CONVERGENCE, CONSISTENCY AND STABILITY IN FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider First-order fuzzy differential equations with initial value conditions. The convergence, consistency and stability of difference method for approximating the solution of fuzzy differential equations involving generalized H-differentiability, are studied. Then the local truncation error is defined and sufficient conditions for convergence, consistency and stability of difference method are provided and fuzzy stiff differential equation and one example are presented to illustrate the accuracy and capability of our proposed concepts.

1. **Introduction**

The topics of fuzzy differential equations, which attracted a growing interest for some time, in particular, in relation with the fuzzy control, have been rapidly developed recent years. The fuzzy derivative was first introduced by S. L. Chang, L. A. Zadeh in [8]. Then, D. Dubois, H. Prade in [9] defined and used the extension principle. Other methods have been discussed by M. L. Puri, D. A. Ralescu in [15] and R. Goetschel, W. Voxman in [11]. The fuzzy differential equation and the initial value problem were treated by O. Kaleva in [12] and [13] and by S. Seikkala in [18]. The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [14] by the standard Euler method and the authors in [1, 17] presented the new methods for solving FODEs. Recently, in [3], fuzzy Laplace transforms for solving first order fuzzy differential equations under generalized H-differentiability is proposed and application of fuzzy Laplace transforms is represented in [16].
Bede and Gal [4, 5, 6] introduced a more general definition of the derivative for fuzzy mappings, enlarging the class of differentiable fuzzy mapping, and Chalco and Flores [7] solved these FDEs.
The structure of this paper would be as follows:
Section 2 gives related basic concepts and classifies fuzzy number, Hausdorff metric, generalized differential. In Section 3, the main section of the paper, the convergence, consistency and stability of one-step method for approximating the solution of fuzzy differential equations are studied. The proposed properties are illustrated by remarking the fuzzy stiff differential equation in the Section 4. The accuracy and

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capability of our proposed concepts are reviewed by solving one example in Section 5. Finally conclusion is drawn in Section 6.

2. **Basic Concepts**

A nonempty subset A of R is called convex if and only if $(1-k)x+ky \in A$ for every $x, y \in A$ and $k \in [0,1]$. By $p_k(R)$, we denote the family of all nonempty compact convex subsets of R.
There are various definitions for the concept of fuzzy numbers ([9, 10])

Definition 2.1. A fuzzy number is a function $u : R \to [0,1]$ satisfying the following properties:
(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$),
(iii) u is upper semi-continuous on R,
(iv) $\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by E. Obviously $R \subset E$. Here $R \subset E$ is understood as $R = \{\chi_x : \chi$ is usual real number$\}$. For $0 < r \leq 1$, denote $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \overline{\{x \in R; u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]_r$ is a bounded closed interval. For $u, v \in E$, and $\lambda \in R$, where sum $u+v$ and the product $\lambda.u$ are defined by $[u+v]_r = [u]_r + [v]_r$, $[\lambda.u]_r = \lambda[u]_r$, $\forall r \in [0,1]$, where $[u]_r + [v]_r = \{x + y : x \in [u]_r, y \in [v]_r\}$ means the conventional addition of two intervals (subsets) of R and $\lambda[u]_r = \{\lambda x : x \in [u]_r\}$ means the conventional product between a scalar and a subset of R(see e.g. [9, 19].
Another definition for a fuzzy number is as follows:

Definition 2.2. An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:
1. $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.
2. $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$, $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b-c)r$ and $\overline{u}(r) = c - (c-b)r$ are the endpoints of the $r-$level sets, for all $r \in [0,1]$. Here $\underline{u}(r) = \overline{u}(r) = b$ and it is denoted by $[u]_1$. For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$, we define addition and multiplication by k as follows:
1. $\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r))$,
2. $\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r))$,
3. $\underline{(ku)}(r) = k\underline{u}(r), \overline{(ku)}(r) = k\overline{u}(r)$, $k \geq 0$,
4. $\underline{(ku)}(r) = k\overline{u}(r), \overline{(ku)}(r) = k\underline{u}(r)$, $k < 0$.
In this paper, we represent an arbitrary fuzzy number with compact support by a pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$. Also, we use the Hausdorff distance

between fuzzy numbers. This fuzzy number space as shown in [5] can be embedded into Banach space $B = \overline{c}[0,1] \times \overline{c}[0,1]$ where the metric is usually defined as follows: Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level sets. Since the $r-$levels of fuzzy numbers are always closed and bounded, the intervals are written as $u[r] = [\underline{u}(r), \overline{u}(r)]$, for all r. We denote by ω the set of all nonempty compact subsets of R and by ω_c the subsets of ω consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in$ R from $A \in \omega$ and the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \omega$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on ω is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The metric d_∞ is defined on E as

$$d_\infty(u, v) = \sup\{d_H(u[r], v[r]) : 0 \leq r \leq 1\}, \quad u, v \in E.$$

For arbitrary $(u, v) \in \overline{c}[0,1] \times \overline{c}[0,1]$. The following properties are well-known. (see e.g. [10, 19])

(i) $d_\infty(u + w, v + w) = d_\infty(u, v)$, $\forall u, v, w \in E$,

(ii) $d_\infty(k.u, k.v) = |k| d_\infty(u, v)$, $\forall k \in R, u, v \in E$,

(iii) $d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e)$, $\forall u, v, w, e \in E$,

(iv) $d_\infty(u, v) = d_\infty(v, u)$, $\forall u, v \in E$

Theorem 2.3. *(i) If we define $\widetilde{0} = \chi_0$, then $\widetilde{0} \in E$ is a neutral element with respect to addition, i.e. $u + \widetilde{0} = \widetilde{0} + u = u$, for all $u \in E$.*
(ii) With respect to $\widetilde{0}$, none of $u \in E \setminus R$, has opposite in E.
(iii) For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in E$, we have $(a + b).u = a.u + b.u$; however, this relation dose not necessarily hold for any $a, b \in R$, in general.
(iv) For any $\lambda \in R$ and any $u, v \in E$, we have $\lambda.(u + v) = \lambda.u + \lambda.v$;
(v) For any $\lambda, \mu \in R$ and any $u \in E$, we have $\lambda.(\mu.u) = (\lambda.\mu).u$.(see [19]*)*

Remark 2.4. $d_\infty(u, 0) = d_\infty(0, u) = \| u \|$.

