

INTERVAL TYPE-2 FUZZY ROUGH SETS AND INTERVAL TYPE-2 FUZZY CLOSURE SPACES

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ABSTRACT. The purpose of the present work is to establish a one-to-one correspondence between the family of interval type-2 fuzzy reflexive/tolerance approximation spaces and the family of interval type-2 fuzzy closure spaces.

1. Introduction

Rough set theory, firstly proposed by Pawlak [14] has now developed significantly due to its importance for the study of intelligent systems having insufficient and incomplete information. In rough sets introduced by Pawlak, the key role is played by equivalence relations. In literature (cf. [7, 21]), several generalizations of rough sets have been made by replacing the equivalence relation by an arbitrary relation. After introduction of fuzzy rough sets by Dubois and Prade [1] as a fuzzy generalization of rough sets by replacing crisp binary relations by fuzzy relations, the relationship between fuzzy rough sets, fuzzy preorders and fuzzy topological spaces were studied in [6, 15, 17, 22]. Also, independent to fuzzy rough set theory, the relationship between fuzzy preorders and fuzzy closure operators were studied in [2, 3, 4], while the relationship between fuzzy preorders and fuzzy topologies were studied in [5, 8]. One central observation in the studies done in [8, 15, 17, 22] is as follows. Given a Kuratowski fuzzy closure operator c on a nonempty set X , there exists a fuzzy preorder (fuzzy reflexive and fuzzy transitive relation) R on X such that for all fuzzy subsets A of X , $c(A) = \overline{R}(A)$, where $\overline{R}(A)(x) = \bigvee \{R(x, y) \wedge A(y) : y \in X\}$, $\forall x \in X$ iff (i) $c(\bigvee_{i \in J} A_i) = \bigvee_{i \in J} c(A_i)$, and (ii) $c(A \wedge \overline{\alpha}) = c(A) \wedge \overline{\alpha}$, $\alpha \in [0, 1]$. This shows that there is a one-to-one correspondence between the family of all fuzzy preorders and the family of all fuzzy topological spaces satisfying (i) and (ii). At this point, we also wish to mention that the condition (ii) has stated differently by different authors, which have been shown equivalent in [19].

After the introduction of the notion of type-2 fuzzy sets by Zadeh [24], the concept of type-2 fuzzy sets are gaining increasing popularity because they can improve certain types of inferences better than ordinary (type-1) fuzzy sets [23] with increasing imprecision, uncertainty, and fuzziness in information (c.f., [11, 13]). Also, the computational complexity of using general type-2 fuzzy sets is very high, which makes it very difficult to deploy them in practical applications. Hence, in

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practical fields, the researcher use only interval type-2 (IT2) fuzzy sets (special case of general type-2 fuzzy sets) [9, 20]. After the advent of IT2 fuzzy sets, Wu, Wu and Luo [20] and Zhang [25], recently introduced the concept of IT2 fuzzy rough sets by combining rough set theory with IT2 fuzzy set theory and also established the relationship between IT2 fuzzy rough sets and IT2 fuzzy topological spaces. Also, inspired from [8, 15, 17, 22], in these studies, it has shown that there is a one-to-one correspondence between the family of IT2 fuzzy preorders (IT2 fuzzy reflexive and IT2 fuzzy transitive relation) and the family of all IT2 fuzzy topological spaces satisfying the conditions similar to (i) and (ii) stated above. In the above cited works, there is a silence on the similar correspondence between the family of IT2 fuzzy reflexive/tolerance approximation spaces and the family of some IT2 fuzzy topological spaces. Through out this paper, we try to fill this gap by introducing the concept of IT2 fuzzy closure spaces (a generalization of the concept of closure spaces [10, 18]).

2. Preliminaries

In this section, we recall some concepts related to type-2 fuzzy sets, IT2 fuzzy sets, IT2 fuzzy relation, IT2 fuzzy rough sets, IT2 fuzzy topology and collect some results, which we need in the subsequent sections.

Throughout this paper, X is a nonempty set, $I = [0, 1]$ and $[I] = \{[a, b] : a \leq b, a, b \in I\}$. Also, for $[a_i, b_i] \in [I], i = 1, 2$,

- i) $[a_1, b_1] = [a_2, b_2] \Leftrightarrow a_1 = a_2, b_1 = b_2$,
- ii) $[a_1, b_1] \leq [a_2, b_2] \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2$,
- iii) $[a_1, b_1] < [a_2, b_2] \Leftrightarrow [a_1, b_1] \leq [a_2, b_2]$ and $[a_1, b_1] \neq [a_2, b_2]$.

Now, we begin with the following.

Definition 2.1. [11] A type-2 fuzzy set \tilde{A} is characterized by a type-2 membership function $\mu_{\tilde{A}} : X \times J_x \rightarrow I, \forall x \in X$ and $J_x \subseteq I$, i.e.,

$$\tilde{A} = \{(x, u), \mu_{\tilde{A}}(x, u) : x \in X, u \in J_x \subseteq I\},$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. \tilde{A} can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u), J_x \subseteq I,$$

where $\int \int$ denotes the union over all admissible x and u .

Definition 2.2. [11] Let \tilde{A} be a type-2 fuzzy set in X . Then for each $x' \in X$, a vertical slice $\mu_{\tilde{A}}(x')$ of \tilde{A} is the intersection between the two-dimensional plane whose axes are $u \in J_{x'}$ and $\mu_{\tilde{A}}(x', u)$ and the three dimensional type-2 membership function \tilde{A} , i.e.,

$$\mu_{\tilde{A}}(x') = \mu_{\tilde{A}}(x = x', u) = \int_{u \in J_{x'}} f_{x'}(u) / u, J_{x'} \subseteq I,$$

in which $0 \leq f_{x'}(u) \leq 1$. In terms of vertical slice, a type-2 fuzzy set \tilde{A} can also be re-expressed as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

or, as the following:

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x)/x = \int_{x \in X} \left[\int_{u \in J_x} f_x(u)/u \right] /x, J_x \subseteq I,$$

where $f_x(u) = \mu_{\tilde{A}}(x, u)$.

Definition 2.3. [20, 25] The footprint of uncertainty, denoted by $D\tilde{A}$, of a type-2 fuzzy set \tilde{A} is given by

$$D\tilde{A} = \bigcup_{x \in X} J_x.$$

Let $D\tilde{A}(x) = J_x, \forall x \in X$. Then a type-2 fuzzy set \tilde{A} can be re-expressed as:

$$\tilde{A} = \int \int_{(x,u) \in D\tilde{A}} \mu_{\tilde{A}}(x, u)/(x, u).$$

For given type-2 fuzzy set \tilde{A} , a lower and an upper membership function are the two type-1 membership functions that are the bounds of $D\tilde{A}$. The lower membership function (denoted as $\underline{D}\tilde{A}$) is associated with the lower bound of $D\tilde{A}$, and the upper membership function (denoted as $\overline{D}\tilde{A}$) is associated with the upper bound of $D\tilde{A}$.

Definition 2.4. [9, 12] A type-2 fuzzy set \tilde{A} is called an interval type-2 fuzzy set (IT2 fuzzy set) if $\mu_{\tilde{A}}(x, u) = 1, \forall x \in X$ and $\forall u \in J_x \subseteq I$.

