SOLVABLE L-SUBGROUP OF AN L-GROUP

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Abstract. In this paper, we study the notion of solvable L-subgroup of an L-group and provide its level subset characterization and this justifies the suitability of this extension. Throughout this work, we have used normality of an L-subgroup of an L-group in the sense of Wu rather than Liu.

1. Introduction

Rosenfeld introduced the notion of a fuzzy group in his pioneering paper [18] which laid the foundation for the development of fuzzy algebraic substructures. The early development of this theory is marked by the emergence of set product of fuzzy sets and the notion of fuzzy normal subgroup of a group introduced by Liu [8]. This notion of fuzzy normal subgroup was defined in an ordinary group and was pursued by almost all the researchers in the development of fuzzy group theory. During eighties several concepts came into existence which were related to this definition of normality of fuzzy subgroups such as fuzzy cosets, fuzzy conjugates and the normalizer of a fuzzy subgroup. Using this notion, even an attempt was made in [13] to introduce the concept of a solvable fuzzy subgroup. However, before the introduction of normality introduced by Liu [8], Wu [22] in 1981, formulated the concept of normality for a fuzzy subgroup of a fuzzy group. For this development, Wu preferred to work in L-setting rather than fuzzy setting. Although this definition of normal fuzzy subgroup in a fuzzy group is more fruitful than the definition of Liu [8], this concept is not much discussed in the literature so far except for few ones like Martinez and Mordeson [9, 10]. In a recent paper [2], a detailed study of a normal fuzzy subgroup and characteristic fuzzy subgroup of a fuzzy group is carried out. Some more researchers [17, 19] also discussed solvability of fuzzy subgroups. In the studies carried out by these authors, also, the concept of normality introduced by Liu [8] is used. Consequently, the parent structure in their studies is an ordinary group not a fuzzy group. In this paper, we introduce the concept of a solvable L-subgroup of an L-group. Throughout our study, we use normality of L-subgroups in the sense of Wu [22]. During this development, we need a modification in the definition of commutator fuzzy subsets introduced in [3]. This modified notion has played a significant role in the development of a consistent theory of nilpotent L-subgroups in [3]. In the present study, also, this modification plays a key role. The concept of trivial L-subgroup modified in [3] is

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used very fruitfully in the development of this paper. Homomorphic images and homomorphic pre-images of solvable L-subgroups are also discussed. Moreover, the level subset characterization of solvable L-subgroup is established which justifies the suitability of this extension.

2. **Preliminaries**

Throughout this paper, the system $\langle L, \leq, \vee, \wedge\rangle$ denotes a completely distributive lattice where '$\leq$' denotes the partial ordering of L, the join (sup) and the meet (inf) of the elements of L are denoted by '$\vee$' and '$\wedge$', respectively. Also, we write 1 and 0 for maximal and minimal elements of L, respectively. In this section, we first introduce some basic definitions and results which are used in the sequel. For details, we refer to [1, 5, 7, 11, 12, 14, 15, 16, 17, 20, 21, 22].

An L-subset of X is a function from X into L. The set of L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) : x \in X\}$ is called the image of μ and is denoted by $Im\ \mu$ and the tip of μ is defined as $\underset{x\in X}{\vee}\{\mu(x)\}$. Also, $\underset{x\in X}{\wedge}\{\mu(x)\}$ is known as the tail of μ. If $\mu, \nu \in L^X$, then we say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted by $\mu \subseteq \nu$. For a family $\{\mu_i : i \in I\}$ of L-subsets in X, where I is a nonempty index set, the union $\underset{i\in I}{\cup}\mu_i$ and the intersection $\underset{i\in I}{\cap}\mu_i$ of a family $\{\mu_i : i \in I\}$ are, respectively, defined by