Definition 2.5. Consider $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" always stands for H-difference and note that $x \ominus y \neq x + (-y)$. Let us recall the definition of strongly generalized differentiability introduced in [5].

Lemma 2.6. [5]*, Let $u, v \in E$ be such that $u(1) - \underline{u}(0) > 0$, $\overline{u}(0) - u(1) > 0$ and $len(v) = (\overline{v}(0) - \underline{v}(0)) \leq \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\}$. Then the H-difference $u \ominus v$ exists.*

Definition 2.7. Let E be a set of all fuzzy numbers, we say that $f(x)$ is a fuzzy valued function if $f : \Re \to E$.

Definition 2.8. [6], Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f^{'}(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small,

$$\exists f(x_0 + h) \ominus f(x_0), \qquad \exists f(x_0) \ominus f(x_0 - h)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0)$$

or

(ii) for all $h > 0$ sufficiently small,

$$\exists f(x_0) \ominus f(x_0 + h), \qquad \exists f(x_0 - h) \ominus f(x_0)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

(iii) for all $h > 0$ sufficiently small,

$$\exists f(x_0 + h) \ominus f(x_0), \qquad \exists f(x_0 - h) \ominus f(x_0)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

(iv) for all $h > 0$ sufficiently small,

$$\exists f(x_0) \ominus f(x_0 + h), \qquad \exists f(x_0) \ominus f(x_0 - h)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively $\forall s = 1 \ldots n$)

Proposition 2.9. [9], *If $f : (a, b) \to E$ is a continuous fuzzy valued function then $g(x) = \int_a^x f(t)d$ is differentiable with derivative $g'(x) = f(x)$.*

Lemma 2.10. [5], *For $x_0 \in R$, the fuzzy differential equation $y^{'} = f(x, y)$, $y(x_0) = y_0 \in E$ where $f : R \times E \longrightarrow E$ is supposed to be continuous, is equivalent to one of the integral equations:*
$y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt, \qquad \forall x \in [x_0, x_1],$
or
$y_0 = y(x) + (-1). \int_{x_0}^x f(t, y(t))dt, \qquad \forall x \in [x_0, x_1],$
on some interval $(x_0, x_1) \subset R$, depending on the strong differentiability considered,

(i) or (ii), respectively.
Here the equivalence between two equations means that any solution of an equation is a solution for the other one,too.

Remark 2.11. [5], In the case of strongly generalized differentiability, to the fuzzy differential equation $y^{'} = f(x, y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the definition of H-differentiable, we may attach only one. The second integral equation in Lemma (2.10) can be written in the form $y(x) = y_0 \ominus (-1).\int_{x_0}^{x} f(t, y(t))dt$.

Proposition 2.12. [6], *If $f, g : (a, b) \to E$ are generalized differentiable at $x \in (a, b)$ in the same case of differentiability, then $f + g$ is generalized differentiable at x and $(f + g)^{'}(x) = f^{'}(x) + g^{'}(x)$.*

The following theorems concern the existence of solutions of a fuzzy initial-value problem under generalized differentiability (see [5]).

Theorem 2.13. *Let us suppose that the following conditions hold: (a)Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $p, q > 0$, $y_0 \in E$, where $\overline{B}(y_0, q) = \{y \in E : D(y, y_0) \leq q\}$ denote a closed ball in E and let $f : R_0 \longrightarrow E$ be a continuous function such that $d_\infty(\widetilde{0}, f(x, y)) = \|f(x, y)\| \leq M$ for all $(x, y) \in R_0$ (b) Let $g : [x_0, x_0 + p] \times [0, q] \longrightarrow E$, such that $g(x, 0) \equiv 0$ and $0 \leq g(x, u) \leq M_1$, $\forall x \in [x_0, x_0 + p]$, $0 \leq u \leq q$, such that $g(x, u)$ is non-decreasing in u and g is such that the initial-value problem $u^{'}(x) = g(x, u(x)), u(x_0) = 0$ has only the solution $u(x) \equiv 0$ on $[x_0, x_0 + p]$. (c) We have $d_\infty(f(x, y), f(x, z)) \leq g(x, d_\infty(y, z))$, $\forall (x, y), (x, z) \in R_0$ and $d_\infty(y, z) \leq q$. (d) There exists $d > 0$ such that for $x \in [x_0, x_0 + d]$ the sequence $\overline{y}_n : [x_0, x_0 + d] \longrightarrow E$ given by $\overline{y}_0(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^{x} f(t, \overline{y}_n)dt$ is defined for any $n \in N$. Then the fuzzy initial -value problem*

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions (one (i)-differentiable and the other one (ii)-differentiable) $y, \hat{y} : [x_0, x_0 + r] \longrightarrow B(y_0, q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ and the successive iterations

$$y_0(x) = y_0, y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

and

$$\hat{y}_0(x) = y_0, \hat{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^{x} f(t, \hat{y}_n(t))dt$$

converge to these two solutions respectively.

According to theorem (2.13), we restrict our attention to functions which are (i)- or (ii)-differentiable on their domain except for a finite number of points (see also [5]).
The following corollary gives simple sufficient condition for the existence of fuzzy differential equations under strongly generalized differentiability.

