DISTINCT FUZZY SUBGROUPS OF A DIHEDRAL GROUP OF ORDER $2pqrs$ FOR DISTINCT PRIMES p, q, r AND s

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Abstract. In this paper we classify fuzzy subgroups of the dihedral group D_{pqrs} for distinct primes p, q, r and s. This follows similar work we have done on distinct fuzzy subgroups of some dihedral groups. We present formulae for the number of (i) distinct maximal chains of subgroups, (ii) distinct fuzzy subgroups and (iii) non-isomorphic classes of fuzzy subgroups under our chosen equivalence and isomorphism. Some results presented here hold for any dihedral group of order $2n$ where n is a product of any number of distinct primes.

1. **Introduction**

In [4] we studied the classification of fuzzy subgroups of the dihedral groups D_{pqr} for distinct primes p, q and r, and any positive integer n. We presented formulae for the number of distinct maximal chains, distinct fuzzy subgroups and non-isomorphic classes, of fuzzy subgroups, under our chosen notions of equivalence and isomorphism. Our main objective is to classify fuzzy subgroups of any dihedral group D_n. As mentioned in [4], this is a difficult task that requires a number of stages. Thus first we look at several types of dihedral groups, and fix a number of primes. Consequently we will present a number of papers that should lead to the main result.

In this paper we study the classification of fuzzy subgroups of D_{pqrs} for distinct primes p, q, r and s. We present formulae for the number of maximal chains of subgroups and the number of distinct fuzzy subgroups. The equivalence we use in this paper is the same as the one used in [4].

There are other notions of equivalence in literature, as mentioned in [4]. Our version of equivalence is stronger than the one studied by Volf [9], Branimir and Tepavcevic [1] and Tarnaucenu and Bentea [8]. Further, our equivalence is a special case of the S*-equivalence relation introduced by Degang et al [2]. So computing the number of distinct fuzzy subgroups of any finite group using the Murali and Makamba definition [3], which is the one used in this paper, yields more distinct fuzzy subgroups than when using the definition of Degang et al [2].

Apart from the classification of fuzzy subgroups according to equivalence, we also

Received: August 2012; Revised: January 2013; Accepted: April 2015

Key words and phrases: Dihedral group, Equivalence, Isomorphism, Fuzzy subgroup, Maximal chain, Keychain, Distinguishing factor.

discuss a classification according to isomorphism. This necessitates looking at isomorphic maximal chains, under a suitable isomorphism, so as to enumerate non-isomorphic fuzzy subgroups of the group. Indeed, as mentioned also in [4], isomorphism cuts across groups, whereas equivalence is confined to a chosen group. Our notion of equivalence is a special case of our isomorphism. Special cases tend to yield more exciting results than general cases. So we use both equivalence and isomorphism, but with more emphasis on equivalence. For more information on isomorphic chains and lattices, see [5] and [6]. A detailed demonstration of our counting techniques for equivalence classes is given in the form of Example 3.4 of [4].

The rest of the paper is organised as follows: Section 2 presents a few basic concepts necessary for our discussion on classification, so that the paper is self-contained. Section 3 discusses maximal chains of subgroups of the dihedral group D_{pqrs}, and these will be used in the classification of fuzzy subgroups. Section 4 discusses distinct fuzzy subgroups of $G = D_{pqrs}$ and formulae are presented for the number of distinct fuzzy subgroups. Section 5 is a shorter discussion on non-isomorphic classes of fuzzy subgroups of D_{pqrs}. Section 6 presents concluding remarks.

2. **Preliminaries**

A fuzzy subset $\mu : G \longrightarrow [0, 1]$ of a group G is a fuzzy subgroup of G if $\mu(xy) \geq min\{\mu(x), \mu(y)\}, \forall x, y \in G$ and $\mu(x^{-1}) = \mu(x), \forall x \in G$, [7]. Clearly for the identity element $e \in G$, $\mu(x) \leq \mu(e)\ \forall x \in G$.

Throughout this paper **we assume $\boldsymbol{\mu(e) = 1}$** for any fuzzy subgroup μ of a group G. We say a fuzzy subgroup μ is equivalent to a fuzzy subgroup ν, written $\mu \sim \nu$, if (i) $\forall x, y \in G$, $\mu(x) > \mu(y) \Longleftrightarrow \nu(x) > \nu(y)$ and (ii) $\mu(x) = 0 \Longleftrightarrow \nu(x) = 0$, [3]. Two fuzzy subgroups μ and ν are said to be distinct iff $[\mu] \neq [\nu]$ where $[\mu]$ and $[\nu]$ are the resulting equivalence classes containing μ and ν respectively.

The support of μ is defined as $supp\,\mu = \{x \in G : \mu(x) > 0\}$.

An increasing chain of a finite number of subgroups of G starting with the trivial subgroup $\{e\}$ and ending up with G is called a flag on G.

In this paper, a subgroup (component) that distinguishes a maximal chain from other maximal chains is referred to as a *distinguishing factor*. If a maximal chain is distinguished by two or more subgroups, such subgroups may also be called distinguishing factors, [4]. In this paper, whenever we use the term maximal chain, we mean a maximal chain of subgroups of G.

Now we recall the following facts for comparison and use in this paper:

Proposition 2.1. [3] *The number of distinct fuzzy subgroups represented by a maximal chain (flag) of length n is $2^n - 1$.*

Proposition 2.2. *The number of maximal chains of D_{pq} is equal to $2! + 1!(p + q) + 2!(pq)$, and the number of distinct fuzzy subgroups of D_{pq} is*

$$2^4 - 1 + 2^3(1 + p + q + pq) + 2^2(pq) \tag{1}$$

.

Proposition 2.3. [4] *The number of maximal chains of* D_{pqr} *is equal to* $3!+2!(p+q+r)+2(pq+pr+qr)+3!(pqr)$, *and the number of distinct fuzzy subgroups of* D_{pqr} *is*

$$\begin{aligned} &2^5-1+4\times 2^4+2^3+(2^4+2^3)[p+q+r+pq+pr+qr] \\ &\quad +(2^4+2^3\times 4+2^2)pqr \\ &= 2^5-1+2^4(4+p+q+r+pq+pr+qr+pqr) \\ &\quad +2^3(1+p+q+r+pq+pr+qr+4pqr)+2^2pqr \end{aligned} \tag{2}$$

The following two propositions are easy to prove:

Proposition 2.4. *In an ascending maximal chain of subgroups of* $G=D_{p_1p_2\ldots p_n}$, *a dihedral subgroup of* G *can only be followed by a dihedral subgroup.*

Proposition 2.5. *The length of each maximal chain of subgroups of* $G=D_{p_1p_2\ldots p_n}$ *is* $n+2$, *where all the* p_i *are distinct primes.*

From the above proposition, it is clear that any maximal chain of subgroups of D_{pqrs} has length $4+2=6$.