$$\left(\underset{i\in I}{\cup}\mu_i\right)(x) = \underset{i\in I}{\vee}\{\mu_i(x)\} \text{ and } \left(\underset{i\in I}{\cap}\mu_i\right)(x) = \underset{i\in I}{\wedge}\{\mu_i(x)\}$$

for each $x \in X$. If $\mu \in L^X$ and $a \in L$, then the level subset μ_a is defined as

$$\mu_a = \{x \in X : \mu(x) \geq a\}.$$

Proposition 2.1. *Let $\mu, \nu \in L^X$. Then,*

(i) *if $\mu \subseteq \nu$, then $\mu_a \subseteq \nu_a$ for each $a \in L$,*
(ii) *if $\mu_a \subseteq \nu_a$ for each $a \in Im\ \mu$, then $\mu \subseteq \nu$.*

Let f be a mapping from a set X to a set Y. If $\mu \in L^X$ and $\nu \in L^Y$, then the image $f(\mu)$ of μ under f and the preimage $f^{-1}(\nu)$ of ν under f are L-subsets of Y and X, respectively, defined by

$$f(\mu)(y) = \underset{x\in f^{-1}(y)}{\vee}\{\mu(x)\} \quad \text{and} \quad f^{-1}(\nu)(x) = \nu(f(x)).$$

Here we point out that in the above definition if $f^{-1}(y) = \phi$, then $f(\mu)(y)$, being the least upper bound of the empty set, is zero.

The following result is well known in literature:

Proposition 2.2. *Let f be a mapping from a set X to a set Y.*

(i) *If $\mu_1, \mu_2 \in L^X$ and $\mu_1 \subseteq \mu_2$, then $f(\mu_1) \subseteq f(\mu_2)$,*
(ii) *if $\nu_1, \nu_2 \in L^Y$ and $\nu_1 \subseteq \nu_2$, then $f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2)$.*

Throughout this paper G denotes an ordinary group with the identity element 'e', and I denotes a nonempty indexing set.

Definition 2.3. Let $\mu \in L^G$. Then, μ is called an L-subgroup of G if for each $x, y \in G$,

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x^{-1}) = \mu(x)$.

The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element of G. Moreover, $\mu \in L(G)$ is said to be a proper L-subgroup of G if μ is non-constant.

Theorem 2.4. *Let $\mu \in L^G$ with tip a_0. Then, $\mu \in L(G)$ if and only if μ_a is a subgroup of G for each $a \leq a_0$.*

Definition 2.5. Let $\mu \in L(G)$. Then, μ is called an L-normal subgroup of G if for all $x, y \in G$, $\mu(xy) = \mu(yx)$.

It is well known in literature that the intersection of an arbitrary family of L-subgroups of a group is an L-subgroup of the given group.

Definition 2.6. Let $\mu \in L^G$. Then, the L-subgroup of G generated by μ, denoted by $\langle \mu \rangle$, is defined as the smallest L-subgroup of G which contains μ. i.e.

$$\langle \mu \rangle = \bigcap \{\mu_i \in L(G) \ : \ \mu \subseteq \mu_i\}.$$

If $\mu, \eta \in L^X$ and $\eta \subseteq \mu$, then we say that η is an L-subset of μ. We denote the set of L-subsets of μ by L^μ. If $\mu, \eta \in L(G)$ and $\eta \subseteq \mu$, then we say that η is an L-subgroup of μ. Further, if η is non-constant and $\eta \neq \mu$, then η is said to be a proper L-subgroup of μ. Also, η is said to be a trivial L-subgroup of μ if its chain of level subgroups contains only $\{e\}$ and G.

Let η be an L-subgroup of μ. Then, we define the following L-subgroup of μ contained in η, denoted by $\eta_{t_0}^{a_0}$, as follows:

$$\eta_{t_0}^{a_0}(x) = \begin{cases} a_0, & \text{if } \ x = e, \\ t_0, & \text{if } \ x \neq e, \end{cases}$$

where $a_0 = \eta(e)$ and $t_0 = \inf \eta$. Here $\eta_{t_0}^{a_0}$, a trivial L-subgroup of μ, is called the trivial L-subgroup of η.