An IT2 fuzzy set \tilde{A} can be expressed as follows:

$$\tilde{A} = \{((x, u), 1) : x \in X, u \in J_x\}$$

or as:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x, u), J_x \subseteq I.$$

For a nonempty set X , $\tilde{F}_{IT2}(X)$ will denote the set of all IT2 fuzzy sets in X .

For $\tilde{A} \in \tilde{F}_{IT2}(X)$ and $x \in X$, $\mu_{\tilde{A}}(x)$ is an interval type-1 fuzzy set on I . Thus $D\tilde{A}(x) = [\underline{D}\tilde{A}(x), \overline{D}\tilde{A}(x)]$, where $\underline{D}\tilde{A}(x)$ and $\overline{D}\tilde{A}(x)$ are lower and upper membership functions (both of which are type-1 fuzzy sets) respectively. For simplicity, let $D\tilde{A}(x) = [l_{\tilde{A}}(x), r_{\tilde{A}}(x)]$. A constant IT2 fuzzy set $[\overline{\alpha}, \overline{\beta}] = \int_{x \in X} \int_{u \in [\alpha, \beta]} 1/(x, u)$ with $0 \leq \alpha \leq \beta \leq 1$. The IT2 fuzzy universe set is $\tilde{1}_X = X = [1, 1]$ and the IT2 fuzzy empty set is $\phi = \int_{x \in X} \int_{u \in I} 0/(x, u)$. For $y \in X$, an IT2 fuzzy singleton set $\tilde{1}_y = \{y\}$ is defined as follows:

$$\mu_{\underline{D}\tilde{1}_y}(x) = \mu_{\overline{D}\tilde{1}_y}(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Now, we recall the following operations on IT2 fuzzy sets from [25].

Let $\tilde{A}, \tilde{B} \in \tilde{F}_{IT2}(X)$, $\mu_{\tilde{A}}(x) = \int_{u \in [l_1, r_1]} 1/u$ and $\mu_{\tilde{B}}(x) = \int_{v \in [l_2, r_2]} 1/v, \forall x \in X$. Then the union, intersection and complement of \tilde{A} and \tilde{B} are respectively defined as follows:

$$\text{i) } \tilde{A} \cup \tilde{B} \Leftrightarrow \mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) = \int_{q \in [(l_1 \vee l_2), (r_1 \vee r_2)]} 1/q, \forall x \in X,$$

- ii) $\tilde{A} \cap \tilde{B} \Leftrightarrow \mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x) = \int_{q \in [(t_1 \wedge t_2), (r_1 \wedge r_2)]} 1/q, \forall x \in X,$
 iii) $\sim \tilde{A} \Leftrightarrow \mu_{\sim \tilde{A}}(x) = \neg \mu_{\tilde{A}}(x) = \int_{q \in [1 - \mu_{\overline{\tilde{A}}}(x), 1 - \mu_{\underline{\tilde{A}}}(x)]} 1/q, \forall x \in X.$

where, \vee represents the *max* t-conorm and \wedge denotes the *min* t-norm.

Definition 2.5. [20] Let $\tilde{A}, \tilde{B} \in \tilde{F}_{IT_2}(X)$. Then $\tilde{A} \preceq \tilde{B}$ if $\underline{D}\tilde{A} \subseteq \underline{D}\tilde{B}$ and $\overline{D}\tilde{A} \subseteq \overline{D}\tilde{B}$. Also, if $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{A}$, then $\tilde{A} = \tilde{B}$.

Proposition 2.6. [25] Let $\tilde{A}, \tilde{B} \in \tilde{F}_{IT_2}(X)$. Then $\tilde{A} \preceq \tilde{B} \Leftrightarrow \underline{D}\tilde{A} \subseteq \underline{D}\tilde{B}$ and $\overline{D}\tilde{A} \subseteq \overline{D}\tilde{B} \Leftrightarrow \mu_{\underline{D}\tilde{A}}(x) \leq \mu_{\underline{D}\tilde{B}}(x)$ and $\mu_{\overline{D}\tilde{A}}(x) \leq \mu_{\overline{D}\tilde{B}}(x), \forall x \in X.$

Next, we recall the concepts of type-2 fuzzy relation and IT2 fuzzy relation.

Definition 2.7. [20] A type-2 fuzzy relation \tilde{R} on X is a type-2 fuzzy set $\tilde{R} \in \tilde{F}(X \times X)$, i.e.,

$$\tilde{R} = \{((x, y), u), \mu_{\tilde{A}}((x, y), u) : x, y \in X, u \in J_{(x, y)} \subseteq I\},$$

in which $0 \leq \mu_{\tilde{A}}((x, y), u) \leq 1$. \tilde{R} can also be expressed as:

$$\tilde{R} = \int_{x, y \in X} \int_{u \in J_{(x, y)}} \mu_{\tilde{R}}((x, y), u) / ((x, y), u), J_{(x, y)} \subseteq I.$$

Definition 2.8. [20] A vertical slice $\mu_{\tilde{R}}(x', y')$ of a type-2 fuzzy relation \tilde{R} is the intersection between the two-dimensional plane whose axes are u and $\mu_{\tilde{R}}((x', y'), u)$ and the three-dimensional type-2 membership function \tilde{R} , given by

$$\mu_{\tilde{R}}(x', y') = \mu_{\tilde{R}}((x, y) = (x', y'), u) = \int_{u \in J_{(x', y')}} f_{(x', y')}(u) / u, J_{(x', y')} \subseteq I,$$

in which $0 \leq f_{(x', y')}(u) \leq 1$. In terms of vertical slice, \tilde{R} can be re-expressed as:

$$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) : x, y \in X\},$$

or, as the following:

$$\begin{aligned} \tilde{R} &= \int_{x, y \in X} \mu_{\tilde{R}}(x, y) / (x, y) \\ &= \int_{x, y \in X} \left[\int_{u \in J_{(x, y)}} f_{(x, y)}(u) / u \right] / (x, y), J_{(x, y)} \subseteq I. \end{aligned}$$

The footprint of uncertainty $D\tilde{R}$ of \tilde{R} is given by

$$D\tilde{R} = \bigcup_{x, y \in X} J_{(x, y)}.$$

Definition 2.9. [20] An IT2 fuzzy relation \tilde{R} on X is an IT-2 fuzzy set $\tilde{R} \in \tilde{F}_{IT_2}(X \times X)$, i.e.,

$$\tilde{R} = \{((x, y), u), 1 : x, y \in X, u \in J_{(x, y)} \subseteq I\}.$$

\tilde{R} can also be expressed as:

$$\begin{aligned} \tilde{R} &= \int_{x, y \in X} \int_{u \in J_{(x, y)}} 1 / ((x, y), u) \\ &= \int_{x, y \in X} \left[\int_{u \in J_{(x, y)}} 1 / u \right] / (x, y), J_{(x, y)} \subseteq I. \end{aligned}$$

Definition 2.10. [20] An IT2 fuzzy relation \tilde{R} on X is called

- i) IT2 fuzzy reflexive if $\forall x \in X, \mu_{\tilde{R}}((x, x), 1) = 1$ and $\mu_{\tilde{R}}((x, x), u) = 0, \forall u \in [0, 1)$;
- ii) IT2 fuzzy symmetric if $\forall x, y \in X, J_{(x, y)} = J_{(y, x)}$ and $\mu_{\tilde{R}}((x, y), u) = \mu_{\tilde{R}}((y, x), u), \forall u \in J_{(x, y)}$;
- iii) IT2 fuzzy transitive if $\forall x, y \in X, l_{\tilde{R}}(x, y) \geq \vee_{z \in X} [l_{\tilde{R}}(x, z) \wedge l_{\tilde{R}}(z, y)]$ and $r_{\tilde{R}}(x, y) \geq \vee_{z \in X} [r_{\tilde{R}}(x, z) \wedge r_{\tilde{R}}(z, y)]$.

