BOUNDEDNESS OF LINEAR ORDER-HOMOMORPHISMS IN L-TOPOLOGICAL VECTOR SPACES

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Abstract. A new definition of boundedness of linear order-homomorphisms (LOH) in L-topological vector spaces is proposed. The new definition is compared with the previous one given by Fang [The continuity of fuzzy linear order-homomorphism, J. Fuzzy Math. 5 (4) (1997) 829−838]. In addition, the relationship between boundedness and continuity of LOHs is discussed. Finally, a new uniform boundedness principle in L-topological vector spaces is established in the sense of a new definition of uniform boundedness for a family of LOHs.

1. **Introduction**

Fang and Yan [5] proposed the notion of L-fuzzy topological vector space in 1997. According to the standardized terminology in [8], more accurately, it should be called lattice-valued topological vector space (L-tvs), which is the extension of both the notion of classical topological vector space ($\{0,1\}$-tvs) and that of $[0,1]$-topological vector space ($[0,1]$-tvs) due to Katsaras [9].

Wang [14] initiated the notion of order-homomorphism, which unifies and generalizes the concept of ordinary mappings and that of Zadeh extension maps [2]. One may refer to [10, 15] for detailed discussion about the theory of order-homomorphism and its applications in fuzzy topology. Combining order-homomorphism with linear structure, Fang [3] proposed the concept of (L-fuzzy) linear order-homomorphism (LOH) and investigated its structure. From the subsequent works [4, 16, 17, 18] about LOHs, we can conclude that LOH is relatively reasonable generalization of the ordinary linear operator. In addition, it is proved by Yan [17] that LOH can be seen as an application of fuzzy powerset operators for variable basis proposed by Rodabaugh [8, 12, 13] to vector structure.

As is well known, boundedness of linear operators is a basic and important concept in the theory of $\{0,1\}$-tvs. So, it is natural and necessary to define rationally the boundedness of LOH in the research of L-tvs. In 1997, Fang [4] introduced the concept of boundedness of LOH and studied the relation between continuity and boundedness of LOH. Moreover, Yan and Fang [18] defined uniform boundedness for a family of LOHs and extended the famous uniform boundedness principle in $\{0,1\}$-tvs to general L-tvs.

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Quite recently, Fang and Zhang [6] proposed a new definition of boundedness of LOH in $[0,1]$-tvs. In this paper, we intend to extend this definition from $[0,1]$-tvs to general L-tvs, that is, an LOH from one L-tvs to another is said to be bounded if it maps bounded L-fuzzy sets into bounded L-fuzzy sets, which coincides with boundedness of linear operators in $\{0,1\}$-tvs in a more natural way compared with that of Fang [4]. Further, a new definition of uniform boundedness for a family of LOHs is introduced and the corresponding uniform boundedness principle in L-tvs is established.

This paper is organized as follows. In Section 2, we recall some basic definitions and notations to be used in the remaining parts of the paper. In Section 3, we propose a new definition of boundedness of LOH and investigate the relationship between the new definition and Fang's. In addition, the relation between boundedness and continuity of LOH is discussed. In Section 4, a new uniform boundedness principle in L-tvs is established in the sense of a new definition of uniform boundedness for a family of LOHs. We draw a conclusion in Section 5.

2. **Preliminaries**

Throughout this paper, L and N denote Hutton algebras [8], i.e., complete and completely distributive lattices equipped with order-reversing involutions $'$. $0_L(0_N)$ and $1_L(1_N)$ are their bottom and top elements, respectively. $M(L)$ denotes the set of all non-zero union-irreducible elements in L. The elements of $M(L)$ are also called molecules [11] in L. L^X denotes the family of all L-fuzzy sets on X. Naturally, L^X is also a Hutton algebra and $M(L^X) = \{x_\lambda \mid x \in X,\ \lambda \in M(L)\}$. An L-fuzzy set which takes the constant value $\lambda \in L$ on X is denoted by $\underline{\lambda}$. An L-fuzzy set on X is called an L-fuzzy point if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda \in L \setminus \{0_L\}$, we denote this L-fuzzy point by x_λ. A lattice-valued topology δ on X is called *stratified*, if it contains all constant L-fuzzy sets on X. We always assume that the lattice-valued topologies referred to in the present paper are all stratified and the lattices are regular (i.e., the intersection of each pair of non-zero elements is not zero, or, equivalently, the top element is a molecule). For other symbols which are not mentioned, we refer to [5, 11].