The set of L-subgroups of an L-group μ is denoted by $L(\mu)$. From now on μ denotes an L-subgroup of G and we shall call the parent L-subgroup μ simply an L-group.

Theorem 2.7. *Let $\eta \in L^\mu$ with tip a_0. Then, $\eta \in L(\mu)$ if and only if η_a is a subgroup of μ_a for each $a \leq a_0$.*

Here we point out that for every subgroup H of G, 1_H is an L-subgroup of the L-group 1_G. If $\eta \in L^\mu$ and $\langle \eta \rangle_\mu$ denotes the L-subgroup of μ generated by η, then it can be easily verified that $\langle \eta \rangle_\mu = \langle \eta \rangle$.

Recall the following results from [12]:

Theorem 2.8. *Let $f : G \to K$ be a group homomorphism. If $\mu \in L(G)$, then $f(\eta)$ is an L-subgroup of $f(\mu)$ for each $\eta \in L(\mu)$.*

Theorem 2.9. *Let $f : G \to K$ be a group homomorphism.If $\nu \in L(K)$, then $f^{-1}(\theta)$ is an L-subgroup of $f^{-1}(\nu)$ for each $\theta \in L(\nu)$.*

We recall the definition of a normal L-subgroup of an L-group.

Definition 2.10. Let $\eta \in L(\mu)$. Then, we say that η is a normal L-subgroup of μ if for all $x, y \in G$

$$\eta(yxy^{-1}) \geq \eta(x) \wedge \mu(y).$$

The set of normal L-subgroups of μ is denoted by $NL(\mu)$.

Theorem 2.11. *Let $\eta \in L(\mu)$ with tip a_0. Then, $\eta \in NL(\mu)$ if and only if η_a is a normal subgroup of μ_a for each $a \leq a_0$.*

Definition 2.12. Let $\eta \in L^\mu$. Then, η is said to have sup-property if for each nonempty subset A of G, there exists $a_0 \in A$ such that $\sup\limits_{a \in A}\{\eta(a)\} = \eta(a_0)$.

If $x, y \in G$, then their commutator $xyx^{-1}y^{-1}$ is denoted by $[x, y]$. Recall the following from [3]:

Definition 2.13. Let $\eta, \theta \in L^\mu$. Then, the commutator of η and θ is an L-subset (η, θ) of G defined as follows:

$$(\eta, \theta)(x) = \begin{cases} \bigvee\limits_{x=[y,z]}\{\eta(y) \wedge \theta(z)\}, & \text{if x is a commutator,} \\ \inf \eta \wedge \inf \theta, & \text{if if x is not a commutator.} \end{cases}$$

The commutator L-subgroup of $\eta, \theta \in L^\mu$ is defined as the L-subgroup of G generated by (η, θ). It is denoted by $[\eta, \theta]$. Clearly, $\inf(\eta, \theta) = \inf \eta \wedge \inf \theta$ and $[\eta, \theta] \in L(\mu)$.

Proposition 2.14. *Let $\eta, \theta \in L(\mu)$. Then,*

$$[\eta, \theta](e) = \eta(e) \wedge \theta(e).$$

Proposition 2.15. *Let $\eta, \theta \in L^\mu$ and $\eta \subseteq \theta$. Then, $[\eta, \sigma] \subseteq [\theta, \sigma]$ for each $\sigma \in L^\mu$.*

Proposition 2.16. *Let $\eta, \theta \in L^\mu$. Then, $[\eta, \theta] = [\theta, \eta]$.*

Theorem 2.17. *Let $\eta, \theta \in NL(\mu)$. Then, $[\eta, \theta] \in NL(\mu)$.*

Next, we recall the notion of nilpotent L-subgroup [3]:

Let $\eta \in \mu$. Then, define inductively the following sequence of L-subgroups of μ:

$$Z_0(\eta) = \eta, \text{ and } Z_i(\eta) = [Z_{i-1}(\eta), \eta] \text{ for each } i.$$

It is easy to verify that $Z_i(\eta) \subseteq Z_{i-1}(\eta)$. Moreover, $Z_i(\eta)$ and η have identical tips and, also, the identical tails.