Definition 2.11. [20] An IT2 fuzzy relation \tilde{R} on X is called an IT2 fuzzy tolerance relation if it is IT2 fuzzy reflexive and IT2 fuzzy symmetric.

Definition 2.12. [25] An IT2 fuzzy approximation space is a pair (X, \tilde{R}) , where X is a nonempty set and \tilde{R} is an IT2 fuzzy relation on X .

An IT2 fuzzy approximation space (X, \tilde{R}) is called an IT2 fuzzy reflexive approximation space if \tilde{R} is an IT2 fuzzy reflexive relation on X . Also, if \tilde{R} is an IT2 fuzzy tolerance relation on X , then (X, \tilde{R}) is called an IT2 fuzzy tolerance approximation space.

Definition 2.13. [25] Let (X, \tilde{R}) be an IT2 fuzzy approximation space and $\tilde{A} \in \tilde{F}_{IT2}(X)$. The lower approximation $\underline{\tilde{R}}(\tilde{A})$ of \tilde{A} and the upper approximation $\overline{\tilde{R}}(\tilde{A})$ of \tilde{A} in (X, \tilde{R}) are respectively defined as follows:

$$\underline{\tilde{R}}(\tilde{A}) = \int_{x \in X} \int_{u \in D_{\underline{\tilde{R}}(\tilde{A})}(x)} 1/(x, u), \text{ and}$$

$$\overline{\tilde{R}}(\tilde{A}) = \int_{x \in X} \int_{u \in D_{\overline{\tilde{R}}(\tilde{A})}(x)} 1/(x, u), \forall x \in X. \text{ Here,}$$

$$D_{\underline{\tilde{R}}(\tilde{A})}(x) = \left[\mu_{\underline{D}_{\underline{\tilde{R}}(\tilde{A})}}(x), \mu_{\overline{D}_{\underline{\tilde{R}}(\tilde{A})}}(x) \right] \text{ and}$$

$$D_{\overline{\tilde{R}}(\tilde{A})}(x) = \left[\mu_{\underline{D}_{\overline{\tilde{R}}(\tilde{A})}}(x), \mu_{\overline{D}_{\overline{\tilde{R}}(\tilde{A})}}(x) \right], \text{ where}$$

$$\mu_{\underline{D}_{\underline{\tilde{R}}(\tilde{A})}}(x) = \wedge \{ (1 - \mu_{\overline{D}_{\tilde{R}}}(x, y)) \vee \mu_{\underline{D}_{\tilde{A}}}(y) : y \in X \},$$

$$\mu_{\overline{D}_{\underline{\tilde{R}}(\tilde{A})}}(x) = \wedge \{ (1 - \mu_{\underline{D}_{\tilde{R}}}(x, y)) \vee \mu_{\overline{D}_{\tilde{A}}}(y) : y \in X \},$$

$$\mu_{\underline{D}_{\overline{\tilde{R}}(\tilde{A})}}(x) = \vee \{ \mu_{\underline{D}_{\tilde{R}}}(x, y) \wedge \mu_{\underline{D}_{\tilde{A}}}(y) : y \in X \},$$

$$\mu_{\overline{D}_{\overline{\tilde{R}}(\tilde{A})}}(x) = \vee \{ \mu_{\overline{D}_{\tilde{R}}}(x, y) \wedge \mu_{\overline{D}_{\tilde{A}}}(y) : y \in X \}.$$

The pair $(\underline{\tilde{R}}(\tilde{A}), \overline{\tilde{R}}(\tilde{A}))$ is called an IT2 fuzzy rough set.

Proposition 2.14. [25] Let (X, \tilde{R}) be an IT2 fuzzy approximation space and $\tilde{A}, \tilde{B}, \tilde{A}_i \in \tilde{F}_{IT2}(X), i \in J, [\alpha, \beta] \subseteq I$. Then

$$i) \underline{\tilde{R}}(\bigcap_{i \in J} \tilde{A}_i) = \bigcap_{i \in J} \underline{\tilde{R}}(\tilde{A}_i); \overline{\tilde{R}}(\bigcup_{i \in J} \tilde{A}_i) = \bigcup_{i \in J} \overline{\tilde{R}}(\tilde{A}_i),$$

$$ii) \underline{\tilde{R}}(\sim \tilde{A}) = \sim \overline{\tilde{R}}(\tilde{A}); \overline{\tilde{R}}(\sim \tilde{A}) = \sim \underline{\tilde{R}}(\tilde{A}),$$

$$iii) \underline{\tilde{R}}(X) = X, \overline{\tilde{R}}(\phi) = \phi,$$

$$iv) \underline{\tilde{R}}(\tilde{A} \cup [\underline{\alpha}, \underline{\beta}]) = \underline{\tilde{R}}(\tilde{A}) \cup [\underline{\alpha}, \underline{\beta}], \overline{\tilde{R}}(\tilde{A} \cap [\overline{\alpha}, \overline{\beta}]) = \overline{\tilde{R}}(\tilde{A}) \cap [\overline{\alpha}, \overline{\beta}]$$

Proposition 2.15. [25] *Let (X, \tilde{R}) be an IT2 fuzzy approximation space and $\tilde{A} \in \tilde{F}_{IT2}(X)$. Then \tilde{R} is an IT2 fuzzy reflexive relation on X iff*

- i) $\tilde{R}(\tilde{A}) \leq \tilde{A}$,
- ii) $\tilde{A} \leq \tilde{R}(\tilde{A})$.

Proposition 2.16. [25] *Let (X, \tilde{R}) be an IT2 fuzzy approximation space and $\tilde{A} \in \tilde{F}_{IT2}(X)$. Then \tilde{R} is an IT2 fuzzy transitive relation on X iff*

- i) $\tilde{R}(\tilde{A}) \leq \tilde{R}(\tilde{R}(\tilde{A}))$,
- ii) $\tilde{R}(\tilde{R}(\tilde{A})) \leq \tilde{R}(\tilde{A})$.

Finally, we recall the following IT2 fuzzy topological concepts introduced in [25].

Definition 2.17. [25] *An IT2 fuzzy topology on a nonempty set X is a family of IT2 fuzzy sets in X such that*

- i) $\phi, X \in \tau$,
- ii) $\forall \tilde{A}, \tilde{B} \in \tau, \tilde{A} \cap \tilde{B} \in \tau$.
- iii) $\forall \tilde{A}_i \in \tau, i \in J, \cup_{i \in J} \tilde{A}_i \in \tau$.

If τ is an IT2 fuzzy topology on X then the pair (X, τ) is called an IT2 fuzzy topological space. As usual, the members of τ are called IT2 fuzzy τ -open sets and their complements are called IT2 fuzzy τ -closed sets.

