $\chi_s^r(A)$is a fuzzy ideal of R if and only if A is an ideal of R.*

Proof. It is easy to see that for $A \in I(R)$, we have

$$\sup_{x \in R} \{\chi_s^r(A)\}(x) = r.$$

If $t \in [0, 1]$ is such that $t \leq r$ then the level subset $(\chi_s^r(A))_t$ is either A or R. Thus by Proposition 3.3, it follows that $\chi_s^r(A)$ is a fuzzy ideal of R if and only if A is an ideal of R. □

Proposition 3.4. *Let A be a subset of R. Then*

$$\langle \chi_s^r(A) \rangle = \chi_s^r(\langle A \rangle),$$

where $\langle A \rangle$ is the ideal generated by A.

Proof. We prove that

$$\langle \chi_s^r(A) \rangle(x) = \begin{cases} r & \text{if} \quad x \in \langle A \rangle \\ s & \text{if} \quad x \notin \langle A \rangle \end{cases}.$$

For $x \in R$, consider

$$\begin{aligned} \langle \chi_s^r(A) \rangle(x) &= \bigcap_{\chi_s^r(A) \subset \theta_i} \theta_i(x) \\ &= \inf_{\chi_s^r(A) \subset \theta_i} \theta_i(x), \end{aligned}$$

where $\theta_i \in \vartheta(R)$. Now since $A \subseteq \langle A \rangle$, we have

$$\chi_s^r(A) \subseteq \chi_s^r(\langle A \rangle).$$

Also, by Proposition 3.4, $\chi_s^r(\langle A \rangle)$ is a fuzzy ideal.

If $x \notin \langle A \rangle$, then $\langle \chi_s^r(A) \rangle(x) = s$ and this implies that

$$\inf_{\chi_s^r(A) \subset \theta_i} \theta_i(x) = s.$$

So

$$\langle \chi_s^r(A) \rangle(x) = s.$$

On the other hand, if $x \in \langle A \rangle$, then

$$x = \sum_{i=1}^{k} a_i^r x_i b_i^s + \sum_{i=k+1}^{p} n_i x_i$$

where $x_i \in A$ for all $i = 1, 2, \ldots, p$; $a_i, b_i \in R$ for all $i = 1, 2, \ldots, k$; and $n_i \in Z$ for all $i = k+1, \ldots, p$. Note that here $r, s = 0$ or 1 such that r and s can be taken as zero only one at a time and $a_i^0 x_i b_i^s = x_i b_i, a_i^r x_i b_i^0 = a_i x_i$ respectively. As $\theta_i \in \vartheta(R)$, we have

$$\theta_i(x) \geq \min \left\{ \theta_i(x_1), \theta_i(x_2), \ldots, \theta_i(x_p) \right\},$$

for all i and since $\chi_s^r(A) \subseteq \theta_i, hence r = \chi_s^r(A)(x_j) \leq \theta_i(x_j)$ for all $j = 1, 2, \ldots, p$. Thus $\theta_i(x) \geq r$. Also, for some $i_0, \theta_{i_0} = \chi_s^r(\langle A \rangle)$ and hence $\theta_{i_0}(x) = r$. Consequently

$$\inf_{\chi_s^r(A) \subset \theta_i} \theta_i(x) = r.$$

In other words,

$$\langle \chi_s^r(A) \rangle(x) = r.$$

This completes the proof. □

For fixed $r, s \in [0, 1]$, such that $r > s$, we denote by $C_s^r(I(R))$ the set of all generalized characteristic functions of the members of $I(R)$.

Proposition 3.5. *For each pair $r, s \in [0, 1], C_s^r(I(R))$ is a complete sublattice of the lattice $\vartheta(R)$.*

Proof. Since $C_s^r(I(R))$ contains a greatest element that maps whole of R to r, it suffices to prove that $C_s^r(I(R))$ is closed under arbitrary infima of the lattice $\vartheta(R)$. Let $\{\chi_s^r(A_i)\}$ be any subfamily of $C_s^r(I(R))$. Then it is easy to verify that

$$\cap \chi_s^r(A_i) = \chi_s^r(\cap A_i).$$

Since $I(R)$ is a complete lattice under set inclusion, it is closed under arbitrary infima and hence $\cap A_i \in I(R)$. Thus $\chi_s^r(\cap A_i) \in C_s^r(I(R))$. Therefore $C_s^r(I(R))$ is also closed under arbitrary infima. □

Now, in order to illustrate the embedding of the lattice $I(R)$ of ideals of R into the lattice $\vartheta(R)$ of fuzzy ideals of R, we show that there exists a morphism of complete lattices between $I(R)$ and $C_s^r(I(R))$.

Proposition 3.6. *For each $r, s \in [0,1]$, $r > s$, define a map $\theta_s^r : I(R) \to \vartheta(R)$ by*

$$\theta_s^r(A) = \chi_s^r(A).$$

Then θ_s^r commutes with arbitrary suprema and infima. That is, θ_s^r is a morphism of complete lattices.

Proof. Let S be any family of ideals of R. The assertion

$$\theta_s^r(\inf\{A : A \in S\}) = \inf \theta_s^r(\{A : A \in S\})$$

states that θ_s^r commutes with arbitrary intersections. This can be verified in view of the following facts that, if

(1) $x \in \cap\{A : A \in S\}$ then
   (a) $\theta_s^r(\cap\{A : A \in S\})(x) = r$,
   (b) $\theta_s^r(A)(x) = r$ for each $A \in S$,
   (c) $\cap\theta_s^r(\{A : A \in S\})(x) = r$, and

(2) $x \notin \cap\{A : A \in S\}$ can be dealt similarly.