Corollary 2.14. *Let $f : R_0 \longrightarrow E$ where $R_0 = [x_0, x_0 + p] \times (\overline{B}(y_0, q) \cap E)$, and $y_0 \in E$ such that $y(0, 1) - \underline{y}(0, 0)$ and $\overline{y}(0, 0) - y(0, 1)$. Let $m = \min\{y(0, 1) -$*

$\underline{y}(0,0), \overline{y}(0,0) - \underline{y}(0,1)\}$. *Under the assumptions (a)-(c) of Theorem (2.13), the fuzzy initial-value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has two solutions $y, \overline{y} : [x_0, x_0 + r] \longrightarrow B(\widetilde{y}_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, \frac{m}{2M}\}$ *and the successive iterations in (2.13) converge to these two solutions.*

Lemma 2.15. [6], *The H-difference is a continuous function in both of argument.*

Definition 2.16. [2], Let $f : (a, b) \times E \to E$ and $x_0 \in (a, b)$. We Define the nth-order differential of f as follow: We say that f is strongly generalized differentiable of the $nth-$order at x_0. If there exists an element $f^{(s)}(x_0) \in E, \qquad \forall s = 1, \ldots, n$, such that

(i) for all $h > 0$ sufficiently small,

$$\exists f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0), \qquad \exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0)}{h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)}{h} = f^{(s)}(x_0)$$

or

(ii) for all $h > 0$ sufficiently small,

$$\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h), \qquad \exists f^{(s-1)}(x_0 - h) \ominus f^{(s-1)}(x_0)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0 - h) \ominus f(x_0)}{-h} = f^{(s)}(x_0)$$

or

(iii) for all $h > 0$ sufficiently small,

$$\exists f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0), \qquad \exists f^{(s-1)}(x_0 - h) \ominus f^{(s-1)}(x_0)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0)}{h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0 - h) \ominus f^{(s-1)}(x_0)}{-h} = f^{(s)}(x_0)$$

or

(iv) for all $h > 0$ sufficiently small,

$$\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h), \qquad \exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)$$

and the limits(in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)}{h} = f^{(s)}(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively $\forall s = 1 \ldots n$)

Lemma 2.17. [6], *for arbitrary* $(u, v) \in \overline{c}[0, 1] \times \overline{c}[0, 1]$ *we have:*

$$d_\infty(u \ominus w, u \ominus v) = d_\infty(w, v), \quad \forall u, v, w \in E$$

Definition 2.18. The fuzzy valued function $f : [a, b] \to E$ is bounded on $[a, b]$ if there exists a real value $M > 0$ such that

$$\|f\| \leq M$$

In this section, we are going to define the convergence, consistency and stability of the one-step method and compute its local truncation error.

3. Convergence, Consistency and Stability of the One-step Method

Consider the first-order fuzzy differential equation with initial value

$$\begin{cases} y^{'} = f(t, y), & 0 \leq t \leq b \\ y(0) = y_0 \end{cases} \tag{1}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, $y_0 \in E$ and y is generalized differentiable of $(n + 1)$-th order. Hence f is generalized differentiable of n-th order.

Definition 3.1. A one-step method for approximating the solution of fuzzy differential equation (1) is a method which can be written in the form

$$\begin{cases} y_{n+1} = y_n + h\phi(t_n, y_n, h), & n = 0, \ldots, N - 1 \\ y(0) = y_0 \end{cases} \tag{2}$$

if y would be (i)-differentiable and

$$\begin{cases} y_{n+1} = y_n \ominus (-1)h\phi(t_n, y_n, h), & n = 0, \ldots, N - 1 \\ y(0) = y_0 \end{cases} \tag{3}$$

if y would be (ii)-differentiable and the function ϕ is determined by h, t_n, y_n.

We have derived the following lemma from the Theorem (2.13).

Lemma 3.2. *If $f(t, y)$ is a continuous fuzzy function of t and satisfies the Lipschitz condition in y in region $0 \leq t \leq b$, $-\infty \leq y \leq \infty$, then equation (1) has the unique differentiable fuzzy solution $Y(t)$ such that*

$$Y^{'}(t) = f(t, Y(t)), \qquad Y(0) = y_0.$$

Thus sufficient conditions for the existence of a unique solution of equation (1) are:

1) Continuity of f,
2) Lipschitz condition

$$d_\infty(f(t, x), f(t, y)) \leq Ld_\infty(x, y), \quad L > 0$$

Definition 3.3. The one-step methods (2) and (3) are said to be convergent with respect to the fuzzy differential equation it approximates if

$$\lim_{n\to\infty} d_\infty(y_n, Y(t_n)) = 0, \qquad 0 \leq t \leq b \tag{4}$$

where $Y(t)$ denotes the exact solution and $y(t_n) := y_n$ is the approximation obtained from the difference method (2) and (3) at the $n-$th step.

Definition 3.4. A one-step method (2) or (3) is stable if for each fuzzy differential equation satisfying a Lipschitz condition with respect to the y, there exist positive constants h_0 and K such that two different solutions y_n and $\overline{y}_n$ each satisfying equation (1) are such that

$$d_\infty(y_n, \overline{y}_n) \le K d_\infty(y_0, \overline{y}_0), \quad 0 \le h \le h_0 \tag{5}$$

Before starting the next theorem, we note the following lemma which is needed in its proof.

Lemma 3.5. *For arbitrary $(u, v), (w, p) \in \overline{c}[0, 1] \times \overline{c}[0, 1]$, we have:*

$$d_\infty(u \ominus w, v \ominus p) \le d_\infty(u, v) + d_\infty(w, p) \quad \forall u, v, w, p \in E \tag{6}$$

Proof.

$$\begin{aligned}
d_\infty(u \ominus w, v \ominus p) &= \sup\{d_H((u \ominus w)[r], (v \ominus p)[r]) : 0 \le r \le 1\} \\
&= \sup\{\max\{|\underline{(u \ominus w)}[r] - \underline{(v \ominus p)}[r]|, \\
&\qquad\qquad |\overline{(u \ominus w)}[r] - \overline{(v \ominus p)}[r]|\} : 0 \le r \le 1\} \\
&= \sup\{\max\{|\underline{u}(r) - \underline{w}(r) - (\underline{v}(r) - \underline{p}(r))|, \\
&\qquad\qquad |\overline{u}(r) - \overline{w}(r) - (\overline{v}(r) - \overline{p}(r))|\} : 0 \le r \le 1\} \\
&\le \sup\{\max\{|\underline{u}(r) - \underline{v}(r)| + |\underline{p}(r) - \underline{w}(r)|, \\
&\qquad\qquad |\overline{u}(r) - \overline{v}(r)| + |\overline{p}(r) - \overline{w}(r))|\} : 0 \le r \le 1\} \\
&\le \sup\{\max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\} \\
&\qquad + \max\{|\underline{p}(r) - \underline{w}(r)|, |\overline{p}(r) - \overline{w}(r))|\} : 0 \le r \le 1\} \\
&\le \sup\{\max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\} : 0 \le r \le 1\} \\
&\quad + \sup\{\max\{|\underline{p}(r) - \underline{w}(r)|, |\overline{p}(r) - \overline{w}(r))|\} : 0 \le r \le 1\} \\
&= d_\infty(u, v) + d_\infty(w, p)
\end{aligned}$$