Definition 2.18. [25] *An IT2 fuzzy topology is called saturated if it is closed under arbitrary intersection.*

Definition 2.19. [25] *A Kuratowski IT2 fuzzy closure operator on a nonempty set X is a map $\tilde{k} : \tilde{F}_{IT2}(X) \rightarrow \tilde{F}_{IT2}(X)$ such that $\forall \tilde{A}, \tilde{B} \in \tilde{F}_{IT2}(X)$,*

- i) $\tilde{k}(\phi) = \phi$,
- ii) $\tilde{A} \leq \tilde{k}(\tilde{A})$,
- iii) $\tilde{k}(\tilde{A} \cup \tilde{B}) = \tilde{k}(\tilde{A}) \cup \tilde{k}(\tilde{B})$,
- iv) $\tilde{k}(\tilde{k}(\tilde{A})) = \tilde{k}(\tilde{A})$.

Proposition 2.20. [25] *Let (X, \tilde{R}) be an IT2 fuzzy reflexive approximation space. Then $\tau_{\tilde{R}} = \{\tilde{A} \in \tilde{F}_{IT2}(X) : \tilde{R}(\tilde{A}) = \tilde{A}\}$ is a saturated IT2 fuzzy topology on X .*

Proposition 2.21. [25] *Let $\tilde{k} : \tilde{F}_{IT2}(X) \rightarrow \tilde{F}_{IT2}(X)$ be a Kuratowski IT2 fuzzy closure operator on a nonempty set X . Then there exists IT2 fuzzy reflexive and IT2 fuzzy transitive relation \tilde{R} on X such that $\tilde{R}(\tilde{A}) = \tilde{k}(\tilde{A})$ iff \tilde{k} satisfies the following conditions:*

- i) $\tilde{k}(\cup_{i \in J} \tilde{A}_i) = \cup_{i \in J} \tilde{k}(\tilde{A}_i), \forall \tilde{A}_i \in \tilde{F}_{IT2}(X), i \in J$,
- ii) $\tilde{k}(\tilde{A} \cap [\alpha, \beta]) = \tilde{k}(\tilde{A}) \cap [\alpha, \beta], \forall [\alpha, \beta] \in [I]$.

3. IT2 Fuzzy Closure Space

In this section, we introduce the concept of IT2 fuzzy closure spaces and investigate their relationship with Kuratowski IT2 fuzzy closure operators. Lastly,

we show that the Kuratowski IT2 fuzzy closure operator associated with a quasi-discrete IT2 fuzzy closure space induces a saturated IT2 fuzzy topology.

We begin by introducing the following concept of IT2 fuzzy closure space.

Definition 3.1. An IT2 fuzzy closure space is a pair (X, \tilde{c}) , where X is a nonempty set and $\tilde{c} : \tilde{F}_{IT2}(X) \rightarrow \tilde{F}_{IT2}(X)$ is a map such that $\forall \tilde{A}, \tilde{B} \in \tilde{F}_{IT2}(X)$,

- i) $\tilde{c}(\phi) = \phi$,
- ii) $\tilde{A} \preceq \tilde{c}(\tilde{A})$,
- iii) $\tilde{c}(\tilde{A} \cup \tilde{B}) = \tilde{c}(\tilde{A}) \cup \tilde{c}(\tilde{B})$.

The following example shows that the IT2 fuzzy closure space do not satisfy, in general, the property of *idempotence*, i.e., $\tilde{c}(\tilde{c}(\tilde{A})) \neq \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$.

Example 3.2. Let X be a set containing at least two points. Define a map $\tilde{c} : \tilde{F}_{IT2}(X) \rightarrow \tilde{F}_{IT2}(X)$ such that $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$,

$$\tilde{c}(\tilde{A})(x) = \begin{cases} 2\tilde{A}(x), & \text{if } [\mu_{\underline{D}\tilde{A}}(x), \mu_{\overline{D}\tilde{A}}(x)] < [l/2, r/2], \\ & \text{where } [l, r] = \bigvee_{z \in X} [\mu_{\underline{D}\tilde{A}}(z), \mu_{\overline{D}\tilde{A}}(z)] \\ [l, r], & \text{if } [\mu_{\underline{D}\tilde{A}}(x), \mu_{\overline{D}\tilde{A}}(x)] \geq [l/2, r/2]. \end{cases}$$

Then the conditions (i) $\tilde{c}(\phi) = \phi$ and (ii) $\tilde{A} \preceq \tilde{c}(\tilde{A})$ are trivial. Now, we check the condition (iii) $\tilde{c}(\tilde{A} \cup \tilde{B}) = \tilde{c}(\tilde{A}) \cup \tilde{c}(\tilde{B})$ as follows:

Let $\bigvee_{z \in X} [\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(z), \mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(z)] = [m, n]$. Then for $[\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x), \mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x)] < [m/2, n/2]$, $\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x) < m/2$ and $\mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x) < n/2$, or that $\mu_{\underline{D}\tilde{A}}(x) < m/2$, $\mu_{\underline{D}\tilde{B}}(x) < m/2$ and $\mu_{\overline{D}\tilde{A}}(x) < n/2$, $\mu_{\overline{D}\tilde{B}}(x) < n/2$, i.e., $[\mu_{\underline{D}\tilde{A}}(x), \mu_{\overline{D}\tilde{A}}(x)] < [m/2, n/2]$ and $[\mu_{\underline{D}\tilde{B}}(x), \mu_{\overline{D}\tilde{B}}(x)] < [m/2, n/2]$. Thus $\tilde{c}(\tilde{A})(x) = 2\tilde{A}(x)$ and $\tilde{c}(\tilde{B})(x) = 2\tilde{B}(x)$, and so $\tilde{c}(\tilde{A} \cup \tilde{B})(x) = 2(\tilde{A} \cup \tilde{B})(x) = 2\tilde{A}(x) \cup 2\tilde{B}(x) = \tilde{c}(\tilde{A})(x) \cup \tilde{c}(\tilde{B})(x) = (\tilde{c}(\tilde{A}) \cup \tilde{c}(\tilde{B}))(x)$. Hence for $[\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x), \mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x)] < [m/2, n/2]$, $\tilde{c}(\tilde{A} \cup \tilde{B}) = \tilde{c}(\tilde{A}) \cup \tilde{c}(\tilde{B})$. Further, if $[\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x), \mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x)] \geq [m/2, n/2]$, then $\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x) \geq m/2$ and $\mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x) \geq n/2$. Thus $\mu_{\underline{D}\tilde{A}}(x) \geq m/2$, or $\mu_{\underline{D}\tilde{B}}(x) \geq m/2$ and $\mu_{\overline{D}\tilde{A}}(x) \geq n/2$, or $\mu_{\overline{D}\tilde{B}}(x) \geq n/2$. In any case, it can be seen that either $\tilde{c}(\tilde{A})(x) = [m, n]$ or $\tilde{c}(\tilde{B})(x) = [m, n]$, and therefore, for $[\mu_{\underline{D}(\tilde{A} \cup \tilde{B})}(x), \mu_{\overline{D}(\tilde{A} \cup \tilde{B})}(x)] \geq [m/2, n/2]$, $\tilde{c}(\tilde{A} \cup \tilde{B}) = \tilde{c}(\tilde{A}) \cup \tilde{c}(\tilde{B})$. Hence \tilde{c} is an IT2 fuzzy closure space. On the other hand, it is not a Kuratowski IT2 fuzzy closure operator, since if we take a non-constant IT2 fuzzy set $\tilde{A} \in \tilde{F}_{IT2}(X)$ such that $\tilde{A}(x_0) = [m/8, n/8]$, for some $x_0 \in X$, where $[m, n] = \bigvee_{z \in X} [\mu_{\underline{D}\tilde{A}}(z), \mu_{\overline{D}\tilde{A}}(z)]$. Then $\tilde{c}(\tilde{A})(x_0) = [m/4, n/4]$ and $\tilde{c}(\tilde{c}(\tilde{A}))(x_0) = [m/2, n/2]$, implying that $\tilde{c}(\tilde{c}(\tilde{A})) \neq \tilde{c}(\tilde{A})$.