Definition 2.6. A maximal chain of subgroups of G is called cyclic if all its proper subgroups are cyclic.

Definition 2.7. (a) A maximal chain of subgroups of $G=D_{p_1p_2\ldots p_n}=<a,b:a^{p_1p_2\cdots p_n}=e=b^2=(ab)^2>$ for $n\geq 2$, is $d-cyclic$ if (i) it has a cyclic subgroup $<a^k>$, $k>1$ and (ii) it has exactly one non-trivial proper dihedral subgroup, where all the p_i are distinct primes.
(b) A maximal chain of subgroups of $G=D_{p_1p_2\ldots p_n}=<a,b:a^{p_1p_2\cdots p_n}=e=b^2=(ab)^2>$ for $n\geq 2$, is $2d-cyclic$ if (i) it has a cyclic subgroup $<a^k>$, $k>1$ and (ii) it has exactly two non-trivial proper dihedral subgroups, where all the p_i are distinct primes.
(c) A maximal chain of subgroups of $G=D_{p_1p_2\ldots p_n}=<a,b:a^{p_1p_2\cdots p_n}=e=b^2=(ab)^2>$ for $n\geq 2$, is $3d-cyclic$ if (i) it has a cyclic subgroup $<a^k>$, $k>1$ and (ii) it has exactly three non-trivial proper dihedral subgroups, where all the p_i are distinct primes.
(d) A maximal chain of subgroups of $G=D_{p_1p_2\ldots p_n}=<a,b:a^{p_1p_2\cdots p_n}=e=b^2=(ab)^2>$ for $n\geq 2$, is $b-cyclic$ if it has exactly one non-trivial proper subgroup of the form $<a^kb>$, for $k\geq 0$, where all the p_i are distinct primes.

In [5] and [6], versions of fuzzy isomorphism and isomorphism of finite chains are presented. In this paper we use the following definitions:

Definition 2.8. [3]. Let μ and ν be two fuzzy subgroups of G and G_1 respectively. We say μ is fuzzy isomorphic to ν, denoted $\mu\cong\nu$, iff there exists an isomorphism $f:G\to G_1$ such that for $x,y\in G$, $\mu(x)>\mu(y)\iff\nu(f(x))>\nu(f(y))$ and $\mu(x)=0\iff\nu(f(x))=0$.

Definition 2.9. Two or more maximal chains are isomorphic if their lengths are equal and their corresponding components (subgroups) are isomorphic subgroups.

3. Maximal Chains of the Dihedral Group D_{pqrs}

In this section we study and compute the number of maximal chains of $G = D_{pqrs}$, for distinct primes p, q, r and s. We also make some generalisations for the case $G = D_{p_1p_2\cdots p_n}$, for distinct primes p_i and any positive integer n.

Proposition 3.1. *Let $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$, $n \geq 2$. Then the number of cyclic maximal chains of G is $n!$, where the p_i are distinct primes.*

Proof. Apply induction on n. For $n = 2$, $G = D_{p_1p_2}$ has the cyclic maximal chains $\{e\} \subset < a^{p_1} > \subset < a > \subset D_{p_1p_2}$ and $\{e\} \subset < a^{p_2} > \subset < a > \subset D_{p_1p_2}$. It is clear that all other maximal chains are non-cyclic. Thus there are 2! cyclic maximal chains for G.
Assume now that the proposition is true for $n = k > 2$. Thus $D_{p_1p_2\cdots p_k}$ has $k!$ cyclic maximal chains. We show that $G = D_{p_1p_2\cdots p_kp_{k+1}}$ has $(k+1)!$ cyclic maximal chains. All cyclic maximal chains of $D_{p_1p_2\cdots p_kp_{k+1}}$ must have the maximal subgroup $< a >$ of $D_{p_1p_2\cdots p_kp_{k+1}}$ as a component. Consider the cyclic maximal chain $< e > \subset < a^{p_1p_2\cdots p_k} > \subset < a^{p_1p_2\cdots p_{k-1}} > \subset \cdots \subset < a^{p_1} > \subset < a > \subset G \cdots$ (1) of $G = D_{p_1p_2\cdots p_kp_{k+1}}$.
In this chain (1), $p_1p_2\cdots p_{k-1}$ can be replaced by any product of $k-1$ primes from $\{p_1, p_2, \cdots, p_k\}$. The chain $< e > \subset < a^{p_1p_2\cdots p_{k-1}} > \subset \cdots \subset < a^{p_1} > \subset < a > \subset D_{p_1p_2\cdots p_k}$ is a maximal chain in $D_{p_1p_2\cdots p_k}$, and by assumption, there are $k!$ cyclic maximal chains in $D_{p_1p_2\cdots p_k}$, thus there are $k!$ cyclic maximal chains corresponding to $< a^{p_1p_2\cdots p_k} >$ in the group $D_{p_1p_2\cdots p_kp_{k+1}}$. The primes in $p_1p_2\cdots p_k$ can be varied in $k+1$ ways by replacing each one of them by p_{k+1}, including nil replacement, giving $k+1$ maximal dihedral subgroups of $D_{p_1p_2\cdots p_kp_{k+1}}$, including $D_{p_1p_2\cdots p_k}$. By the multiplication principle, there are $(k+1)k! = (k+1)!$ cyclic maximal chains of $D_{p_1p_2\cdots p_kp_{k+1}}$. □

Corollary 3.2. *The number of cyclic maximal chains of $G = D_{pqrs} =< a, b : a^{pqrs} = e = b^2 = (ab)^2 >$ is $4! = 24$, where p, q, r and s are distinct primes.*