□

Theorem 3.6. *If $\phi(t, y, h)$ satisfy a Lipschitz condition in y, then the methods given by (2) and (3) are stable.*

Proof. First, let y is (i)-differentiable, therefore

$$\begin{aligned}
d_\infty(y_{n+1}, \overline{y}_{n+1}) &= d_\infty(y_n + h\phi(t_n, y_n, h), \overline{y}_n + h\phi(t_n, \overline{y}_n, h)) \\
&\le d_\infty(y_n, \overline{y}_n) + h d_\infty(\phi(t_n, y_n, h), \phi(t_n, \overline{y}_n, h)) \\
&\le d_\infty(y_n, \overline{y}_n) + hL d_\infty(y_n, \overline{y}_n) \\
&= (1 + hL) d_\infty(y_n, \overline{y}_n) = k_1 d_\infty(y_n, \overline{y}_n) \\
&\le k_1 k_2 d_\infty(y_{n-1}, \overline{y}_{n-1}) \le \cdots \le K d_\infty(y_0, \overline{y}_0)
\end{aligned} \tag{7}$$

Now, let y is (ii)-differentiable, by using lemma(3.5), we have

$$\begin{aligned} d_\infty(y_{n+1},\overline{y}_{n+1}) &= d_\infty(y_n \ominus (-1)h\phi(t_n,y_n,h), \overline{y}_n \ominus (-1)h\phi(t_n,\overline{y}_n,h)) \\ &\leq d_\infty(y_n,\overline{y}_n) + hd_\infty(\phi(t_n,y_n,h),\phi(t_n,\overline{y}_n,h)) \\ &\leq d_\infty(y_n,\overline{y}_n) + hLd_\infty(y_n,\overline{y}_n) \\ &= (1+hL)d_\infty(y_n,\overline{y}_n) = k_1 d_\infty(y_n,\overline{y}_n) \\ &\leq k_1k_2 d_\infty(y_{n-1},\overline{y}_{n-1}) \leq \cdots \leq K d_\infty(y_0,\overline{y}_0). \end{aligned} \tag{8}$$

□

Definition 3.7. The one-step method (2) has the local truncation error

$$\begin{cases} \frac{Y(t_n+h)\ominus Y(t_n)}{h} \ominus \phi(t_n,Y(t_n),h) = \tau_n(h), \\ \\ or \\ \\ \frac{Y(t_n)\ominus Y(t_n-h)}{h} \ominus \phi(t_n-h,Y(t_n-h),h) = \gamma_n(h) \end{cases} \tag{9}$$

for each $n = 0,\ldots,N-1$ and the one-step method (3) has the local truncation error

$$\begin{cases} \frac{Y(t_n)\ominus Y(t_n+h)}{-h} \ominus \phi(t_n+h,Y(t_n+h),h) = \tau_n(h), \\ \\ or \\ \\ \frac{Y(t_n-h)\ominus Y(t_n)}{-h} \ominus \phi(t_n,Y(t_n),h) = \gamma_n(h) \end{cases} \tag{10}$$

for each $n = 0,\ldots,N-1$.

Definition 3.8. A one-step method (2) or (3) with the local truncation error $\tau_n(h)$ and $\gamma_n(h)$ at the n-th step said to be consistent with the differential equation that it approximates if

$$\begin{aligned} &\lim_{h\to 0}\tau_n(h) = \tilde{0}, \\ &\lim_{h\to 0}\gamma_n(h) = \tilde{0}, \qquad n = 1,\ldots,N. \end{aligned} \tag{11}$$

Definition 3.9. The local truncation error of one-step method (2) is in the order (h^α) where $\alpha = \min\{q,p\}$ and

$$\tau_n(h) = O(h^q), \qquad \gamma_n(h) = O(h^p).$$

If we suppose that

$$\phi(t_n,y_n,h) = f(t_n,y_n) + \frac{h}{2}f^{'}(t_n,y_n) + \cdots + \frac{h^{n-1}}{n!}f^{(n)}(\xi_n,y(\xi_n)) \tag{12}$$

so, the difference method corresponding to (2) and (3) are called the Taylor method of order n and in particular, the Taylor method of order 1 is called the Euler method. Therefore, we have

Corollary 3.10. *If the Taylor method of order n is used to approximate the solution equation (1) with step size h and if $Y \in C^{n+1}[0,b]$, then the local truncation error is $O(h^n)$.*