Definition 3.3. An IT2 fuzzy closure space (X, \tilde{c}) is called

- i) quasi-discrete if $\tilde{c}(\bigcup\{\tilde{A}_i : i \in J\}) = \bigcup\{\tilde{c}(\tilde{A}_i) : i \in J\}$, $\forall \tilde{A}_i \in \tilde{F}_{IT2}(X)$, $i \in J$, and
- ii) symmetric if $\tilde{c}(\tilde{I}_y)(x) = \tilde{c}(\tilde{I}_x)(y)$, $\forall x, y \in X$.

Example 3.4. The IT2 fuzzy closure space given in Example 3.2 is both quasi-discrete and symmetric.

Definition 3.5. Let (X, \tilde{c}) be an IT2 fuzzy closure space and $\tilde{A} \in \tilde{F}_{IT2}(X)$. Then

- i) the IT2 fuzzy interior of \tilde{A} is defined by $int(\tilde{A}) = \sim \tilde{c}(\sim \tilde{A})$, and
- ii) \tilde{A} is an IT2 fuzzy closed if $\tilde{c}(\tilde{A}) = \tilde{A}$.

Lemma 3.6. Let (X, \tilde{c}) be an IT2 fuzzy closure space. Then

- i) $\tilde{c}(\cap_{i \in J} \tilde{A}_i) \preceq \cap_{i \in J} \tilde{c}(\tilde{A}_i)$, $\forall \tilde{A}_i \in \tilde{F}_{IT2}(X), i \in J$,
- ii) if $\tilde{A} \preceq \tilde{B}$, then $\tilde{c}(\tilde{A}) \preceq \tilde{c}(\tilde{B})$, $\forall \tilde{A}, \tilde{B} \in \tilde{F}_{IT2}(X)$.

Proof. The proofs are straightforward. \square

Definition 3.7. For an IT2 fuzzy closure space (X, \tilde{c}) , a Kuratowski IT2 fuzzy closure $\tilde{c} : \tilde{F}_{IT2}(X) \rightarrow \tilde{F}_{IT2}(X)$ is defined by $\tilde{c}(\tilde{A}) = \cap \{\tilde{B} \in \tilde{F}_{IT2}(X) : \tilde{A} \preceq \tilde{B} \text{ and } \tilde{c}(\tilde{B}) = \tilde{B}\}$.

It follows immediately from the above definition that $\tilde{A} \preceq \tilde{c}(\tilde{A})$.

We shall denote by $\tau_{\tilde{c}}$, the IT2 fuzzy topology induced by \tilde{c} , which is given by $\tau_{\tilde{c}} = \{\tilde{A} \in \tilde{F}_{IT2}(X) : \tilde{c}(\sim \tilde{A}) = \sim \tilde{A}\}$.

Proposition 3.8. Let (X, \tilde{c}) be an IT2 fuzzy closure space and $\tilde{A} \in \tilde{F}_{IT2}(X)$. Then

- i) $\tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{A})$ (i.e., $\tilde{c}(\tilde{A})$ is an IT2 fuzzy $\tau_{\tilde{c}}$ -closed set),
- ii) $\tilde{c}(\tilde{A}) \preceq \tilde{c}(\tilde{A})$,
- iii) $\tilde{c}(\tilde{A}) = \tilde{A} \Leftrightarrow \tilde{c}(\tilde{A}) = \tilde{A}$.

Proof. (i) Let $\tilde{A} \in \tilde{F}_{IT2}(X)$. Then from Definition 3.7, $\tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\cap \{\tilde{B} : \tilde{A} \preceq \tilde{B} \text{ and } \tilde{c}(\tilde{B}) = \tilde{B}\}) \preceq \cap \{\tilde{c}(\tilde{B}) : \tilde{A} \preceq \tilde{B} \text{ and } \tilde{c}(\tilde{B}) = \tilde{B}\} = \cap \{\tilde{B} : \tilde{A} \preceq \tilde{B} \text{ and } \tilde{c}(\tilde{B}) = \tilde{B}\} = \tilde{c}(\tilde{A})$.

(ii) $\tilde{A} \preceq \tilde{c}(\tilde{A}) \Rightarrow \tilde{c}(\tilde{A}) \preceq \tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{A})$.

(iii) Let $\tilde{c}(\tilde{A}) = \tilde{A}$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$. Then \tilde{A} is an IT2 fuzzy $\tau_{\tilde{c}}$ -closed. Therefore, $\tilde{c}(\tilde{A}) \preceq \tilde{A}$. Also, $\tilde{A} \preceq \tilde{c}(\tilde{A})$. Thus $\tilde{c}(\tilde{A}) = \tilde{A}$. Conversely, let $\tilde{c}(\tilde{A}) = \tilde{A}$. Then from (ii), $\tilde{A} \preceq \tilde{c}(\tilde{A}) \preceq \tilde{c}(\tilde{A})$, whereby $\tilde{c}(\tilde{A}) = \tilde{A}$. Hence $\tilde{c}(\tilde{A}) = \tilde{A}$. \square

Proposition 3.9. Let (X, \tilde{c}) be an IT2 fuzzy closure space. Then $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, $\tilde{c}(\tilde{A}) = \tilde{c}(\tilde{A}) \Leftrightarrow \tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{A})$.

Proof. Let $\tilde{c}(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$. Then $\tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{A}) = \tilde{c}(\tilde{A})$.

Conversely, let $\tilde{c}(\tilde{c}(\tilde{A})) = \tilde{c}(\tilde{A})$. Then $\tilde{c}(\tilde{A})$ is an IT2 fuzzy $\tau_{\tilde{c}}$ -closed set. Hence from Proposition 3.8(iii), $\tilde{c}(\tilde{A}) = \tilde{c}(\tilde{A})$. \square

Lastly, we have the following result.

Proposition 3.10. Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space. Then the IT2 fuzzy topology $\tau_{\tilde{c}}$ induced by \tilde{c} on X is a saturated IT2 fuzzy topology.

Proof. It follows from Definitions 3.3, 3.7 and Proposition 3.9. \square

4. Equivalence Between IT2 Fuzzy Rough Sets and IT2 Fuzzy Closure Spaces

In this section, we show that there exists a one-to-one correspondence between the family of all IT2 fuzzy reflexive/tolerance approximation spaces and the family of all IT2 fuzzy closure spaces satisfying certain extra conditions. We also show that the IT2 fuzzy topology introduced in the previous section by Kuratowski IT2 fuzzy closure operator is precisely the same IT2 fuzzy topology associated with an IT2 fuzzy reflexive approximation space.