Proposition 3.3. *The number of d-cyclic maximal chains of subgroups of $G = D_{p_1p_2\cdots p_n} =< a, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ is*

$$(n-1)![p_1 + p_2 + \cdots + p_n], n \geq 2 \qquad (3)$$

Proof. Apply induction on n. For $n = 2$, $G = D_{p_1p_2}$. Consider the d-cyclic maximal chain $\{e\} \subset < a^{p_1} > \subset D^b_{p_2} \subset D_{p_1p_2}$ where $D^b_{p_2} =< a^{p_1}, b : a^{p_1p_2} = e = b^2 = (ab)^2 >$. There are p_1 d-cyclic maximal chains having $< a^{p_1} >$ as a component, obtained by replacing b with a^kb in $D^b_{p_2}$. Similarly swopping the roles of the two primes yields p_2 d-cyclic maximal chains. Hence there are $(p_1 + p_2)$ d-cyclic maximal chains.
Now assume the proposition is true for $k > 2$. Consider $G = D_{p_1p_2\cdots p_{k+1}}$. Then $D^b_{p_1p_2\cdots p_k} =$
$< a^{k+1}, b : (a^{k+1})^{p_1p_2\cdots p_k} = e = b^2 = (a^{k+1}b)^2 >$ is a proper maximal dihedral subgroup of G having $(k-1)![p_1 + p_2 + \cdots + p_k]$ d-cyclic maximal chains by assumption.

There are $k+1$ proper maximal dihedral subgroups of G obtained by replacing each p_i by p_{k+1} in $p_1p_2\cdots p_k$, including nil replacement. For example $D_{p_1p_2\cdots p_{k-1}p_{k+1}}$ is a proper maximal dihedral subgroup of G having $(k-1)![p_1+p_2+\cdots+p_{k-1}+p_{k+1}]$ d-cyclic maximal chains by assumption. So each p_i appears k times (in the maximal chain formulae obtained by replacing p_i by p_{k+1}) for each $i=1,2,\cdots,k+1$ since each p_j in $p_1p_2\cdots p_k$ must be removed once and at the same time p_{k+1} is introduced. Hence the total number of maximal chains is $k(k-1)![p_1+p_2+\cdots+p_k+p_{k+1}] = k![p_1+p_2+\cdots+p_k+p_{k+1}]$. This completes the proof. □

Corollary 3.4. *The number of d-cyclic maximal chains of subgroups of* $G = D_{pqrs} =< a,\, b : a^{pqrs} = e = b^2 = (ab)^2 >$ *is*

$$3![p+q+r+s] \tag{4}$$

Proposition 3.5. *The number of 2d-cyclic maximal chains of subgroups of* $G = D_{p_1p_2\cdots p_n} =< a,\, b : a^{p_1p_2\cdots p_n} = e = b^2 = (ab)^2 >$ *is*

$$\begin{aligned} &2(n-2)![p_1p_2+p_1p_3+\cdots+p_1p_n+p_2p_3+p_2p_4+\cdots+p_2p_n+\cdots+p_{n-1}p_n] \\ &\quad = 2(n-2)!\sum_{i<j} p_ip_j, i,j \in \{1,2,\cdots,n\}, n>2. \end{aligned} \tag{5}$$

Proof. Apply induction on n. For $n=3$, $G = D_{p_1p_2p_3}$. Now consider the 2d-cyclic chain $\{e\} \subset < a^{p_1p_2} > \subset D^b_{p_2} \subset D^b_{p_1p_2} \subset D_{p_1p_2p_3}$ where $D^b_{p_2} =< a^{p_1p_3},\, b : a^{p_1p_2p_3} = e = b^2 = (ab)^2 >$ and $D^b_{p_1p_2} =< a^{p_3},\, b : a^{p_1p_2p_3} = e = b^2 = (ab)^2 >$. Since b in $D^b_{p_1p_2}$ may be replaced by $< a^kb >$ for $k = 1,2,\cdots,p_1p_2-1$, and p_2 and p_1 may be swopped, it follows that there are $p_1p_2 \times 2$ maximal chains having $< a^{p_1p_2} >$ as a component. Similarly swopping the roles of the three primes yields a further $2(p_1p_3+p_2p_3)$ 2d-cyclic maximal chains. Hence there are $2(p_1p_2+p_1p_3+p_2p_3)$ 2d-cyclic maximal chains for $G = D_{p_1p_2p_3}$.

Now assume the proposition is true for $k>3$. Consider $G = D_{p_1p_2\cdots p_{k+1}}$. Then $D^b_{p_1p_2\cdots p_k}$ is a proper dihedral subgroup of G giving rise to $2(k-2)![p_1p_2+p_1p_3+\cdots+p_1p_n+p_2p_3+p_2p_4+\cdots+p_2p_n+\cdots+p_{k-1}p_k] = 2(k-2)!\sum_{i<j} p_ip_j$, $i,j \in \{1,2,\cdots,k\}$, 2d-cyclic maximal chains by assumption. There are $k+1$ proper maximal dihedral subgroups of G obtained by replacing each p_i by p_{k+1} in $p_1p_2\cdots p_k$. For example $D_{p_1p_2\cdots p_{k-1}p_{k+1}}$ is a proper maximal dihedral subgroup of G having $2(k-2)![p_1p_2+\cdots+p_1p_{k+1}+\cdots+p_{k-1}p_{k+1}] = 2(k-2)!\sum_{i<j} p_ip_j$, $i,j \in \{1,2,\cdots,k-1,k+1\}$, 2d-cyclic maximal chains by assumption. So each p_i appears k times (in the maximal chain formulae obtained by replacing p_i by p_{k+1}) for each $i=1,2,\cdots,k+1$ since each p_j in $p_1p_2\cdots p_k$ must be removed once and at the same time p_{k+1} is introduced. Hence, by counting, the total number of maximal chains is $2(k-1)(k-2)![p_1p_2+p_1p_3+\cdots+p_1p_{k+1}+p_2p_3+\cdots+p_2p_{k+1}+\cdots+p_kp_{k+1}] = 2(k+1-2)!\sum_{i<j} p_ip_j$, $i,j \in \{1,2,\cdots,k+1\}$ since each p_ip_j can appear only $k-1$ times from the replacements of both p_i and p_j by p_{k+1} (at different times). This completes the proof. □

Corollary 3.6. *The number of 2d-cyclic maximal chains of subgroups of* $G = D_{pqrs} =< a,\, b\, : a^{pqrs} = e = b^2 = (ab)^2 >$ *is*

$$4[pq + pr + ps + qr + qs + rs] \tag{6}$$

It is now compelling to have

Proposition 3.7. *The number of md-cyclic maximal chains of subgroups of* $G = D_{p_1 p_2 \cdots p_n} =< a,\, b\, : a^{p_1 p_2 \cdots p_n} = e = b^2 = (ab)^2 >$ *is*

$$2(n-m)! \sum_{k_1 < k_2 < \cdots < k_m} [p_{k_1} p_{k_2} \cdots p_{k_m}],$$
$$k_i \in \{1,\, 2,\, \cdots,\, n\}, m > 1, i \in \{1, 2, \cdots, m\} \tag{7}$$

It is cumbersome to give a detailed proof of this proposition, however, the proof follows a pattern similar to the proofs of the above propositions.