Proof. Y is generalized differentiable of $(n+1)th$-order and since we just need the kind of the differentiability of Y at t_n (the kind of differentiability of the $Y^{(s)}$, $s = 1, \ldots, n+1$, is not important) then without loss of generality we can consider Y is (i)-differentiable at t_n, so by using the Taylor polynomial, we have:

$$\begin{aligned} Y(t_{n+1}) = Y(t_n + h) &= Y(t_n) + hY^{'}(t_n) + \tfrac{h^2}{2}Y^{''}(t_n) + \cdots + \tfrac{h^n}{n!}Y^{(n)}(t_n) \\ &+ \tfrac{h^{n+1}}{(n+1)!}Y^{(n+1)}(\xi_n), \end{aligned} \tag{13}$$

where $\xi_n \in (t_n, t_n + h)$ and

$$\begin{aligned} Y(t_n) &= Y(t_n - h) + hY^{'}(t_n - h) + \tfrac{h^2}{2}Y^{''}(t_n - h) + \cdots + \tfrac{h^n}{n!}Y^{(n)}(t_n - h) \\ &+ \tfrac{h^{n+1}}{(n+1)!}Y^{(n+1)}(\xi_{n-1}), \end{aligned} \tag{14}$$

where $\xi_{n-1} \in (t_n - h, t_n)$. If we define:

$$\begin{aligned} &\phi(t_n, Y(t_n), h) = Y^{'}(t_n) + \tfrac{h}{2}Y^{''}(t_n) + \cdots + \tfrac{h^{n-1}}{n!}Y^{(n)}(t_n), \\ &\phi(t_n - h, Y(t_n - h), h) = Y^{'}(t_n - h) + \tfrac{h}{2}Y^{''}(t_n - h) + \cdots + \tfrac{h^{n-1}}{n!}Y^{(n)}(t_n - h), \end{aligned} \tag{15}$$

therefore we have:

$$\begin{aligned} &Y(t_n + h) = Y(t_n) + h\phi(t_n, Y(t_n), h) + \tfrac{h^{n+1}}{(n+1)!}Y^{(n+1)}(\xi_n), \\ &Y(t_n) = Y(t_n - h) + h\phi(t_n - h, Y(t_n - h), h) + \tfrac{h^{n+1}}{(n+1)!}Y^{(n+1)}(\xi_{n-1}), \end{aligned} \tag{16}$$

thus the local truncation error is obtained as follows:

$$\begin{cases} \tau_n(h) = \frac{Y(t_n+h) \ominus Y(t_n)}{h} \ominus \phi(t_n, Y(t_n), h), \\ \gamma_n(h) = \frac{Y(t_n) \ominus Y(t_n - h)}{h} \ominus \phi(t_n - h, Y(t_n - h), h). \end{cases} \tag{17}$$

Since $Y \in C^{n+1}[0,b]$, we have $Y^{n+1}(t) = f^n(t, Y(t))$ bounded on $[0,b]$ and

$$\tau_n(h) = O(h^n), \qquad \gamma_n(h) = O(h^n).$$

Now, we suppose that Y is (ii)-differentiable at t_n (we just need the kind of the differentiability of Y at t_n and the kind of differentiability of the $Y^{(s)}$, $s = 1, \ldots, n+1$, is not important) and we define

$$(\ominus 1)^n = \begin{cases} +1, & if\ n\ is\ even \\ -1, & if\ n\ is\ odd \end{cases} \tag{18}$$

In this paper, the sign " $\ominus b$ " always stands for " $\tilde{0} \ominus b$ " and notes that $\ominus b = \tilde{0} \ominus b \neq \tilde{0} + (-b)$.
So by using the Taylor polynomial, we have:

$$\begin{aligned} Y(t_n) &= Y(t_n + h) + (\ominus 1)hY^{'}(t_n + h) + \tfrac{(\ominus 1)^2 h^2}{2}Y^{''}(t_n + h) + \cdots \\ &+ \tfrac{(\ominus 1)^n h^n}{n!}Y^{(n)}(t_n + h) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!}Y^{(n+1)}(\upsilon_n) + \cdots \end{aligned} \tag{19}$$

where $\upsilon_n \in (t_n, t_n + h)$ and

$$\begin{aligned} Y(t_n - h) &= Y(t_n) + (\ominus 1)hY'(t_n) + \tfrac{(\ominus 1)^2 h^2}{2} Y''(t_n) + \cdots \\ &+ \tfrac{(\ominus 1)^n h^n}{n!} Y^{(n)}(t_n) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!} Y^{(n+1)}(\upsilon_{n-1}) + \cdots \end{aligned} \tag{20}$$

where $\upsilon_{n-1} \in (t_n - h, t_n)$. If we define:

$$\begin{aligned} \phi(t_n + h, Y(t_n + h), h) &= Y'(t_n + h) + \tfrac{(\ominus 1)h}{2} Y''(t_n + h) \\ &+ \cdots + \tfrac{(\ominus 1)^{n-1} h^{n-1}}{n!} Y^{(n)}(t_n + h), \end{aligned}$$

$$\phi(t_n, Y(t_n), h) = Y'(t_n) \quad + \tfrac{(\ominus 1)h}{2} Y''(t_n) + \cdots + \tfrac{(\ominus 1)^{n-1} h^{n-1}}{n!} Y^{(n)}(t_n - h) \tag{21}$$

therefore, we have:

$$\begin{aligned} Y(t_n) &= Y(t_n + h) + (\ominus 1)h\phi(t_n + h, Y(t_n + h), h) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!} Y^{(n+1)}(\upsilon_n) + \cdots \\ &= Y(t_n + h) + (-1)h\phi(t_n + h, Y(t_n + h), h) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!} Y^{(n+1)}(\upsilon_n) + \cdots \end{aligned} \tag{22}$$

and

$$\begin{aligned} Y(t_n - h) &= Y(t_n) + (\ominus 1)h\phi(t_n, Y(t_n), h) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!} Y^{(n+1)}(\upsilon_{n-1}) + \cdots \\ &= Y(t_n) + (-1)h\phi(t_n, Y(t_n), h) + \tfrac{(\ominus 1)^{n+1} h^{n+1}}{(n+1)!} Y^{(n+1)}(\upsilon_{n-1}) + \cdots \end{aligned} \tag{23}$$

thus the local truncation error is obtained as follows:

$$\begin{cases} \tau_n(h) = \frac{Y(t_n) \ominus Y(t_n + h)}{-h} \ominus \phi(t_n + h, Y(t_n + h), h), \\ \\ \gamma_n(h) = \frac{Y(t_n - h) \ominus Y(t_n)}{-h} \ominus \phi(t_n, Y(t_n), h). \end{cases} \tag{24}$$