Definition 2.18. Let $\eta \in L(\mu)$ with tip a_0 and tail t_0 and $a_0 \neq t_0$. If the descending central chain

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_i(\eta) \supseteq \cdots$$

terminates finitely to the trivial L-subgroup $\eta_{t_0}^{a_0}$, then η is known as a nilpotent L-subgroup of μ. More precisely, η is said to be nilpotent of class c if c is the least non-negative integer such that $Z_c(\eta) = \eta_{t_0}^{a_0}$. In this case, the series

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_c(\eta) = \eta_{t_0}^{a_0}$$

is called the descending central series of η. If η is a nilpotent L-subgroup of μ, then we simply write η is nilpotent.

Definition 2.19. Let $\eta \in L(\mu)$ with tip a_0 and tail t_0. Then, the chain

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n \supseteq \cdots$$

of L-subgroups of G is called a central chain of η if

$$[\eta_{i-1}, \eta] \subseteq \eta_i \text{ for each } i.$$

If $a_0 \neq t_0$ and there exists a positive integer m such that $\eta_m = \eta_{t_0}^{a_0}$, where $\eta_{t_0}^{a_0}$ is the trivial L-subgroup of η, then

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_m = \eta_{t_0}^{a_0}$$

is known as a central series of η.

Theorem 2.20. *Let $\eta \in L(\mu)$ be a proper L-subgroup. Then, η is nilpotent if and only if η has a central series.*

3. Solvable L-subgroups

We begin this section by recalling the following results from [5] for generating an L-subgroup by a given L-subset of an L-group and study its relationship with other notions of L-group theory.

Theorem 3.1. *Let $\eta \in L^{\mu}$. Let $a_0 = \underset{x \in G}{\vee} \{\eta(x)\}$ and define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \underset{a \leq a_0}{\vee} \{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$.

Corollary 3.2. *Let $\eta \in L^{\mu}$. Then, $\langle \eta \rangle (e) = \underset{x \in G}{\vee} \{\eta(x)\}$.*

Theorem 3.3. *Let $\eta \in L^{\mu}$ and possesses sup-property. Then, define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \underset{a \in Im\mu}{\vee} \{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$. Moreover, $\hat{\eta}$ possesses sup-property and Im $\hat{\eta} \subseteq$ Im μ.

In order to discuss the solvable L-subgroup, we begin with:
Let $\eta \in L(\mu)$. Then, define inductively the following sequence of L-subgroups of μ:

$$\eta_0 = \eta \text{ and } \eta^{(i)} = [\eta^{(i-1)}, \eta^{(i-1)}] \text{ for each } i.$$

Proposition 3.4. *Let $\eta \in L(\mu)$. Then, $\eta^{(i)}$ and η have the identical tips and, also, the identical tails.*

Theorem 3.5. *Let $\eta \in L(\mu)$. Then, $\eta^{(i)} \subseteq \eta^{(i-1)}$ for each i.*

Proof. We prove the result by induction on i. Let $x \in G$. If x is not a commutator, then

$$(\eta, \eta)(x) = \inf \eta \leq \eta(x).$$

Hence the result is true. Suppose that $x = [y, z]$, where $y, z \in G$. Then, as $\eta \in L(\mu)$, we have

$$(\eta, \eta)\,(x) = \bigvee_{x=[y,z]} \{\eta(y) \wedge \eta(z)\} \leq \bigvee_{x=[y,z]} \eta([y, z]) = \eta(x).$$