Proposition 3.8. *The number of b-cyclic maximal chains of subgroups of* $G = D_{p_1 p_2 \cdots p_n} =< a,\, b\, : a^{p_1 p_2 \cdots p_n} = e = b^2 = (ab)^2 >$ *is*

$$n![p_1 p_2 p_3 \cdots p_n]. \tag{8}$$

Proof. All non-trivial subgroups in a b-cyclic maximal chain, other than the subgroup $< a^k b >$, are dihedral. Fix $< b >$ and look at all maximal chains involving $< b >$. These are all the maximal chains involving $D^b_{p_i}$, $i = 1, 2, \cdots, n$. Fix i, and look at all maximal chains involving $D^b_{p_i}$. By counting, there are $(n-1)!$ maximal chains having $D^b_{p_i}$ as a component. Varying i from 1 to n, gives $n(n-1)! = n!$ maximal chains corresponding to $< b >$. Now there are $p_1 p_2 p_3 \cdots p_n$ subgroups of the form $< a^k b >$, corresponding to the subgroups $D^{a^k b}_{p_i}$ which can be used in the place of $D^b_{p_i}$. Thus the total number of maximal chains is $n![p_1 p_2 p_3 \cdots p_n]$. □

Theorem 3.9. *The number of maximal chains of subgroups of* $G = D_{p_1 p_2 \cdots p_n} =< a,\, b\, : a^{p_1 p_2 \cdots p_n} = e = b^2 = (ab)^2 >$*, is equal to*

$$n! + (n-1)! \sum_{i=1}^{n} [p_i] + 2(n-2)! \sum_{i<j} [p_i p_j] + 2(n-3)! \sum_{i<j<k} [p_i p_j p_k] + \cdots +$$
$$2(n-3)! \sum_{i_1 < i_2 < \cdots < i_{n-3}} [p_{i_1} p_{i_2} \cdots p_{i_{n-3}}] + 2(n-2)! \sum_{i_1 < i_2 < \cdots < i_{n-2}} [p_{i_1} p_{i_2} \cdots p_{i_{n-2}}]$$
$$+(n-1)! \sum_{i_1 < i_2 < \cdots < i_{n-1}} [p_{i_1} p_{i_2} \cdots p_{i_{n-1}}] + n![p_1 p_2 p_3 \cdots p_n]. \tag{9}$$

Proof. This result is a combination of the above propositions. □

Corollary 3.10. *The number of maximal chains of subgroups of* $G = D_{pqrs}$*, is equal to*

$$4! + 3![p + q + r + s] + 4[pq + pr + ps + qr + qs + rs]$$
$$+3![pqr + pqs + prs + qrs] + 4!(pqrs). \tag{10}$$

As a verification of the above theorem, we computed manually and laboriously the number of maximal chains of D_{pqrst} for distinct primes p, q, r, s, t and this agrees with the number given by Theorem 3.9, viz $5! + 4![p+q+r+s+t] + 12[pq + pr + ps + pt + qr + qs + qt + rs + rt + st] + 12[pqr + pqs + pqt + prs + prt + pst + qrs + qrt + qst + rst] + 4!(pqrs + pqrt + pqst + prst + qrst) + 5!(pqrst)$.

4. **Fuzzy Subgroups of the Dihedral Group** $G = D_{pqrs}$

In this section we compute the number of distinct fuzzy subgroups of $G = D_{pqrs}$ in stages. First we compute the number of distinct fuzzy subgroups represented by the cyclic maximal chains. We list the 4! maximal chains:

$< e > \subset < a^{pqr} > \subset < a^{pq} > \subset < a^{p} > \subset < a > \subset G \cdots (1)$
$< e > \subset < a^{pqr} > \subset < a^{pq} > \subset < a^{q} > \subset < a > \subset G \cdots (2)$
$< e > \subset < a^{pqr} > \subset < a^{pr} > \subset < a^{p} > \subset < a > \subset G \cdots (3)$
$< e > \subset < a^{pqr} > \subset < a^{pr} > \subset < a^{r} > \subset < a > \subset G \cdots (4)$
$< e > \subset < a^{pqr} > \subset < a^{qr} > \subset < a^{q} > \subset < a > \subset G \cdots (5)$
$< e > \subset < a^{pqr} > \subset < a^{qr} > \subset < a^{r} > \subset < a > \subset G \cdots (6)$
$< e > \subset < a^{pqs} > \subset < a^{pq} > \subset < a^{p} > \subset < a > \subset G \cdots (7)$
$< e > \subset < a^{pqs} > \subset < a^{pq} > \subset < a^{q} > \subset < a > \subset G \cdots (8)$
$< e > \subset < a^{pqs} > \subset < a^{ps} > \subset < a^{p} > \subset < a > \subset G \cdots (9)$
$< e > \subset < a^{pqs} > \subset < a^{ps} > \subset < a^{s} > \subset < a > \subset G \cdots (10)$
$< e > \subset < a^{pqs} > \subset < a^{qs} > \subset < a^{q} > \subset < a > \subset G \cdots (11)$
$< e > \subset < a^{pqs} > \subset < a^{qs} > \subset < a^{s} > \subset < a > \subset G \cdots (12)$
$< e > \subset < a^{prs} > \subset < a^{pr} > \subset < a^{p} > \subset < a > \subset G \cdots (13)$
$< e > \subset < a^{prs} > \subset < a^{pr} > \subset < a^{r} > \subset < a > \subset G \cdots (14)$
$< e > \subset < a^{prs} > \subset < a^{ps} > \subset < a^{p} > \subset < a > \subset G \cdots (15)$
$< e > \subset < a^{prs} > \subset < a^{ps} > \subset < a^{s} > \subset < a > \subset G \cdots (16)$
$< e > \subset < a^{prs} > \subset < a^{rs} > \subset < a^{r} > \subset < a > \subset G \cdots (17)$
$< e > \subset < a^{prs} > \subset < a^{rs} > \subset < a^{s} > \subset < a > \subset G \cdots (18)$
$< e > \subset < a^{qrs} > \subset < a^{qr} > \subset < a^{r} > \subset < a > \subset G \cdots (19)$
$< e > \subset < a^{qrs} > \subset < a^{qr} > \subset < a^{q} > \subset < a > \subset G \cdots (20)$
$< e > \subset < a^{qrs} > \subset < a^{rs} > \subset < a^{r} > \subset < a > \subset G \cdots (21)$
$< e > \subset < a^{qrs} > \subset < a^{rs} > \subset < a^{s} > \subset < a > \subset G \cdots (22)$
$< e > \subset < a^{qrs} > \subset < a^{qs} > \subset < a^{q} > \subset < a > \subset G \cdots (23)$
$< e > \subset < a^{qrs} > \subset < a^{qs} > \subset < a^{s} > \subset < a > \subset G \cdots (24)$