Since $Y \in C^{n+1}[0, b]$, we have $Y^{n+1}(t) = f^n(t, Y(t))$ bounded on $[0, b]$ and

$$\tau_n(h) = O(h^n), \qquad \gamma_n(h) = O(h^n).$$

□

Lemma 3.11. [1], *Let the sequence of numbers* $\{w_n\}_{n=0}^N$ *satisfies*

$$|w_{n+1}| \leq A|w_n| + B, \qquad 0 \leq n \leq N - 1,$$

for the given positive constants A *and* B*. Then*

$$|w_n| \leq A^n |w_0| + B\frac{A^n - 1}{A - 1}, \qquad 0 \leq n \leq N.$$

Theorem 3.12. *If* $\phi(t, y, h)$ *is continuous in* y, t *and* h *for* $0 \leq h \leq h_0$, $0 \leq t \leq b$ *and all* y *and if it satisfies a Lipschitz condition in* y, t *and* h *in that region, a necessary and sufficient condition for convergence is that*

$$\phi(t, y, 0) = f(t, y). \tag{25}$$

Equation (25) is called the condition of consistency.

Proof. Let $\phi(t, y, 0) = g(t, y)$. Since g satisfies the conditions of lemma (3.2), the fuzzy differential equation

$$z^{'} = g(t, z), \qquad z_0 = y_0, \tag{26}$$

has a unique differentiable solution.
We will show that, if the numerical solution given by (2) is (i)-differentiable then it converges to $z(t)$ and if the numerical solution given by (3) is (ii)-differentiable then it converges to $z(t)$ and hence that $f = g$ is a necessary and sufficient condition. Let the numerical solution is (i)-differentiable and it satisfies

$$y_{n+1} = y_n + h\phi(t_n, y_n, h). \tag{27}$$

By the mean value theorem, we have

$$z(t_{n+1}) = z(t_n) + hg(t_n + \theta_n h, z(t_n + \theta_n h)), \quad 0 < \theta_n < 1. \tag{28}$$

Denoting $e_n = d_\infty(y_n, z(t_n))$, hence

$$\begin{aligned}
e_{n+1} \quad & \leq e_n + hd_\infty(\phi(t_n, y_n, h), g(t_n + \theta_n h, z(t_n + \theta_n h))) \\
& = e_n + hd_\infty[\phi(t_n, y_n, h) + \phi(t_n, z(t_n), h) + \phi(t_n, z(t_n), 0) \\
& \qquad , g(t_n + \theta_n h, z(t_n + \theta_n h)) + \phi(t_n, z(t_n), h) + \phi(t_n, z(t_n), 0)] \\
& \leq e_n + hd_\infty(\phi(t_n, y_n, h), \phi(t_n, z(t_n), h)) \\
& + hd_\infty(\phi(t_n, z(t_n), 0), \phi(t_n, z(t_n), h)) \\
& + hd_\infty(\phi(t_n, z(t_n), 0), g(t_n + \theta_n h, z(t_n + \theta_n h))) \\
& \leq e_n + hLe_n + h^2 L_1 + hd_\infty(g(t_n, z(t_n)), g(t_n + \theta_n h, z(t_n + \theta_n h))) \\
& = e_n(1 + Lh) + h^2 L_1 \\
& + hd_\infty[g(t_n, z(t_n)) + g(t_n, z(t_n + \theta_n h)) \\
& \qquad \qquad , g(t_n + \theta_n h, z(t_n + \theta_n h)) + g(t_n, z(t_n + \theta_n h))] \\
& \leq e_n(1 + Lh) + h^2 L_1 + hd_\infty(g(t_n, z(t_n)), g(t_n, z(t_n + \theta_n h))) \\
& + hd_\infty(g(t_n, z(t_n + \theta_n h)), g(t_n + \theta_n h, z(t_n + \theta_n h))) \\
& \leq e_n(1 + Lh) + h^2 L_1 + hLd_\infty(z(t_n), z(t_n + \theta_n h)) + L_3 \theta_n h^2 \\
& \leq e_n(1 + Lh) + h^2(L_1 + L_2).
\end{aligned} \tag{29}$$

From lemma (3.11), we obtain

$$e_N \leq (L_1 + L_2)h\frac{e^{Lb} - 1}{L} + e^{Lb}e_0. \tag{30}$$

This converges to zero as h and $e_0 \to 0$, so the numerical solution converges to the solution of (26). Sufficiently of the condition $g(t, y) = f(t, y)$ follows immediately. If, on the other hand, we have convergence, then $z(t)$, the solution of (26), is identical to $y(t)$, the solution of $y^{'}(t) = f(t, y(t))$. Suppose also that f and g are different at some point (t_α, y_α). If we consider the initial fuzzy value problem starting from (t_α, y_α), we have

$$y^{'}(t_\alpha) = f(t_\alpha, y(t_\alpha)) \neq g(t_\alpha, y(t_\alpha)) = g(t_\alpha, z(t_\alpha)) = z^{'}(t_\alpha),$$

leading to a contradiction. Now, let the numerical solution is (ii)-differentiable and it satisfies

$$y_{n+1} = y_n \ominus (-1)h\phi(t_n, y_n, h). \tag{31}$$

By the mean value theorem, we have

$$z(t_{n+1}) = z(t_n) \ominus (-1)hg(t_n + \theta_n h, z(t_n + \theta_n h)), \quad 0 < \theta_n < 1. \tag{32}$$

therefore,

$$\begin{aligned} d_\infty(y_{n+1}, z(t_{n+1})) = d_\infty(\quad & y_n \ominus (-1)h\phi(t_n, y_n, h), \\ & z(t_n) \ominus (-1)hg(t_n + \theta_n h, z(t_n + \theta_n h))). \end{aligned} \tag{33}$$

Denoting $e_n = d_\infty(y_n, z(t_n))$ and by using lemma (3.5), we have

$$\begin{aligned} e_{n+1} \quad & \leq e_n + hd_\infty(\phi(t_n, y_n, h), g(t_n + \theta_n h, z(t_n + \theta_n h))) \\ & \leq e_n(1 + Lh) + h^2(L_1 + L_2) \end{aligned} \tag{34}$$

and the remainder of proof is similar to the previous case. □

Lemma 3.13. *For arbitrary $u \in \overline{c}[0,1]$, if $au \ominus bu$ exists, then we have*

$$au \ominus bu = (a - b)u, \qquad \forall a, b \in R \ \& \ a > b > 0$$

Proof.