Consequently,

$$\eta^{(1)} = [\eta, \eta] = \langle(\eta, \eta)\rangle \subseteq \eta = \eta^{(0)}.$$

Now, suppose $\eta^{(k)} \subseteq \eta^{(k-1)}$. Then, by using Proposition 2.15 and Proposition 2.16, we have

$$\begin{aligned}\eta^{(k+1)} &= [\eta^{(k)}, \eta^{(k)}] \\ &\subseteq [\eta^{(k-1)}, \eta^{(k)}] \\ &= [\eta^{(k)}, \eta^{(k-1)}] \\ &\subseteq [\eta^{(k-1)}, \eta^{(k-1)}] \\ &= \eta^{(k)}.\end{aligned}$$

Thus, the result follows by the Principle of Mathematical Induction. □

In the following proposition we prove that each member of the above defined sequence is a normal L-subgroup of its preceding member in the sense of Wu[22].

Proposition 3.6. *Let $\eta \in NL(\mu)$. Then, $\eta^{(i)} \in NL(\eta^{(i-1)})$.*

Proof. Let $x, g \in G$. Then,

$$\begin{aligned}\eta^{(i)}(gxg^{-1}) &= \eta^{(i)}([g, x]x) \\ &\geq \eta^{(i)}([g, x]) \wedge \eta^{(i)}(x) \\ &\geq (\eta^{(i-1)}, \eta^{(i-1)})([g, x]) \wedge \eta^{(i)}(x) \\ &\geq \eta^{(i-1)}(g) \wedge \eta^{(i-1)}(x) \wedge \eta^{(i)}(x) \\ &= \eta^{(i-1)}(g) \wedge \eta^{(i)}(x). (\mathit{As}\ \eta^{(i)} \subseteq \eta^{(i-1)})\end{aligned}$$

This establishes the result. □

Proposition 3.7. *Let $\eta \in NL(\mu)$. Then, $\eta^{(i)} \in NL(\mu)$.*

Proof. As $\eta \in NL(\mu)$, by Theorem 2.17, $\eta^{(1)} = [\eta, \eta] \in NL(\mu)$. Thus, the result follows by a routine application of the Principle of Mathematical Induction and again by using Theorem 2.17. □

Here we redefine the concepts of derived series and derived chain in L-setting and that of a solvable L-subgroup of an L-group by using the notion of normality in the sense of Wu [22]. Note that we use the modified notion of a trivial L-subgroup of an L-group in the definition of a solvable L-subgroup. This notion has been utilized fruitfully in our previous works on nilpotent L-subgroups [3].

Definition 3.8. Let $\eta \in L(\mu)$. Then, the chain

$$\eta = \eta^{(0)} \supseteq \eta^{(1)} \supseteq \cdots \supseteq \eta^{(n)} \supseteq \cdots$$

of L-subgroups of η is called the derived chain of η. If η is a proper L-subgroup with tip a_0 and tail t_0 and the derived series terminates finitely to the trivial L-subgroup $\eta_{t_0}^{a_0}$, then η is known as a solvable$L$-subgroup of μ. If m is the least non negative integer such that $\eta^{(m)} = \eta_{t_0}^{a_0}$. Then, the series

$$\eta = \eta^{(0)} \supseteq \eta^{(1)} \supseteq \cdots \supseteq \eta^{(m)} = \eta_{t_0}^{a_0}$$

is called the derived series of η and m is said to be the solvable length of η .