We begin with the following result.

Proposition 4.1. *Let (X, \tilde{R}) be an IT2 fuzzy reflexive approximation space. Then (X, \tilde{R}) is a quasi-discrete IT2 fuzzy closure space satisfying $\overline{\tilde{R}}(\tilde{A} \cap [\alpha, \beta]) = \overline{\tilde{R}}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$.*

Proof. The proof follows from Propositions 2.14 and 2.15. □

Proposition 4.2. *For a quasi-discrete IT2 fuzzy closure space $(X, \overline{\tilde{R}})$, $\tilde{R} : \tilde{F}_{IT2} \rightarrow \tilde{F}_{IT2}$ is an IT2 fuzzy interior operator on X .*

Proof. It follows from Proposition 2.14 and Definition 3.5(i). □

Before stating the next, we now introduce the following concept of IT2 fuzzy points, which is a natural generalization of the concept of fuzzy point (cf., [15]).

Definition 4.3. For $y \in X$ and $[\alpha, \beta] \subseteq I$, an IT2 fuzzy subset $\tilde{I}_y \cap [\alpha, \beta]$ of X is called an IT2 fuzzy point in X , and is denoted as $y_{[\alpha, \beta]}$.

Proposition 4.4. *Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space satisfying $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$. Then there exists a unique IT2 fuzzy reflexive relation $\tilde{R}_{\tilde{c}}$ on X such that $\overline{\tilde{R}_{\tilde{c}}}(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$.*

Proof. Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space satisfying $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$. Also, let $\tilde{R}_{\tilde{c}}$ be an IT2 fuzzy relation on X given by $\tilde{R}_{\tilde{c}}(x, y) = \tilde{c}(\tilde{I}_y)(x)$, $\forall x, y \in X$. Then $1 = \mu_{\underline{D}\tilde{I}_x}(x) \leq \mu_{\underline{D}\tilde{c}(\tilde{I}_x)}(x)$. Similarly, $1 \leq \mu_{\overline{D}\tilde{c}(\tilde{I}_x)}(x)$. Thus $\tilde{I}_x(x) \leq \tilde{c}(\tilde{I}_x)(x)$, i.e., $\tilde{c}(\tilde{I}_x)(x) = 1$, whereby $\tilde{R}_{\tilde{c}}$ is an IT2 fuzzy reflexive relation on X . Now, we need to show that $\overline{\tilde{R}_{\tilde{c}}}(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, for which it suffices to show that $\mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(\tilde{A})}(x) = \mu_{\overline{D}\tilde{c}(\tilde{A})}(x)$ and $\mu_{\underline{D}\overline{\tilde{R}_{\tilde{c}}}(\tilde{A})}(x) = \mu_{\underline{D}\tilde{c}(\tilde{A})}(x)$, $\forall x \in X$. Now, $\mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(\tilde{A})}(x) = \mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(\cup_{y \in X} y_{[\alpha, \beta]})}(x) = \vee_{y \in X} \mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(y_{[\alpha, \beta]})}(x)$, whereby $\mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(y_{[\alpha, \beta]})}(x) = \vee \{ \mu_{\overline{D}\tilde{R}_{\tilde{c}}}(x, u) \wedge y_{[\alpha, \beta]}(u) : u \in X \} = \mu_{\overline{D}\tilde{R}_{\tilde{c}}}(x, y) \wedge [\alpha, \beta] = \mu_{\overline{D}\tilde{c}(\tilde{I}_y)}(x) \wedge [\alpha, \beta] = (\mu_{\overline{D}\tilde{c}(\tilde{I}_y)} \wedge [\alpha, \beta])(x) = \mu_{\overline{D}\tilde{c}(y_{[\alpha, \beta]})}(x)$. Thus $\mu_{\overline{D}\overline{\tilde{R}_{\tilde{c}}}(\tilde{A})}(x) = \mu_{\overline{D}\tilde{c}(y_{[\alpha, \beta]})}(x)$. Similarly, we can show that $\mu_{\underline{D}\overline{\tilde{R}_{\tilde{c}}}(\tilde{A})}(x) = \mu_{\underline{D}\tilde{c}(y_{[\alpha, \beta]})}(x)$. Hence $\overline{\tilde{R}_{\tilde{c}}}(\tilde{A}) = \tilde{c}(\tilde{A})$. Finally, to show the uniqueness of IT2 fuzzy relation $\tilde{R}_{\tilde{c}}$, let \tilde{R}' be another IT2 fuzzy reflexive relation on X such that $\overline{\tilde{R}'}(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$. Then $\mu_{\overline{D}\tilde{R}_{\tilde{c}}}(x, y) = \mu_{\overline{D}\tilde{c}(\tilde{I}_y)}(x) = \mu_{\overline{D}\tilde{R}'}(\tilde{I}_y)(x) =$

$\vee\{\mu_{\overline{D}\tilde{R}'}(x, z) \wedge \mu_{\overline{D}\tilde{I}_y}(z) : z \in X\} = \mu_{\overline{D}\tilde{R}'}(x, y)$, whereby $\mu_{\overline{D}\tilde{R}_c}(x, y) = \mu_{\overline{D}\tilde{R}'}(x, y)$. Similarly, $\mu_{\overline{D}\tilde{R}_c}(x, y) = \mu_{\overline{D}\tilde{R}'}(x, y)$. Thus $\tilde{R}_c = \tilde{R}'$. Hence the IT2 fuzzy relation \tilde{R}_c on X is unique. \square

Proposition 4.5. *Let \mathcal{F} be the family of all IT2 fuzzy reflexive approximation spaces and \mathcal{T} be the family of all quasi-discrete IT2 fuzzy closure spaces satisfying $\tilde{c}(\tilde{A} \cap \overline{[\alpha, \beta]}) = \tilde{c}(\tilde{A}) \cap \overline{[\alpha, \beta]}$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$. Then there exists a one-to-one correspondence between \mathcal{F} and \mathcal{T} .*

Proof. It follows from Propositions 4.1 and 4.4. \square

Proposition 4.6. *Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space satisfying $\tilde{c}(\tilde{A} \cap \overline{[\alpha, \beta]}) = \tilde{c}(\tilde{A}) \cap \overline{[\alpha, \beta]}$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$ and \tilde{R}_c be an IT2 fuzzy reflexive relation on X such that $\tilde{R}_c(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$. Then $\tau_{\tilde{R}_c} = \tau_{\tilde{c}}$.*

Proof. Let $\tilde{A} \in \tau_{\tilde{c}}$. Then $\tilde{c}(\sim \tilde{A}) = \sim \tilde{A}$. As from Proposition 3.8, $\tilde{c}(\tilde{A}) \preceq \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, we have $\tilde{c}(\sim \tilde{A}) \preceq \sim \tilde{A}$, or that $\tilde{A} \preceq \sim \tilde{c}(\sim \tilde{A})$, i.e., $\tilde{A} \preceq \tilde{R}_c(\tilde{A})$. Also, as $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, $\tilde{R}_c(\tilde{A}) \preceq \tilde{A}$, we have $\tilde{R}_c(\tilde{A}) = \tilde{A}$. Thus $\tau_{\tilde{c}} \preceq \tau_{\tilde{R}_c}$.