Starting from the top, chain (1) gives $2^6 - 1$ distinct fuzzy subgroups by Proposition 2.1 [3]. In the next discussion we use the counting technique discussed in our previous papers and extensively exploited in [4]. Each of the chains (2) - (5), (7), (9) - (11), (13), (17), (19) gives 2^5 distinct fuzzy subgroups since they are distinguished by $< a^q >$, $< a^{pr} >$, $< a^r >$, $< a^{qr} >$, $< a^{pqs} >$, $< a^{ps} >$, $< a^s >$, $< a^{qs} >$, $< a^{prs} >$, $< a^{rs} >$, $< a^{qrs} >$ respectively. Each of the chains (6), (8), (12), (14) - (16), (18), (20) - (23) gives 2^4 distinct fuzzy subgroups since each can only have a pair of distinguishing factors. For example (6) cannot have a single distinguishing factor but can be distinguished by $(< a^{qr} >, < a^r >)$. Chain (24) can only have a triple of disinguishing factors $(< a^{qrs} >, < a^{qs} >, < a^s >)$ since all the single and pairs of components have been used elsewhere as distinguishing factors, thus it yields 2^3 distinct fuzzy subgroups. Hence the cyclic chains yield $2^6 - 1 + 2^5 \times 11 + 2^4 \times 11 + 2^3$ distinct fuzzy subgroups.

We note that the listing of maximal chains is cumbersome and takes up too much space. Hence, from now on, in each cluster of isomorphic maximal chains, we list only one representative. Since we have already dealt with cyclic maximal chains, next we consider d-cyclic maximal chains. We obtain these from the cyclic chains by replacing $< a >$ by a dihedral subgroup, for example $D^b_{qrs} =< a^{qrs}, b : (a^{qrs})^p = e = b^2 = (a^{qrs}b)^2 >$. Thus we obtain the following d-cyclic chain:
$< e >\subset< a^{pqr} >\subset< a^{pq} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (1)$
There are $p - 1$ maximal chains isomorphic to this chain, obtained by replacing b in D^b_{qrs} by ab, a^2b, $\cdots$, $a^{p-1}b$, respectively.
Replacing D^b_{qrs} in the above argument by D^b_{prs}, we may have $< e >\subset< a^{pqr} >\subset< a^{pq} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (2)$, yielding q isomorphic maximal chains. Continuing the process, we obtain the following d-cyclic non-isomorphic maximal chains
$< e >\subset< a^{pqr} >\subset< a^{pr} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (3)$
$< e >\subset< a^{pqr} >\subset< a^{pr} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (4)$
$< e >\subset< a^{pqr} >\subset< a^{qr} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (5)$
$< e >\subset< a^{pqr} >\subset< a^{qr} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (6)$
$< e >\subset< a^{pqs} >\subset< a^{pq} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (7)$
$< e >\subset< a^{pqs} >\subset< a^{pq} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (8)$
$< e >\subset< a^{pqs} >\subset< a^{ps} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (9)$
$< e >\subset< a^{pqs} >\subset< a^{ps} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (10)$
$< e >\subset< a^{pqs} >\subset< a^{qs} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (11)$
$< e >\subset< a^{pqs} >\subset< a^{qs} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (12)$
$< e >\subset< a^{prs} >\subset< a^{pr} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (13)$
$< e >\subset< a^{prs} >\subset< a^{pr} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (14)$
$< e >\subset< a^{prs} >\subset< a^{ps} >\subset< a^p >\subset D^b_{qrs} \subset G \cdots (15)$
$< e >\subset< a^{prs} >\subset< a^{ps} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (16)$
$< e >\subset< a^{prs} >\subset< a^{rs} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (17)$
$< e >\subset< a^{prs} >\subset< a^{rs} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (18)$
$< e >\subset< a^{qrs} >\subset< a^{qr} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (19)$
$< e >\subset< a^{qrs} >\subset< a^{qr} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (20)$
$< e >\subset< a^{qrs} >\subset< a^{rs} >\subset< a^r >\subset D^b_{pqs} \subset G \cdots (21)$
$< e >\subset< a^{qrs} >\subset< a^{rs} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (22)$
$< e >\subset< a^{qrs} >\subset< a^{qs} >\subset< a^q >\subset D^b_{prs} \subset G \cdots (23)$
$< e >\subset< a^{qrs} >\subset< a^{qs} >\subset< a^s >\subset D^b_{pqr} \subset G \cdots (24)$
Each of the chains represented by chains (1), (2) and (4) is distinguished by a single dihedral group, thus each yields 2^5 distinct fuzzy subgroups. Using pairs and triples of distinguishing factors for the other maximal chains listed above, we establish that the number of distinct fuzzy subgroups resulting from d-cyclic maximal chains is

$2^5(p+q+r+s)+2^4(4p+4q+4r+4s)+2^3(p+q+r+s)$.