$$\begin{aligned} d_\infty(au \ominus bu, (a-b)u) \quad & = \sup\{d_H((au \ominus bu)[r], ((a-b)u)[r]) : 0 \leq r \leq 1\} \\ & = \sup\{\max\{|\underline{(au \ominus bu)}[r] - (\underline{(a-b)u})[r]|, \\ & \qquad |\overline{(au \ominus bu)}[r] - (\overline{(a-b)u})[r]|\} : 0 \leq r \leq 1\} \\ & = \sup\{\max\{|a\underline{u}[r] - b\underline{u}[r] - (a-b)\underline{u}[r]|, \\ & \qquad |a\overline{u}[r] - b\overline{u}[r] - (a-b)\overline{u}[r]|\} : 0 \leq r \leq 1\} \\ & = 0 \end{aligned} \tag{35}$$

□

4. **Fuzzy Stiff Differential Equations**

In this section, we are going to study the convergence, the consistency and the stability of Euler method and Taylor method of order 2 for approximating the solution of the fuzzy stiff differential equation.

The fuzzy stiff differential equations are characterized as those whose exact solution has a term of the form $e^{-\lambda t}$, where λ is a large positive crisp number. This is usually only a part of the solution which is called transient solution. The transient portion of a stiff equation will rapidly decreases to zero as t increases. Let us consider the fuzzy stiff differential equation

$$y^{'}(t) = -\lambda y(t), \qquad y(0) = y_0, \tag{36}$$

where y_0 is a fuzzy number.

An exact unique fuzzy solution is given by the $r-$level intervals $[Y_1^r(t), Y_2^r(t)]$, $0 \leq r \leq 1$, $0 \leq t \leq b$, where

$$\begin{aligned} Y_1^r(t) \quad & = \tfrac{y_{01}^r + y_{02}^r}{2} e^{-\lambda t} + \tfrac{y_{01}^r - y_{02}^r}{2} e^{\lambda t}, \\ Y_2^r(t) \quad & = \tfrac{y_{01}^r + y_{02}^r}{2} e^{-\lambda t} - \tfrac{y_{01}^r - y_{02}^r}{2} e^{\lambda t}. \end{aligned} \tag{37}$$

By theorem (6) in [6], $Y(t) = y_0 e^{-\lambda t}$ is (ii)-differentiable and it is the solution of the equation (36). obviously, it has a decreasing length of its support.
If we solve equation (36) by using the approach based on the Hukuhara differentiability concept and $y_0 = (1, 2, 8)$, we get

$$Y(t) = (cosh\lambda t - 8sinh\lambda t, 2cosh\lambda t - 2sinh\lambda t, 8cosh\lambda t - sinh\lambda t) \tag{38}$$

that it has an increasing length of its support. Now let $\lambda = 2$, then

$$Y(t) = (cosh2t - 8sinh2t, 2cosh2t - 2sinh2t, 8cosh2t - sinh2t) \tag{39}$$

and it has the representation in Figure 1.

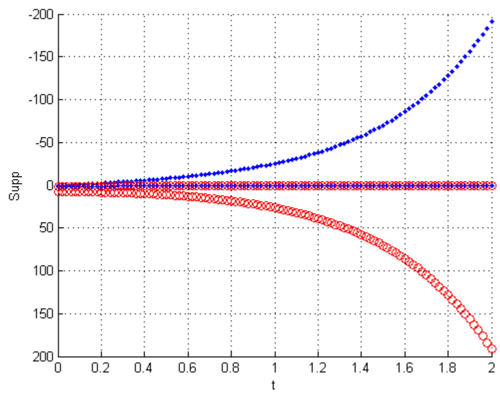

Figure 1

First consider Euler method applied to approximate equation (36). Letting $h = \frac{b}{N}$ and $t_j = jh$, $j = 0, \ldots, N$, equation (3) implies that

$$\begin{aligned} y_{n+1} &= y_n \ominus (-1)h(-\lambda)y_n, \quad n = 0, \ldots, N-1, \\ &= y_n \ominus \lambda h y_n. \end{aligned} \tag{40}$$

If $\lambda h < 1$, then by using lemma (3.13), we have

$$\begin{aligned} y_{n+1} &= y_n \ominus \lambda h y_n, \quad n = 0, \ldots, N-1, \\ &= (1 - \lambda h)y_n = \cdots = (1 - \lambda h)^{n+1} y_0, \end{aligned} \tag{41}$$

therefore,

$$\begin{aligned} \lim_{n\to\infty} d_\infty(Y(t_n), y_n) &= \lim_{n\to\infty} d_\infty(y_0 e^{-\lambda nh}, y_0(1-\lambda h)^n) \\ &= \lim_{n\to\infty} \sup\{\max\{|e^{-\lambda nh}\underline{y}_0[r] - (1-\lambda h)^n \underline{y}_0[r]|, \\ &\qquad |e^{-\lambda nh}\overline{y}_0[r] - (1-\lambda h)^n \overline{y}_0[r]|\} : 0 \le r \le 1\} \\ &= \lim_{n\to\infty} \sup\{\max\{|e^{-\lambda nh} - (1-\lambda h)^n||\underline{y}_0[r]|, \\ &\qquad |e^{-\lambda nh} - (1-\lambda h)^n||\overline{y}_0[r]|\} : 0 \le r \le 1\} = 0, \end{aligned} \tag{42}$$

thus, if $h < \frac{1}{\lambda}$, the one-step method (3) is convergent. Suppose now that initial condition is