Definition 2.1. [15] Let (L^X, δ) be a lattice-valued topological space and $x_\lambda \in M(L^X)$. $P \in L^X$ is called a closed R-neighborhood of x_λ, if $P \in \delta'$ and $x_\lambda \nleqslant P$. The set of all closed R-neighborhoods of x_λ is denoted by $\eta^-(x_\lambda)$.

$A \in L^X$ is called an R-neighborhood of x_λ, if there exists $P \in \eta^-(x_\lambda)$ such that $A \leqslant P$. The set of all R-neighborhoods of x_λ is denoted by $\eta(x_\lambda)$.

$\mathscr{U} \subseteq \eta(x_\lambda)$ is said to be an R-neighborhood base of x_λ if for each $P \in \eta(x_\lambda)$, there exists $Q \in \mathscr{U}$ such that $P \leqslant Q$.

Definition 2.2. [14, 15] A mapping $\varphi : L \to N$ is called an order-homomorphism if the following conditions hold:

(1) $\varphi(0_L) = 0_N$;

(2) φ is union-preserving, i.e., $\varphi(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \varphi(a_i)$ for all $\{a_i\}_{i\in I} \subseteq L$;

(3) $\varphi^{\vee}$ is complement-preserving, i.e., for each $b \in N$, $\varphi^{\vee}(b') = (\varphi^{\vee}(b))'$, where $\varphi^{\vee}$ is the right adjoint of φ (see [8, 12, 13]), i.e., $\varphi^{\vee}(b) = \bigvee\{a \in L \mid \varphi(a) \leqslant b\}$.

Obviously, if $\varphi : L \to N$ is an order-homomorphism, then

$\varphi^{\vee}(1_N) = 1_L$ and $\varphi^{\vee}(0_N) = \varphi^{\vee}((1_N)') = (\varphi^{\vee}(1_N))' = (1_L)' = 0_L$.

Definition 2.3. [15] Let (L^X, δ_X) and (N^Y, δ_Y) be two lattice-valued topological spaces and $F : L^X \to N^Y$ be an order-homomorphism. Then F is said to be continuous, if $F^{\vee}(B) \in \delta_X$ for every $B \in \delta_Y$.

When $L = N$, a mapping $f : X \to Y$ is said to be continuous, if the Zadeh extension map [2] $F : L^X \to L^Y$ of f is a continuous order-homomorphism.

F is said to be continuous at the molecule $e \in M(L^X)$, if $F^{\vee}(Q)$ is an R-neighborhood of e in (L^X, δ_X) for each R-neighborhood Q of $F(e)$ in (N^Y, δ_Y).

Remark 2.4. It is not difficult to show that an order-homomorphism $F : L^X \to N^Y$ is continuous iff it is continuous at e for each $e \in M(L^X)$ (see [15] for detail).

In the sequel, X and Y always denote vector spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). For simplicity, θ denotes the zero elements of both X and Y.

Definition 2.5. [1, 5] Applying the extension principle of Zadeh to addition and scalar multiplication, we define the addition and scalar multiplication of L-fuzzy sets on X as follows. For $A,\ B \in L^X$ and $k \in \mathbb{K}$,

$$(A+B)(x) = \bigvee_{y+z=x} (A(y) \wedge B(z)),$$

$$(kA)(x) = A(x/k), \text{ whenever } k \neq 0,$$

$$(0A)(x) = \begin{cases} \bigvee\limits_{y \in X} A(y), & x = \theta, \\ 0, & x \neq \theta. \end{cases}$$

In particular, for L-fuzzy points x_λ, y_μ and $k \in \mathbb{K}$, we have

$x_\lambda + y_\mu = (x+y)_{\lambda \wedge \mu}$ and $kx_\lambda = (kx)_\lambda$.