Next we consider the 2d-cyclic maximal chains. This is done by replacing $< a^k >$, k prime, with a dihedral subgroup in the d-cyclic chains. Thus for example in (1) we replace $< a^p >$ by D^b_{rs} to obtain the 2d-cyclic maximal chain
$< e > \subset < a^{pqr} > \subset < a^{pq} > \subset D^b_{rs} \subset D^b_{qrs} \subset G \cdots (1)$
As before, we replace b in D^b_{rs} by ab, a^2b, $\cdots$, $a^{pq-1}b$, respectively, to obtain pq isomorphic maximal chains. Continuing the process, by replacing rs in D^b_{rs} and/or qrs in D^b_{qrs}, we obtain the following non-isomorphic $2d$-cyclic maximal chains

$< e > \subset < a^{pqr} > \subset < a^{pq} > \subset D^b_{rs} \subset D^b_{prs} \subset G \cdots (2)$
$< e > \subset < a^{pqr} > \subset < a^{pr} > \subset D^b_{qs} \subset D^b_{qrs} \subset G \cdots (3)$
$< e > \subset < a^{pqr} > \subset < a^{pr} > \subset D^b_{qs} \subset D^b_{pqs} \subset G \cdots (4)$
$< e > \subset < a^{pqr} > \subset < a^{qr} > \subset D^b_{ps} \subset D^b_{prs} \subset G \cdots (5)$
$< e > \subset < a^{pqr} > \subset < a^{qr} > \subset D^b_{ps} \subset D^b_{pqs} \subset G \cdots (6)$
$< e > \subset < a^{pqs} > \subset < a^{pq} > \subset D^b_{rs} \subset D^b_{qrs} \subset G \cdots (7)$
$< e > \subset < a^{pqs} > \subset < a^{pq} > \subset D^b_{rs} \subset D^b_{prs} \subset G \cdots (8)$
$< e > \subset < a^{pqs} > \subset < a^{ps} > \subset D^b_{qr} \subset D^b_{qrs} \subset G \cdots (9)$
$< e > \subset < a^{pqs} > \subset < a^{ps} > \subset D^b_{qr} \subset D^b_{pqr} \subset G \cdots (10)$
$< e > \subset < a^{pqs} > \subset < a^{qs} > \subset D^b_{pr} \subset D^b_{prs} \subset G \cdots (11)$
$< e > \subset < a^{pqs} > \subset < a^{qs} > \subset D^b_{pr} \subset D^b_{pqr} \subset G \cdots (12)$
$< e > \subset < a^{prs} > \subset < a^{pr} > \subset D^b_{qs} \subset D^b_{qrs} \subset G \cdots (13)$
$< e > \subset < a^{prs} > \subset < a^{pr} > \subset D^b_{qs} \subset D^b_{pqs} \subset G \cdots (14)$
$< e > \subset < a^{prs} > \subset < a^{ps} > \subset D^b_{qr} \subset D^b_{qrs} \subset G \cdots (15)$
$< e > \subset < a^{prs} > \subset < a^{ps} > \subset D^b_{qr} \subset D^b_{pqr} \subset G \cdots (16)$
$< e > \subset < a^{prs} > \subset < a^{rs} > \subset D^b_{pq} \subset D^b_{pqs} \subset G \cdots (17)$
$< e > \subset < a^{prs} > \subset < a^{rs} > \subset D^b_{pq} \subset D^b_{pqr} \subset G \cdots (18)$
$< e > \subset < a^{qrs} > \subset < a^{qr} > \subset D^b_{ps} \subset D^b_{pqs} \subset G \cdots (19)$
$< e > \subset < a^{qrs} > \subset < a^{qr} > \subset D^b_{ps} \subset D^b_{prs} \subset G \cdots (20)$
$< e > \subset < a^{qrs} > \subset < a^{rs} > \subset D^b_{pq} \subset D^b_{pqs} \subset G \cdots (21)$
$< e > \subset < a^{qrs} > \subset < a^{rs} > \subset D^b_{pq} \subset D^b_{pqr} \subset G \cdots (22)$
$< e > \subset < a^{qrs} > \subset < a^{qs} > \subset D^b_{pr} \subset D^b_{prs} \subset G \cdots (23)$
$< e > \subset < a^{qrs} > \subset < a^{qs} > \subset D^b_{pr} \subset D^b_{pqr} \subset G \cdots (24)$

Using singles, pairs and triples of distinguishing factors, we establish that the number of distinct fuzzy subgroups corresponding to $2d$-cyclic maximal chains is $2^5(pq+qs+ps+qs+pr+rs)+2^4(2pq+2pr+2qr+2ps+2qs+2rs)+2^3(pq+pr+qr+ps+rs+qs)$.

Next we consider the 3d-cyclic maximal chains of G. We replace $< a^{kt} >$, k, t prime, with a dihedral subgroup in the 2d-cyclic chains. Thus for example in (1) we replace $< a^{pq} >$ by D^b_s. Continuing as in the case of $2d$-cyclic maximal chains, we obtain the following 3d-cyclic non-isomorphic maximal chains:

$< e > \subset < a^{pqr} > \subset D^b_s \subset D^b_{rs} \subset D^b_{qrs} \subset G \cdots (1)$
$< e > \subset < a^{pqr} > \subset D^b_s \subset D^b_{rs} \subset D^b_{prs} \subset G \cdots (2)$

$< e > \subset < a^{pqr} > \subset D_s^b \subset D_{qs}^b \subset D_{qrs}^b \subset G \cdots (3)$
$< e > \subset < a^{pqr} > \subset D_s^b \subset D_{qs}^b \subset D_{pqs}^b \subset G \cdots (4)$
$< e > \subset < a^{pqr} > \subset D_s^b \subset D_{ps}^b \subset D_{prs}^b \subset G \cdots (5)$
$< e > \subset < a^{pqr} > \subset D_s^b \subset D_{ps}^b \subset D_{pqs}^b \subset G \cdots (6)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{rs}^b \subset D_{qrs}^b \subset G \cdots (7)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{rs}^b \subset D_{prs}^b \subset G \cdots (8)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{qr}^b \subset D_{qrs}^b \subset G \cdots (9)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{qr}^b \subset D_{pqr}^b \subset G \cdots (10)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{pr}^b \subset D_{prs}^b \subset G \cdots (11)$
$< e > \subset < a^{pqs} > \subset D_r^b \subset D_{pr}^b \subset D_{pqr}^b \subset G \cdots (12)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{qs}^b \subset D_{qrs}^b \subset G \cdots (13)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{qs}^b \subset D_{pqs}^b \subset G \cdots (14)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{qr}^b \subset D_{qrs}^b \subset G \cdots (15)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{qr}^b \subset D_{pqr}^b \subset G \cdots (16)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{pq}^b \subset D_{pqs}^b \subset G \cdots (17)$
$< e > \subset < a^{prs} > \subset D_q^b \subset D_{pq}^b \subset D_{pqr}^b \subset G \cdots (18)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{ps}^b \subset D_{pqs}^b \subset G \cdots (19)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{ps}^b \subset D_{prs}^b \subset G \cdots (20)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{pq}^b \subset D_{pqs}^b \subset G \cdots (21)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{pq}^b \subset D_{pqr}^b \subset G \cdots (22)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{pr}^b \subset D_{prs}^b \subset G \cdots (23)$
$< e > \subset < a^{qrs} > \subset D_p^b \subset D_{pr}^b \subset D_{pqr}^b \subset G \cdots (24)$