$$y(0) = y_0 + \delta_0 = \tilde{y}_0, \qquad \delta_0 \in E.$$

By considering $\lambda h < 1$, at the nth step, we have

$$\tilde{y}_n = (1-\lambda h)^n \tilde{y}_0 = (1-\lambda h)^n(\delta_0 + y_0),$$

therefore

$$\begin{aligned} d_\infty(y_n, \tilde{y}_n) &= d_\infty((1-\lambda h)^n y_0, (1-\lambda h)^n \tilde{y}_0) \\ &= d_\infty((1-\lambda h)^n y_0, (1-\lambda h)^n(\delta_0 + y_0)) \\ &= d_\infty((1-\lambda h)^n y_0, (1-\lambda h)^n \delta_0 + (1-\lambda h)^n y_0)) \\ &\leq d_\infty(\tilde{0}, (1-\lambda h)^n \delta_0) \\ &= \|(1-\lambda h)^n \delta_0\| = (1-\lambda h)^n \|\delta_0\| = K d_\infty(y_0, \tilde{y}_0), \end{aligned} \tag{43}$$

also, if $h < \frac{1}{\lambda}$ then the one-step method (3) is stable.
If suppose the Taylor method of order 2, $h = \frac{b}{N}$ and $t_n = nh$, $j = 0, \ldots, N$, equation (3) implies that

$$\begin{aligned} y_{n+1} &= y_n \ominus ((-1)h(-\lambda)y_n + (-1)\tfrac{h^2}{2}\lambda^2 y_n), \quad n = 0, \ldots, N-1 \\ &= y_n \ominus (\lambda h - \tfrac{\lambda^2 h^2}{2})y_n. \end{aligned} \tag{44}$$

If $h < \frac{1}{\lambda}$, then $0 < \lambda h - \frac{\lambda^2 h^2}{2} < 1$, thus the one-step method (3) is convergent, stable and

$$\phi(t_n, y_n, 0) = f(t_n, y_n),$$

therefore, it is also consistent.

5. **Example**

Example 5.1. Consider the following fuzzy differential equation

$$y^{'}(t) = ty(t), \qquad y(t_0) = y_0, \tag{45}$$

where y_0 is a fuzzy number.
By Theorem (6) in [6], $Y(t) = y_0 t e^{\frac{t^2}{2}}$ is (i)-differentiable and it is the solution of the equation (45).
Consider Euler method applied to approximate equation (45). Letting $h = \frac{b}{N}$ and $t_j = jh$, $j = 0, \ldots, N$, equation (45) implies that

$$y_{n+1} = y_n + ht_n y_n, \quad n = 0, \ldots, N-1.$$

Then using Lemma (3.13), we have

$$\begin{aligned} y_{n+1} &= y_n + ht_n y_n, \quad n = 0, \ldots, N-1, \\ &= (1 + nh^2)y_n \\ &\vdots \\ &= (1+h^2)(1+2h^2)\cdots(1+nh^2)y_0. \end{aligned}$$

Since $h = \frac{b-a}{n}$, so $n \to \infty$ equivalent to $h \to 0$. Therefore,

$$\lim_{n\to\infty} d_\infty(Y(t_n), y_n) = \lim_{n\to\infty} d_\infty(y_0 e^{\frac{n^2h^2}{2}}, y_0(1+h^2)(1+2h^2)\cdots(1+nh^2))$$

$$= \lim_{h\to 0} \sup\{\max\{|e^{\frac{n^2h^2}{2}} \underline{y}_0[r] - (1+h^2)(1+2h^2)\cdots(1+nh^2)\underline{y}_0[r]|,$$

$$|e^{\frac{n^2h^2}{2}} \overline{y}_0[r] - (1+h^2)(1+2h^2)\cdots(1+nh^2)\overline{y}_0[r]|\} : 0 \le r \le 1\}$$

$$= \lim_{h\to 0} \sup\{\max\{|e^{\frac{n^2h^2}{2}} - (1+h^2)(1+2h^2)\cdots(1+nh^2)||\underline{y}_0[r]|,$$

$$|e^{\frac{n^2h^2}{2}} - (1+h^2)(1+2h^2)\cdots(1+nh^2)||\overline{y}_0[r]|\} : 0 \le r \le 1\}$$

$$= 0$$

then for each sufficient small h, the one-step method (3) is convergent.
Suppose now that initial condition is

$$y(0) = y_0 + \delta_0 = \tilde{y}_0, \qquad \delta_0 \in E.$$

At the nth step, we have

$$\tilde{y}_n = (1+h^2)(1+2h^2)\cdots(1+nh^2)\tilde{y}_0 = (1+h^2)(1+2h^2)\cdots(1+nh^2)(\delta_0 + y_0),$$

therefore

$$d_\infty(y_n, \tilde{y}_n)$$

$$= d_\infty((1+h^2)(1+2h^2)\cdots(1+nh^2)y_0, (1+h^2)(1+2h^2)\cdots(1+nh^2)\tilde{y}_0)$$

$$= d_\infty((1+h^2)(1+2h^2)\cdots(1+nh^2)y_0, (1+h^2)(1+2h^2)\cdots(1+nh^2)(\delta_0 + y_0))$$

$$= d_\infty((1+h^2)(1+2h^2)\cdots(1+nh^2)y_0,$$

$$(1+h^2)(1+2h^2)\cdots(1+nh^2)\delta_0 + (1+h^2)(1+2h^2)\cdots(1+nh^2)y_0))$$

$$\le d_\infty(\tilde{0}, (1+h^2)(1+2h^2)\cdots(1+nh^2)\delta_0)$$

$$= \|(1+h^2)(1+2h^2)\cdots(1+nh^2)\delta_0\|$$

$$= (1+h^2)(1+2h^2)\cdots(1+nh^2)\|\delta_0\|$$

$$= K d_\infty(y_0, \tilde{y}_0)$$

then for each sufficient small h, the one-step method (3) is stable. Also, since

$$\phi(t_n, y_n, 0) = f(t_n, y_n)$$

therefore, it is also consistent.

6. **Conclusions**

We considered fuzzy initial value problems and studied involving generalized H-differentiability and defined local truncation error. Then we provided sufficient conditions for convergence, consistency and stability of difference method. Finally we illustrated the proposed properties by remarking the fuzzy stiff differential equation.

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