Conversely, let $\tilde{A} \in \tau_{\tilde{R}_c}$. Then $\tilde{R}_c(\tilde{A}) = \tilde{A}$, or that $\sim \tilde{R}_c(\sim \tilde{A}) = \tilde{A}$, i.e., $\tilde{R}_c(\sim \tilde{A}) = \sim \tilde{A}$, whereby $\tilde{c}(\sim \tilde{A}) = \sim \tilde{A}$. Thus $\tilde{c}(\sim \tilde{A}) = \sim \tilde{A}$ (c.f., Proposition 3.8), whereby $\tilde{A} \in \tau_{\tilde{c}}$, or that $\tau_{\tilde{R}_c} \preceq \tau_{\tilde{c}}$. Hence $\tau_{\tilde{R}_c} = \tau_{\tilde{c}}$. \square

The following two results are required to establish a one-to-one correspondence between the family of all IT2 fuzzy tolerance approximation spaces and the family of all IT2 fuzzy closure spaces.

Proposition 4.7. *Let (X, \tilde{R}) be an IT2 fuzzy tolerance approximation space. Then $(X, \overline{\tilde{R}})$ is a symmetric quasi-discrete IT2 fuzzy closure space such that $\overline{\tilde{R}}(\tilde{A} \cap \overline{[\alpha, \beta]}) = \overline{\tilde{R}}(\tilde{A}) \cap \overline{[\alpha, \beta]}$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$.*

Proof. From Propositions 2.14 and 2.15, it follows that (X, \tilde{R}) is a quasi-discrete IT2 fuzzy closure space. Also, for $x, y \in X$, $\mu_{\overline{D}\overline{\tilde{R}}(\tilde{I}_y)}(x) = \vee\{\mu_{\overline{D}\tilde{R}}(x, u) \wedge \mu_{\overline{D}\tilde{I}_y}(u) : u \in X\} = \mu_{\overline{D}\tilde{R}}(x, y) = \mu_{\overline{D}\tilde{R}}(y, x) = \mu_{\overline{D}\tilde{R}(\tilde{I}_x)}(y)$. Therefore $\mu_{\overline{D}\overline{\tilde{R}}(\tilde{I}_y)}(x) = \mu_{\overline{D}\tilde{R}(\tilde{I}_x)}(y)$. Similarly, $\mu_{\overline{D}\overline{\tilde{R}}(\tilde{I}_x)}(y) = \mu_{\overline{D}\tilde{R}(\tilde{I}_y)}(x)$. Thus $\overline{\tilde{R}}(\tilde{I}_y)(x) = \overline{\tilde{R}}(\tilde{I}_x)(y)$, showing that $(X, \overline{\tilde{R}})$ is also a symmetric. Finally, for $\tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$, $\overline{\tilde{R}}(\tilde{A} \cap \overline{[\alpha, \beta]}) = \overline{\tilde{R}}(\tilde{A}) \cap \overline{[\alpha, \beta]}$ follows trivially. \square

Proposition 4.8. *Let (X, \tilde{c}) be a symmetric quasi-discrete IT2 fuzzy closure space such that $\tilde{c}(\tilde{A} \cap \overline{[\alpha, \beta]}) = \tilde{c}(\tilde{A}) \cap \overline{[\alpha, \beta]}$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$. Then there exists a unique IT2 fuzzy tolerance relation \tilde{R}_c on X such that $\tilde{R}_c(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$.*

Proof. It follows from Proposition 4.4 and the fact that (X, \tilde{c}) is a symmetric IT2 fuzzy closure space. \square

Proposition 4.9. *Let \mathcal{F} be the family of all IT2 fuzzy tolerance approximation spaces and \mathcal{T} be the family of all symmetric quasi-discrete IT2 fuzzy closure spaces satisfying $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$. Then there exists a one-to-one correspondence between \mathcal{F} and \mathcal{T} .*

Proof. It follows from Propositions 4.7 and 4.8. \square

Remark 4.10. Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space satisfying $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, $\forall [\alpha, \beta] \subseteq I$ and \tilde{c} be its associated Kuratowski IT2 fuzzy closure operator. Then (X, \tilde{c}) is obviously a quasi-discrete IT2 fuzzy closure space such that $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, and hence Proposition 4.4, will induce an IT2 fuzzy reflexive relation on X , say, $\tilde{S}_{\tilde{c}}$, given by $\tilde{S}_{\tilde{c}}(x, y) = \tilde{c}(\tilde{1}_y)(x)$, $\forall x, y \in X$.

Inspired from the concept of fuzzy transitive closure [16], we now introduce the concept of IT2 fuzzy transitive closure.

Definition 4.11. Let \tilde{R} be an IT2 fuzzy relations on X . An IT2 fuzzy relation \tilde{T} on X is called IT2 fuzzy transitive closure of \tilde{R} if

- i) \tilde{T} is an IT2 transitive,
- ii) $\tilde{R} \preceq \tilde{T}$, and
- iii) if \tilde{S} is an IT2 transitive with $\tilde{R} \preceq \tilde{S}$, then $\tilde{T} \preceq \tilde{S}$,

i.e., \tilde{T} is the smallest IT2 fuzzy transitive relation containing \tilde{R} .

Before stating the next proposition we need to prove the following lemma.

Lemma 4.12. *Let (X, \tilde{R}) and (X, \tilde{S}) be two IT2 fuzzy approximation spaces. Then $\tilde{R} \preceq \tilde{S}$ iff $\overline{\underline{D}}\tilde{R}(\tilde{A}) \preceq \overline{\underline{D}}\tilde{S}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$.*

Proof. Let $\overline{\underline{D}}\tilde{R}(\tilde{A}) \preceq \overline{\underline{D}}\tilde{S}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$. Then $\underline{D}\overline{\tilde{R}}(\tilde{A}) \subseteq \underline{D}\overline{\tilde{R}}(\tilde{B})$ and $\overline{\underline{D}}\tilde{R}(\tilde{A}) \subseteq \overline{\underline{D}}\tilde{R}(\tilde{B})$, i.e., $\mu_{\underline{D}\overline{\tilde{R}}(\tilde{A})}(x) \leq \mu_{\underline{D}\overline{\tilde{R}}(\tilde{B})}(x)$ and $\mu_{\overline{\underline{D}}\tilde{R}(\tilde{A})}(x) \leq \mu_{\overline{\underline{D}}\tilde{R}(\tilde{B})}(x)$, $\forall x \in X$, whereby $\mu_{\underline{D}\tilde{R}}(x, y) \wedge \mu_{\underline{D}\tilde{A}}(y) \leq \mu_{\underline{D}\tilde{S}}(x, y) \wedge \mu_{\underline{D}\tilde{A}}(y)$ and $\mu_{\overline{\underline{D}}\tilde{R}}(x, y) \wedge \mu_{\overline{\underline{D}}\tilde{A}}(y) \leq \mu_{\overline{\underline{D}}\tilde{S}}(x, y) \wedge \mu_{\overline{\underline{D}}\tilde{A}}(y)$, for some $y \in X$. Thus $\mu_{\underline{D}\tilde{R}}(x, y) \leq \mu_{\underline{D}\tilde{S}}(x, y)$ and $\mu_{\overline{\underline{D}}\tilde{R}}(x, y) \leq \mu_{\overline{\underline{D}}\tilde{S}}(x, y)$, $\forall x, y \in X$. Hence $\underline{D}\tilde{R} \subseteq \underline{D}\tilde{S}$ and $\overline{\underline{D}}\tilde{R} \subseteq \overline{\underline{D}}\tilde{S} \Rightarrow \tilde{R} \preceq \tilde{S}$.