Definition 2.6. [5] Let δ be a lattice-valued topology on X. The pair (L^X, δ) is called an L-tvs if the following two mappings (the addition and the scalar multiplication on X):

(1) $f : X \times X \to X,\ (x, y) \mapsto x + y$ and

(2) $g : \mathbb{K} \times X \to X,\ (k, x) \mapsto kx$

are both continuous, where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product lattice-valued topologies $\delta \times \delta$ and $J_{\mathbb{K}} \times \delta$, respectively, and $J_{\mathbb{K}}$ denotes the usual topology on $\mathbb{K}$.

Definition 2.7. [4] Let (L^X, δ) be an L-tvs. An L-fuzzy set B on X is said to be λ-bounded ($\lambda \in M(L)$), if for each $Q \in \eta(\theta_\lambda)$, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant tQ'$. B is said to be bounded if it is λ-bounded for each $\lambda \in M(L)$.

Definition 2.8. [3] A mapping $F : L^X \to N^Y$ is called an LOH, if it is an order-homomorphism satisfying the following linearity condition:

$F(k_1 A + k_2 B) = k_1 F(A) + k_2 F(B)$ for each $A, B \in L^X$, $k_1, k_2 \in \mathbb{K}$.

Remark 2.9. A typical example of LOH is the Zadeh extension map $F : L^X \to L^Y$ of an ordinary linear operator $f : X \to Y$.

The following is a decomposition theorem of LOH.

Lemma 2.10. [3] *The mapping $F : L^X \to N^Y$ is an LOH iff there exist an ordinary linear operator $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi : L \to N$ such that F is the bi-induced mapping* [7] *of f and φ, i.e.,*
$F(A)(y) = \bigvee_{f(x)=y} \varphi(A(x))$ *for all $A \in L^X, y \in Y$.*

Remark 2.11. (1) By Lemma 2.10, in the sequel, we shall use $(f,\varphi)^{\to}$ and $(f,\varphi)^{\leftarrow}$ instead of F and $F^{\vee}$, respectively. These notations are taken from [8].

(2) It is not difficult to see that the Zadeh extension map $F : L^X \to L^Y$ of an ordinary linear operator $f : X \to Y$ is exactly the LOH $(f, id_L)^{\to}$, where id_L denotes the identity mapping on L.

3. New Definition of Boundedness of LOH

In this section, we first propose a new definition of boundedness of LOH, then investigate the relationship between the new definition and that given by Fang [4]. In addition, the relation between boundedness and continuity of LOH is discussed.

In [4], Fang gave the following definition of boundedness of LOH.

Definition 3.1. [4] Let (L^X, δ_X) and (N^Y, δ_Y) be two L-tvses and $(f,\varphi)^{\to} : L^X \to N^Y$ an LOH. $(f,\varphi)^{\to}$ is said to be λ-bounded ($\lambda \in M(L)$), if it maps every λ-bounded L-fuzzy set in (L^X, δ_X) into a $\varphi(\lambda)$-bounded L-fuzzy set in (N^Y, δ_Y).

$(f,\varphi)^{\to}$ is called bounded if it is λ-bounded for each $\lambda \in M(L)$.

Now, we propose a new definition of boundedness of LOH, which coincides with boundedness of linear operators in a more natural way.

Definition 3.2. Let (L^X, δ_X) and (N^Y, δ_Y) be two L-tvses and $(f,\varphi)^{\to} : L^X \to N^Y$ an LOH. $(f,\varphi)^{\to}$ is said to be bounded, if it maps every bounded L-fuzzy set in (L^X, δ_X) into a bounded L-fuzzy set in (N^Y, δ_Y).

Remark 3.3. In the sequel, for distinction, we rename bounded LOH in the sense of Definition 3.1 as LOH bounded on each layer and when we speak about bounded LOH, it will be in the sense of Definition 3.2.