Using single, double and triple distinguishing factors, we establish that the number of distinct fuzzy subgroups corresponding to $3d$-cyclic maximal chains is $2^5(pqr + pqs + prs + qrs) + 2^4(4pqr + 4pqs + 4prs + 4qrs) + 2^3(pqr + pqs + prs + qrs)$.

Finally, we consider the b-cyclic maximal chains. In the $3d$-cyclic maximal chains, we replace a cyclic subgroup such as $< a^{qrs} >$ with $< b >, < ab >, < a^2b >, \cdots, < a^{pqrs-1}b >$, respectively. We list only the non-isomorphic b-cyclic maximal chains as follows:

$< e > \subset < b > \subset D_s^b \subset D_{rs}^b \subset D_{qrs}^b \subset G \cdots (1)$
$< e > \subset < b > \subset D_s^b \subset D_{rs}^b \subset D_{prs}^b \subset G \cdots (2)$
$< e > \subset < b > \subset D_s^b \subset D_{qs}^b \subset D_{qrs}^b \subset G \cdots (3)$
$< e > \subset < b > \subset D_s^b \subset D_{qs}^b \subset D_{pqs}^b \subset G \cdots (4)$
$< e > \subset < b > \subset D_s^b \subset D_{ps}^b \subset D_{prs}^b \subset G \cdots (5)$
$< e > \subset < b > \subset D_s^b \subset D_{ps}^b \subset D_{pqs}^b \subset G \cdots (6)$
$< e > \subset < b > \subset D_r^b \subset D_{rs}^b \subset D_{qrs}^b \subset G \cdots (7)$
$< e > \subset < b > \subset D_r^b \subset D_{rs}^b \subset D_{prs}^b \subset G \cdots (8)$
$< e > \subset < b > \subset D_r^b \subset D_{qr}^b \subset D_{qrs}^b \subset G \cdots (9)$
$< e > \subset < b > \subset D_r^b \subset D_{qr}^b \subset D_{pqr}^b \subset G \cdots (10)$
$< e > \subset < b > \subset D_r^b \subset D_{pr}^b \subset D_{prs}^b \subset G \cdots (11)$

$< e >\subset< b >\subset D_r^b \subset D_{pr}^b \subset D_{pqr}^b \subset G \cdots (12)$
$< e >\subset< b >\subset D_q^b \subset D_{qs}^b \subset D_{qrs}^b \subset G \cdots (13)$
$< e >\subset< b >\subset D_q^b \subset D_{qs}^b \subset D_{pqs}^b \subset G \cdots (14)$
$< e >\subset< b >\subset D_q^b \subset D_{qr}^b \subset D_{qrs}^b \subset G \cdots (15)$
$< e >\subset< b >\subset D_q^b \subset D_{qr}^b \subset D_{pqr}^b \subset G \cdots (16)$
$< e >\subset< b >\subset D_q^b \subset D_{pq}^b \subset D_{pqs}^b \subset G \cdots (17)$
$< e >\subset< b >\subset D_q^b \subset D_{pq}^b \subset D_{pqr}^b \subset G \cdots (18)$
$< e >\subset< b >\subset D_p^b \subset D_{ps}^b \subset D_{pqs}^b \subset G \cdots (19)$
$< e >\subset< b >\subset D_p^b \subset D_{ps}^b \subset D_{prs}^b \subset G \cdots (20)$
$< e >\subset< b >\subset D_p^b \subset D_{pq}^b \subset D_{pqs}^b \subset G \cdots (21)$
$< e >\subset< b >\subset D_p^b \subset D_{pq}^b \subset D_{pqr}^b \subset G \cdots (22)$
$< e >\subset< b >\subset D_p^b \subset D_{pr}^b \subset D_{prs}^b \subset G \cdots (23)$
$< e >\subset< b >\subset D_p^b \subset D_{pr}^b \subset D_{pqr}^b \subset G \cdots (24)$

Using single, double and triple distinguishing factors, we establish that the number of distinct fuzzy subgroups corresponding to b-cyclic maximal chains is $2^5(pqrs) + 2^4(11 \times pqrs) + 2^3(12 \times pqrs)$.

Thus the total number of distinct fuzzy subgroups is $2^6-1+2^5\times 11+2^4\times 11+2^3$ $+ [2^5(p+q+r+s)+2^4(4p+4q+4r+4s)+2^3(p+q+r+s)] + [2^5(pq+qs+ps+qs+pr+rs)+2^4(2pq+2pr+2qr+2ps+2qs+2rs)+2^3(pq+pr+qr+ps+rs+qs)] + [2^5(pqr+pqs+prs+qrs)+2^4(4pqr+4pqs+4prs+4qrs)+2^3(pqr+pqs+prs+qrs)] + [2^5(pqrs) + 2^4(11 \times pqrs) + 2^3(12 \times pqrs)]$. Hence we have proved

Theorem 4.1. *The number of distinct fuzzy subgroups of $G = D_{pqrs}$ for distinct primes p, q, r and s is equal to*

$$\begin{aligned} &2^6 - 1 + 2^5[11 + (p+q+r+s) \\ &+(pq+qs+ps+qs+pr+rs) + (pqr+pqs+prs+qrs) + pqrs] \\ &+2^4[11 + (4p+4q+4r+4s) + (2pq+2pr+2qr+2ps+2qs+2rs) \\ &+(4pqr+4pqs+4prs+4qrs) + 11pqrs] \\ &+2^3[1 + (p+q+r+s) + (pq+pr+qr+ps+rs+qs) \\ &+(pqr+pqs+prs+qrs) + 12pqrs] \end{aligned} \tag{11}$$

Remark 4.2. In the above theorem, adding all the coefficients of powers of 2, viz. 2^6, 2^5, 2^4 and 2^3, gives the number of maximal chains of subgroups of G. The sum of those coefficients not depending on p, q, r and s, viz. 1, 11, 11 and 1, is equal to the number of cyclic maximal chains of G.