Now, the equality $\theta_s^r(\sup\{A : A \in S\}) = \sup \theta_s^r(\{A : A \in S\})\theta_s^r$ follows since $\sup\{A : A \in S\}$ is $\left\langle \bigcup_{A\in S} A \right\rangle$, and by Proposition 3.5, we have $\theta_s^r \left\langle \bigcup_{A\in S} A \right\rangle = \left\langle \theta_s^r \left(\bigcup_{A\in S} A \right) \right\rangle$. The equality $\theta_s^r \left(\bigcup_{A\in S} A \right) = \bigcup_{A\in S} \theta_s^r(A)$ can be proved similarly.

Now by Proposition 3.7 and the fact that θ_s^r is an injective map with image $C_s^r(I(R))$, we have:

Corollary 3.7. *The lattice $I(R)$ of ideals of R can be embedded in the lattice $\vartheta(R)$ of fuzzy ideals of R through the identification map θ_s^r for each $r, s \in [0,1]$.*

Remark 3.8. The lattice $\vartheta(R)$ contains uncountably many isomorphic copies $C_s^r(I(R))$ of the lattice $I(R)$.

In the following theorem $\left\langle \bigcup_{r,s\in[0,1]} C_s^r(I(R)) \right\rangle$ denotes the sublattice of $\vartheta(R)$ generated by the union $\bigcup_{r,s\in[0,1]} C_s^r(I(R))$.

Theorem 3.9. *Let R be a ring. Then the lattice of all fuzzy ideals $\vartheta(R)$ is generated by the union of all its complete sublattices $C_s^r(I(R))$. In other words, we have*

$$\vartheta(R) = \left\langle \bigcup_{r,s\in[0,1]} C_s^r(I(R)) \right\rangle.$$

Proof. Since $C_s^r(I(R)) \subseteq \vartheta(R)\infty$ for every $r, s \in [0.1]$, we have

$$\left\langle \bigcup_{r,s\in[0,1]} C_s^r(I(R)) \right\rangle \subseteq \vartheta(R).$$

To show the reverse inclusion, let $\theta \in \vartheta(R)$. Now we provide a construction of fuzzy ideals whose union gives us the ideal θ. For this purpose, let $t' = \inf_{x\in R}\{\theta(x)\}$.

Next, for $t_j \in \operatorname{Im}\theta$ such that $t_j > t'$, we define the generalized characteristic function as follows:

$$\chi_{t'}^{t_j}(\theta_{t_j})(x) = \begin{cases} t_j & \text{if } x \in \theta_{t_j}, \\ t' & \text{if } x \notin \theta_{t_j}. \end{cases}$$

Since $\theta \in \vartheta(R)$, it follows that $\chi_{t'}^{t_j}(\theta_{t_j}) \in C_t^{t_j}(I(R))$. We claim that

$$\bigcup_{t_j \in \operatorname{Im}\theta} \chi_{t'}^{t_j}(\theta_{t_j}) = \theta.$$

For $x \in R$, assume that $t = \theta(x)$. We observe that $x \in \theta_{t_j}$ for all $t_j \leq t$ and $x \notin \theta_{t_j}$ for all $t_j > t$. This implies

$$\bigcup_{t_j \leq t} (\chi_{t'}^{t_j}(\theta_{t_j}))(x) = t, \tag{1}$$

and

$$\bigcup_{t_j > t} (\chi_{t'}^{t_j}(\theta_{t_j}))(x) = t'. \tag{2}$$

Thus by (1) and (2), we have

$$\bigcup_{t_j \in \operatorname{Im}\theta} (\chi_{t'}^{t_j}(\theta_{t_j}))(x) = t.$$

So

$$\bigcup_{t_j \in \operatorname{Im}\theta} (\chi_{t'}^{t_j}(\theta_{t_j}))(x) = \theta(x).$$

Next we show that the fuzzy ideal θ of R is generated by the union of the family of fuzzy ideals $\left\{ \bigcup_{t_j \in \operatorname{Im}\theta} (\chi_{t'}^{t_j}(\theta_{t_j})) \right\}$. Since θ is a fuzzy ideal of R, it follows that $\langle\theta\rangle = \theta$. Therefore we have

$$\left\langle \bigcup_{t_j \in \operatorname{Im}\theta} (\chi_{t'}^{t_j}(\theta_{t_j})) \right\rangle = \theta$$

Hence $\theta \in \left\langle \bigcup_{r,s \in [0,1]} C_s^r(I(R)) \right\rangle$. Consequently

$$\vartheta(R) \subseteq \left\langle \bigcup_{r,s \in [0,1]} C_s^r(I(R)) \right\rangle.$$

The theorem is proved. □

4. Conclusion

We first introduced the notion of generalized characteristic functions which behave very much like characteristic functions and using this notion produced uncountably many isomorphic copies of the lattice of ideals $I(R)$ of a ring in the lattice of fuzzy ideals $\mathcal{I}(R)$. We then demonstrated that the lattice of fuzzy ideals of a ring is generated by the union of all these sublattices each of which is isomorphic to the lattice of crisp ideals. This result is interesting as well as revealing and can find wide applications in the area of fuzzy algebra. Theorem 3.9 poses an interesting question of classical lattice theory : if a lattice is generated by the union of certain sublattices which are modular, then is the given lattice L also modular?

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