Remark 3.4. Examples 3.6 and 3.7 in [6] show that the two kinds of boundedness of LOH do not imply each other even in $[0,1]$-tvs. So, they are different in general L-tvs. However, under certain condition, boundedness on each layer implies boundedness (see Theorem 3.7).

Lemma 3.5. [10] *Let $\varphi : L \to N$ be an order-homomorphism. If φ is surjective, then $\varphi\varphi^{\vee} = id_N$, where id_N is the identity mapping on N.*

Lemma 3.6. *Let $\varphi : L \to N$ be an order-homomorphism. If φ is surjective and finitely meet-preserving, then $\varphi^{\vee}(\lambda) \in M(L)$ for each $\lambda \in M(N)$.*

Proof. Since φ is surjective, for each $\lambda \in M(N)$, there exists $\mu \in L \setminus \{0_L\}$ such that $\varphi(\mu) = \lambda$. Hence $\varphi^\vee(\lambda) \geqslant \mu > 0_L$.

Let $\alpha, \beta \in L$ with $\varphi^\vee(\lambda) \leqslant \alpha \vee \beta$, i.e., $\alpha' \wedge \beta' \leqslant \varphi^\vee(\lambda')$. Hence, since φ is finitely meet-preserving, $\varphi(\alpha') \wedge \varphi(\beta') = \varphi(\alpha' \wedge \beta') \leqslant \varphi(\varphi^\vee(\lambda')) \leqslant \lambda'$, i.e., $\lambda \leqslant (\varphi(\alpha'))' \vee (\varphi(\beta'))'$. Since $\lambda \in M(N)$, we have $\lambda \leqslant (\varphi(\alpha'))'$ or $\lambda \leqslant (\varphi(\beta'))'$, i.e., $\varphi(\alpha') \leqslant \lambda'$ or $\varphi(\beta') \leqslant \lambda' \Longleftrightarrow \alpha' \leqslant \varphi^\vee(\lambda')$ or $\beta' \leqslant \varphi^\vee(\lambda') \Longleftrightarrow \varphi^\vee(\lambda) \leqslant \alpha$ or $\varphi^\vee(\lambda) \leqslant \beta$. So $\varphi^\vee(\lambda)$ is a non-zero union-irreducible element in L, i.e., $\varphi^\vee(\lambda) \in M(L)$. □

Theorem 3.7. *Let (L^X, δ_X) and (N^Y, δ_Y) be two L-tvses and $(f, \varphi)^\to : L^X \to N^Y$ an LOH. If $(f, \varphi)^\to$ is bounded on each layer and φ is surjective. Then $(f, \varphi)^\to$ is bounded.*

Proof. Let $A \in L^X$ be an arbitrary bounded L-fuzzy set in (L^X, δ_X). It suffices to prove that $(f, \varphi)^\to(A)$ is λ-bounded in (N^Y, δ_Y) for each $\lambda \in M(N)$.

In fact, let $\lambda \in M(N)$. Then $\varphi^\vee(\lambda) \in M(L)$ by Lemma 3.6. Hence A is $\varphi^\vee(\lambda)$-bounded since it is bounded, and so $(f, \varphi)^\to(A)$ is $\varphi(\varphi^\vee(\lambda))$-bounded, i.e., λ-bounded by Lemma 3.5. This completes the proof. □

Corollary 3.8. *Let (L^X, δ_X) and (L^Y, δ_Y) be two L-tvses and $f : X \to Y$ be an ordinary linear operator. If f (i.e., $(f, id_L)^\to$) is bounded on each layer. Then f is bounded.*

Finally, we discuss the relation between boundedness and continuity of LOH.