5. Non-isomorphic Chains and Classes of Fuzzy Subgroups of D_{pqrs}

In this section we use the notions of isomorphic maximal chains and isomorphic fuzzy subgroups, given in the definitions 2.9 and 2.8 respectively, to classify fuzzy subgroups of D_{pqrs}. We count the number of non-isomorphic fuzzy subgroups using the number of non-isomorphic maximal chains of subgroups of D_{pqrs}. Since in the previous section we listed only the non-isomorphic maximal chains, it is easy to observe that there 24 non-isomorphic md-cyclic maximal chains for each $m = 1, 2, 3$, and that this is also the number of cyclic maximal chains and also the number of non-isomorphic b-cyclic maximal chains. For the d-cyclic chains, we collapsed all isomorphic chains into one, thus the cluster of chains represented by (i) is collapsed into the chain i for each $i = 1, 2, \cdots, 24$.

In the formula $2^5(p+q+r+s)+2^4(4p+4q+4r+4s)+2^3(p+q+r+s)$ for the number of distinct fuzzy subgroups arising from the d-cyclic chains, the numbers p, q, r, s reflect the number of chains that are isomorphic in a cluster (i) of chains. For instance $p+4p+p$ shows that there are 6 clusters of chains, each with p maximal chains. Since these isomorphic chains have now been collapsed, it follows that each p now collapses to 1. Thus $p+4p+p$ becomes $1+4+1 = 6$. Similarly for the other primes. Therefore the number of non-isomorphic fuzzy subgroups arising from the d-cyclic maximal chains is $2^5(1+1+1+1)+2^4(4+4+4+4)+2^3(1+1+1+1) = 2^5(4)+2^4(16)+2^3(4)$.

Next we look at the 2d-cyclic non-isomorphic maximal chains. As in the d-cyclic case, in the formula $2^5(pq+qs+ps+qs+pr+rs)+2^4(2pq+2pr+2qr+2ps+2qs+2rs)+2^3(pq+pr+qr+ps+rs+qs)$ for the number of distinct fuzzy subgroups arising from the 2d-cyclic chains, each of the numbers pq, pr, ps, qr, qs, rs reflects the number of maximal chains that are isomorphic in a cluster (i) of chains. Since these isomorphic chains have now been collapsed, it follows that each of these six numbers collapses to 1. Thus the number of non-isomorphic fuzzy subgroups for the 2d-cyclic maximal chains is $2^5(1+1+1+1+1+1)+2^4(2+2+2+2+2+2)+2^3(1+1+1+1+1+1) = 2^5(6) + 2^4(12) + 2^3(6)$.

Next we consider the 3d-cyclic maximal chains, and since the pattern is now clear, the formula $2^5(pqr+pqs+prs+qrs)+2^4(4pqr+4pqs+4prs+4qrs)+2^3(pqr+pqs+prs+qrs)$ for the number of distinct fuzzy subgroups arising from $3d$-cyclic maximal chains gives $2^5(1+1+1+1)+2^4(4+4+4+4)+2^3(1+1+1+1) = 2^5(4)+2^4(16)+2^3(4)$ non-isomorphic fuzzy subgroups.

Finally, we consider the b-cyclic non-isomorphic maximal chains. These are the numbered b-cyclic chains listed in the previous section.

From the formula $2^5(pqrs) + 2^4(11 \times pqrs) + 2^3(12 \times pqrs)$ for distinct fuzzy subgroups arising from b-cyclic maximal chains, the number of non-isomorphic fuzzy subgroups for the b-cyclic maximal chains is $2^5(1) + 2^4(11) + 2^3(12)$.

So the total number of non-isomorphic fuzzy subgroups of D_{pqrs} is $2^6-1+2^5\times 11+2^4\times 11+2^3 + 2^5(4)+2^4(16)+2^3(4) + 2^5(6)+2^4(12)+2^3(6) + 2^5(4)+2^4(16)+2^3(4) + 2^5(1)+2^4(11)+2^3(12) = 2^6-1+2^5(11+4+6+4+1)+2^4(11+16+12+16+11)+2^3(1+4+6+4+12)$. Thus we have proved

Theorem 5.1. *The number of non-isomorphic fuzzy subgroups of $G = D_{pqrs}$ is*

$$2^6 - 1 + 2^5(26) + 2^4(66) + 2^3(27) \tag{12}$$

Observe that this number depends only on the number of primes in D_{pqrs} and not on the specific primes. Also it seems clear that the sum of the coefficients of the powers of 2 is the number of non-isomorphic maximal chains. Finally the following proposition is easily established.

Proposition 5.2. *The number of non-isomorphic maximal chains of subgroups of D_{pqrs} is* $5! = 5 \times 4!$.

Hence we have

Conjecture. The number of non-isomorphic maximal chains of subgroups of $D_{p_1p_2\cdots p_n}$ is $(n+1)!$ where all the p_i are distinct primes.

6. Conclusion

(1) By not simplifying further some of the results and formulae in this work, we hope to start a pattern that may lead to the classification of fuzzy subgroups of the dihedral group $D_{p_1p_2\ldots p_n}$ where the p_i are distinct primes, and thereafter the general dihedral group D_n for any positive integer n.

(2) We have successfully presented some general results for the number of maximal chains of the group $D_{p_1p_2\ldots p_n}$.

(3) It is still not easy to conjecture the number of distinct fuzzy subgroups of $D_{p_1p_2\ldots p_n}$. However, the number of non-isomorphic fuzzy subgroups of $D_{p_1p_2\ldots p_n}$ is becoming clearer and it will be taken up in another paper.

Acknowledgements. The second author wishes to thank the National Research Foundation of South Africa (NRF) for financial support.

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