Conversely, let $\tilde{R} \preceq \tilde{S}$. Then $\underline{D}\tilde{R} \subseteq \underline{D}\tilde{S}$ and $\overline{\underline{D}}\tilde{R} \subseteq \overline{\underline{D}}\tilde{S}$. We have to show that $\overline{\underline{D}}\tilde{R}(\tilde{A}) \preceq \overline{\underline{D}}\tilde{S}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$, for which it suffices to show that $\mu_{\underline{D}\overline{\tilde{R}}(\tilde{A})}(x) \leq \mu_{\underline{D}\overline{\tilde{R}}(\tilde{B})}(x)$ and $\mu_{\overline{\underline{D}}\tilde{R}(\tilde{A})}(x) \leq \mu_{\overline{\underline{D}}\tilde{R}(\tilde{B})}(x)$, $\forall x \in X$. Now, for $x \in X$, $\mu_{\underline{D}\overline{\tilde{R}}(\tilde{A})}(x) = \vee \{ \mu_{\underline{D}\tilde{R}}(x, y) \wedge \mu_{\underline{D}\tilde{A}}(y) : y \in X \} \leq \vee \{ \mu_{\underline{D}\tilde{S}}(x, y) \wedge \mu_{\underline{D}\tilde{A}}(y) : y \in X \} \leq \mu_{\underline{D}\overline{\tilde{S}}(\tilde{A})}(x)$.

Similarly, we can show that $\mu_{\overline{\underline{D}}\tilde{R}(\tilde{A})}(x) \leq \mu_{\overline{\underline{D}}\tilde{S}(\tilde{A})}(x)$. Hence $\overline{\underline{D}}\tilde{R}(\tilde{A}) \preceq \overline{\underline{D}}\tilde{S}(\tilde{A})$. \square

Proposition 4.13. *Let (X, \tilde{c}) be a quasi-discrete IT2 fuzzy closure space such that $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT2}(X)$ and $\forall [\alpha, \beta] \subseteq I$ and \tilde{c} be the associated Kuratowski IT2 fuzzy closure operator. Then IT2 fuzzy relation $\tilde{S}_{\tilde{c}}$ is the IT2 fuzzy transitive closure of $\tilde{R}_{\tilde{c}}$.*

Proof. Let $\tilde{S}_{\tilde{c}}(x, y) = \tilde{c}(1_y)(x)$, $\forall x, y \in X$. Then from Propositions 2.16 and 4.4, $\tilde{S}_{\tilde{c}}$ is an IT2 fuzzy transitive relation on X . Also, from Proposition 3.8 it follows that $\tilde{R}_{\tilde{c}} \preceq \tilde{S}_{\tilde{c}}$. To show the relation $\tilde{S}_{\tilde{c}}$ is the IT2 fuzzy transitive closure of $\tilde{R}_{\tilde{c}}$, it only remains to show that $\tilde{S}_{\tilde{c}}$ is the smallest IT2 fuzzy reflexive transitive relation containing $\tilde{R}_{\tilde{c}}$. So, let \tilde{T} be another IT2 fuzzy reflexive transitive relation on X such that $\tilde{R}_{\tilde{c}} \preceq \tilde{T}$. Then from the reflexivity of \tilde{T} , (X, \tilde{T}) is a quasi-discrete IT2 fuzzy closure space. Now, by using the fact that \tilde{T} is IT2 fuzzy transitive also together with Proposition 2.21 followed by Proposition 3.9, $\tilde{T}(\tilde{A}) = \cap\{\tilde{B} \in \tilde{F}_{IT_2}(X) : \tilde{A} \preceq \tilde{B}, \tilde{T}(\tilde{B}) = \tilde{B}\}$, $\forall \tilde{A} \in \tilde{F}_{IT_2}(X)$. Also, $\tilde{S}_{\tilde{c}}$ being an IT2 fuzzy reflexive and IT2 fuzzy transitive relation associated with Kuratowski IT2 fuzzy closure operator \tilde{c} , it follows from Proposition 3.9, that $\tilde{S}_{\tilde{c}}(\tilde{A}) = \tilde{c}(\tilde{A})$, $\forall \tilde{A} \in \tilde{F}_{IT_2}(X)$ and $\tilde{c}(\tilde{A}) = \cap\{\tilde{B} \in \tilde{F}_{IT_2}(X) : \tilde{A} \preceq \tilde{B}, \tilde{c}(\tilde{B}) = \tilde{B}\} = \cap\{\tilde{B} \in \tilde{F}_{IT_2}(X) : \tilde{A} \preceq \tilde{B}, \tilde{R}_{\tilde{c}}(\tilde{B}) = \tilde{B}\}$, $\forall \tilde{A} \in \tilde{F}_{IT_2}(X)$ (from Proposition 4.4). Thus from Lemma 4.12, $\tilde{S}_{\tilde{c}}(\tilde{A}) = \cap\{\tilde{B} \in \tilde{F}_{IT_2}(X) : \tilde{A} \preceq \tilde{B}, \tilde{R}_{\tilde{c}}(\tilde{B}) = \tilde{B}\} \subseteq \cap\{\tilde{B} \in \tilde{F}_{IT_2}(X) : \tilde{A} \preceq \tilde{B}, \tilde{T}(\tilde{B}) = \tilde{B}\} = \tilde{T}(\tilde{A})$, whereby $\tilde{S}_{\tilde{c}}(\tilde{A}) \preceq \tilde{T}(\tilde{A})$, showing that $\tilde{S}_{\tilde{c}} \preceq \tilde{T}$. \square

Proposition 4.14. *Let (X, \tilde{c}) be a symmetric quasi-discrete IT2 fuzzy closure space such that $\tilde{c}(\tilde{A} \cap [\alpha, \beta]) = \tilde{c}(\tilde{A}) \cap [\alpha, \beta]$, $\forall \tilde{A} \in \tilde{F}_{IT_2}(X)$ and $\forall [\alpha, \beta] \subseteq I$ and \tilde{c} be the associated Kuratowski IT2 fuzzy closure operator. Then IT2 fuzzy relation $\tilde{S}_{\tilde{c}}$ is the IT2 fuzzy transitive closure of $\tilde{R}_{\tilde{c}}$.*

Proof. The proof is similar to that of Proposition 4.13. \square

5. Conclusions

We have introduced the concept of IT2 fuzzy closure spaces and establish their equivalence with IT2 fuzzy approximation spaces. In particular, we established a one-to-one correspondence between the family of all IT2 fuzzy reflexive approximation spaces and the family of all quasi-discrete IT2 fuzzy closure spaces satisfying a certain extra condition. We have also shown the similar correspondence between the family of all tolerance approximation spaces and the family of all quasi-discrete IT2 fuzzy symmetric closure spaces satisfying a certain extra condition.

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