Lemma 3.9. [4] *Let (L^X, δ_X) and (N^Y, δ_Y) be two L-tvses and $(f, \varphi)^\to : L^X \to N^Y$ an LOH. If $(f, \varphi)^\to$ is continuous. Then $(f, \varphi)^\to$ is bounded on each layer.*

Theorem 3.10. *Let (L^X, δ_X) and (N^Y, δ_Y) be two L-tvses and $(f, \varphi)^\to : L^X \to N^Y$ an LOH. If $(f, \varphi)^\to$ is continuous and φ is surjective. Then $(f, \varphi)^\to$ is bounded.*

Proof. It follows directly from Lemma 3.9 and Theorem 3.7. □

Corollary 3.11. *Let (L^X, δ_X) and (L^Y, δ_Y) be two L-tvses and $f : X \to Y$ be an ordinary linear operator. If f (i.e., $(f, id_L)^\to$) is continuous. Then f is bounded.*

Remark 3.12. Theorem 3.10 indicates that, under certain condition, continuity of LOH implies boundedness of LOH. Thus, a natural question arises: *when does boundedness of an LOH imply its continuity?* Note that this question has been solved in $[0, 1]$-tvs (see Theorem 4.11 in [6]).

The answer in $\{0, 1\}$-tvs is that if the first space is first-countable, i.e., satisfies the first axiom of countability, then boundedness of a linear operator implies its continuity. Unfortunately, this is not the case in general L-tvs (even $L = [0, 1]$) if we define the first axiom of countability as follows:

An L-tvs (L^X, δ) is said to satisfy the first axiom of countability if there exists a countable R-neighborhood base of θ_λ for each $\lambda \in M(L)$.

To see this, let's refer to Example 3.6 in [6]. It is not difficult to prove that the first space is first-countable.

In the example, $(T_0, \phi_0)^\to$ is bounded, but it is not bounded on each layer. Hence, it is not continuous by Lemma 3.9.

At the end of this section, we pose the above unsolved question as an open one.
Question: In general L-tvs, when does boundedness of an LOH imply its continuity?

4. New Uniform Boundedness Principle in L-tvs

Yan and Fang [18] extended the famous uniform boundedness principle in $\{0,1\}$-tvs to L-tvs by introducing the concepts of uniform boundedness and equicontinuity for a family of LOHs. In this section, we first propose a new definition of uniform boundedness for a family of LOHs, which coincides with uniform boundedness for a family of linear operators in the classical sense. Then, we establish a new uniform boundedness principle in L-tvs in the sense of the new definition of uniform boundedness for a family of LOHs.

Definition 4.1. [18] Let (L^X, δ) be an L-tvs. An L-fuzzy set B is said to be strongly λ-bounded ($\lambda \in M(L)$), if for each $Q \in \eta(\theta_\lambda)$, there exists $t > 0$ such that $B \leqslant tQ'$. B is said to be strongly bounded if it is strongly λ-bounded for each $\lambda \in M(L)$.

Remark 4.2. (1) Comparing the above definition with Definition 2.7, it is obvious that, for an L-fuzzy set, strong λ-boundedness implies λ-boundedness for each $\lambda \in M(L)$, hence strong boundedness implies boundedness.

(2) Since $\lambda \leqslant \mu$ implies $\eta(\theta_\lambda) \subseteq \eta(\theta_\mu)$, strong μ-boundedness implies strong λ-boundedness whenever $\lambda, \mu \in M(L)$ with $\lambda \leqslant \mu$. As a result, strong boundedness is equivalent to strong 1_L-boundedness (Note that $1_L \in M(L)$ by the regularity assumption on L).

Definition 4.3. [18] Let (L^X, δ_X) and (N^Y, δ_Y) be two L-topological vector spaces. A family $\{(f_\alpha, \varphi_\alpha)^\to\}_{\alpha\in\Gamma}$ of LOHs from L^X to N^Y is said to be equicontinuous, if there exists an order-homomorphism $\varphi : L \to N$ satisfying $\varphi \geqslant \bigvee\limits_{\alpha\in\Gamma} \varphi_\alpha$ such that the following condition holds: for each $\lambda \in M(L)$ and each R-neighborhood P of $\theta_{\varphi(\lambda)}$ in (N^Y, δ_Y), there exists an R-neighborhood W of θ_λ in (L^X, δ_X) such that $(f_\alpha, \varphi_\alpha)^\to(W') \leqslant P'$ for all $\alpha \in \Gamma$.

Remark 4.4. (1) Since $\varphi \geqslant \varphi_\alpha$ for all $\alpha \in \Gamma$, each R-neighborhood of $\theta_{\varphi_\alpha(\lambda)}$ in (N^Y, δ_Y) is also an R-neighborhood of $\theta_{\varphi(\lambda)}$ in (N^Y, δ_Y). Hence, equicontinuity of the family $\{(f_\alpha, \varphi_\alpha)^\to\}_{\alpha\in\Gamma}$ of LOHs implies the continuity of $(f_\alpha, \varphi_\alpha)^\to$ for all $\alpha \in \Gamma$.

(2) By (1), it is not difficult to see that the family $\{(f, \varphi)^\to\}$ consisting of only one LOH is equicontinuous iff $(f, \varphi)^\to$ is continuous.

Definition 4.5. [18] Let (L^X, δ_X) and (N^Y, δ_Y) be two L-topological vector spaces. A family $\{(f_\alpha, \varphi_\alpha)^\to\}_{\alpha\in\Gamma}$ of LOHs from L^X to N^Y is said to be uniformly bounded, if there exists an order-homomorphism $\varphi : L \to N$ satisfying $\varphi \geqslant \bigvee\limits_{\alpha\in\Gamma} \varphi_\alpha$ such that the following condition holds: for each $\lambda \in M(L)$ and each λ-bounded L-fuzzy set $A \in L^X$, there exists a $\varphi(\lambda)$-bounded L-fuzzy set $B \in N^Y$ such that $(f_\alpha, \varphi_\alpha)^\to(A) \leqslant B$ for all $\alpha \in \Gamma$.

Remark 4.6. It is easy to show that the condition "there exists a $\varphi(\lambda)$-bounded L-fuzzy set $B \in N^Y$ such that $(f_\alpha, \varphi_\alpha)^\rightarrow(A) \leqslant B$ for all $\alpha \in \Gamma$." is equivalent to the condition "$\bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(A)$ is $\varphi(\lambda)$-bounded."

Now, we propose the following new definition of uniform boundedness for a family of LOHs, which coincides with uniform boundedness for a family of linear operators in the classical sense.

Definition 4.7. Let (L^X, δ_X) and (N^Y, δ_Y) be two L-topological vector spaces. A family $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ of LOHs from L^X to N^Y is said to be uniformly bounded, if for each bounded L-fuzzy set $A \in L^X$, there exists a bounded L-fuzzy set $B \in N^Y$ such that $(f_\alpha, \varphi_\alpha)^\rightarrow(A) \leqslant B$ for all $\alpha \in \Gamma$.

Remark 4.8. (1) It is easy to show that the condition "there exists a bounded L-fuzzy set $B \in N^Y$ such that $(f_\alpha, \varphi_\alpha)^\rightarrow(A) \leqslant B$ for all $\alpha \in \Gamma$." is equivalent to the condition "$\bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(A)$ is bounded in (N^Y, δ_Y)."

(2) It is easy to see that uniform boundedness of a family $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ of LOHs in the sense of Definition 4.7 implies boundedness of each of the LOH $(f_\alpha, \varphi_\alpha)^\rightarrow$.

Remark 4.9. For distinction, we rename uniform boundedness in the sense of Definition 4.5 as φ-uniform boundedness and when we speak about uniform boundedness, it will be in the sense of Definition 4.7.

The following theorem indicates that, under certain condition, φ-uniform boundedness implies uniform boundedness.

Theorem 4.10. *Let (L^X, δ_X) and (N^Y, δ_Y) be two L-topological vector spaces. Suppose that $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ is φ-uniformly bounded and φ is surjective and finitely meet-preserving. Then $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ is uniformly bounded.*

Proof. By (1) of Remark 4.8, it suffices to show that for each bounded L-fuzzy set $A \in L^X$, $\bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(A)$ is bounded in (N^Y, δ_Y).

In fact, for each $\mu \in M(N)$, $\varphi^\vee(\mu) \in M(L)$ by Lemma 3.6, hence A is $\varphi^\vee(\mu)$-bounded, which implies that $\bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(A)$ is $\varphi(\varphi^\vee(\mu))$-bounded by Remark 4.6, i.e., it is μ-bounded by Lemma 3.5. So $\bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(A)$ is bounded. This completes the proof. □

The following lemma is the uniform boundedness principle for a family of LOHs obtained in [18].

Lemma 4.11. [18] *Let (L^X, δ_X) be an L-tvs and X be of second category on every stratum. Let (N^Y, δ_Y) be a regular L-tvs and $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ be a family of continuous LOHs from (L^X, δ_X) to (N^Y, δ_Y). Suppose that there exists an order-homomorphism $\varphi : L \to N$ satisfying $\varphi \geqslant \bigvee_{\alpha\in\Gamma} \varphi_\alpha$ such that for each $x \in X$ and each $\lambda \in M(L)$, $B_x = \bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\rightarrow(x_{1_L})$ is strongly $\varphi(\lambda)$-bounded in (N^Y, δ_Y). Then $\{(f_\alpha, \varphi_\alpha)^\rightarrow\}_{\alpha\in\Gamma}$ is equicontinuous and φ-uniformly bounded.*

Remark 4.12. By (2) of Remark 4.2, the condition "for each $x \in X$ and each $\lambda \in M(L)$, B_x is strongly $\varphi(\lambda)$-bounded" can be replaced by "for each $x \in X$, B_x is strongly $\varphi(1_L)$-bounded"

The following theorem is the new uniform boundedness principle in L-tvs in the sense of the new definition of uniform boundedness for a family of LOHs.

Theorem 4.13. *Let* (L^X, δ_X) *be an* L*-tvs and* X *be of second category on every stratum. Let* (N^Y, δ_Y) *be a regular* L*-tvs and* $\{(f_\alpha, \varphi_\alpha)^\to\}_{\alpha\in\Gamma}$ *be a family of continuous LOHs from* (L^X, δ_X) *to* (N^Y, δ_Y)*. Suppose that there exists a surjective and finitely meet-preserving order-homomorphism* $\varphi : L \to N$ *satisfying* $\varphi \geqslant \bigvee_{\alpha\in\Gamma} \varphi_\alpha$ *such that for each* $x \in X$*,* $B_x = \bigvee_{\alpha\in\Gamma} (f_\alpha, \varphi_\alpha)^\to(x_{1_L})$ *is strongly* $\varphi(1_L)$*-bounded in* (N^Y, δ_Y)*. Then* $\{(f_\alpha, \varphi_\alpha)^\to\}_{\alpha\in\Gamma}$ *is equicontinuous and uniformly bounded.*

Proof. It follows from Lemma 4.11, Remark 4.12 and Theorem 4.10. □

In Theorem 4.13, letting $L = N$ and $\varphi = \varphi_\alpha = id_L$ for each $\alpha \in \Gamma$, we can obtain the following corollary, which is an extension of uniform boundedness principle in $\{0, 1\}$-tvs to L-tvs.

Corollary 4.14. *Let* (L^X, δ_X) *be an* L*-tvs and* X *be of second category on every stratum. Let* (L^Y, δ_Y) *be a regular* L*-tvs and* $\{(f_\alpha, id_L)^\to\}_{\alpha\in\Gamma}$ *be a family of continuous LOHs from* (L^X, δ_X) *to* (L^Y, δ_Y)*. Suppose that for each* $x \in X$*,* $B_x = \bigvee_{\alpha\in\Gamma} [f_\alpha(x)]_{1_L}$ *is strongly bounded in* (L^Y, δ_Y)*. Then* $\{(f_\alpha, id_L)^\to\}_{\alpha\in\Gamma}$ *is equicontinuous and uniformly bounded.*

5. Conclusion

First, we proposed a new definition of LOH and compared it with that of Fang [4]. Then, we discussed the relation between boundedness and continuity of LOHs. Finally, we established a new uniform boundedness principle in L-tvs in the sense of a new definition of uniform boundedness for a family of LOHs.

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