Definition 3.9. Let $\eta \in L(\mu)$ be a proper L-subgroup with tip a_0 and tail t_0 . If $\eta_{t_0}^{a_0}$ is the trivial L-subgroup of η, then a series

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n = \eta_{t_0}^{a_0}$$

of L-subgroups of η is said to be a solvable series for η, if for each $i = 1, 2, ..., n$

$$[\eta_{i-1}, \eta_{i-1}] \subseteq \eta_i \ .$$

The following result is immediate:

Theorem 3.10. *Let $\eta \in L(\mu)$ be a proper L-subgroup with tip a_0 and tail t_0. If $\eta_{t_0}^{a_0}$ is the trivial L-subgroup of η and*

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n = \eta_{t_0}^{a_0}$$

is a solvable series for η, then

(i) $[\eta_{i-1}, \eta_{i-1},] \subseteq \eta_i$ *if and only if* $\eta_i([x, y]) \geq \eta_{i-1}(x) \wedge \eta_{i-1}(y)$ *for each* $x, y \in G$,

(ii) $\eta_i \in NL(\eta_{i-1})$.

Theorem 3.11. *Let $\eta \in L(\mu)$ be a proper L-subgroup with tip a_0 and tail t_0. Then, η is solvable if and only if η has a solvable series.*

Proof. $\Rightarrow$ Let $\eta = \eta_0 \supseteq \eta_1 \supseteq ... \supseteq \eta_n = \eta_{t_0}^{a_0}$, be a solvable series for η. We claim that $\eta^{(i)} \subseteq \eta_i$ for each i. Note that $\eta^{(0)} = \eta = \eta_0$. Moreover, by the definition of a solvable series

$$\eta^{(1)} = [\eta, \eta] = [\eta_0, \eta_0] \subseteq \eta_1.$$

Next, let $\eta^{(i)} \subseteq \eta_i$. Again, by Proposition 2.15, Proposition 2.16 and by the definition of a solvable series, we have

$$\eta^{(i+1)} = [\eta^{(i)}, \eta^{(i)}] \subseteq [\eta_i, \eta_i] \subseteq \eta_{i+1}.$$

This establishes our claim by the Principle of Mathematical Induction. Thus, we have

$$\eta_{t_0}^{a_0} \subseteq \eta^{(n)} \subseteq \eta_n = \eta_{t_0}^{a_0},$$

which implies $\eta^{(n)} = \eta_{t_0}^{a_0}$. Hence, η is solvable.
$\Leftarrow$ If η is solvable, then the derived series is the required solvable series. $\square$

Theorem 3.12. *Let $\eta, \theta \in L(\mu)$ be proper L-subgroups having identical tails . If η is a solvable L-subgroup and $\theta \subseteq \eta$, then θ is also solvable.*

Proof. Let η be a solvable L-subgroup with the tip a_0 and the tail t_0. Then, there exists a positive integer m such that

$$\eta^{(m)} = \eta_{t_0}^{a_0},$$

where $\eta_{t_0}^{a_0}$ is the trivial L-subgroup of η . Let $\theta \subseteq \eta$. Then, in view of Proposition 2.15 and Proposition 2.16, it can be verified easily that for each i, $\theta^{(i)} \subseteq \eta^{(i)}$. Thus,

$$\theta^{(m)} \subseteq \eta^{(m)} = \eta_{t_0}^{a_0}.$$

By Proposition 3.4, $\theta^{(m)}$ and θ have identical tails and, by the hypothesis, η and θ have identical tails. Thus, $inf\ \theta^{(m)} = t_0$. This implies that if $b_0 = \theta(e)$, then

$$\theta_{t_0}^{b_0} \subseteq \theta^{(m)} \subseteq \eta_{t_0}^{a_0}.$$

Consequently,

$$\theta^{(m)}(x) = \begin{cases} b_0, & \text{if } \ x = e, \\ t_0, & \text{if } \ x \neq e. \end{cases}$$

Thus, θ has the following derived series:

$$\theta = \theta^{(0)} \supseteq \theta^{(1)} \supseteq ... \supseteq \theta^{(n)} = \theta_{t_0}^{b_0},$$

which implies θ is a solvable L-subgroup. $\square$

Lemma 3.13. *Let $\eta \in L(\mu)$ be a proper L-subgroup. Then, every central series of η is a solvable series.*

Proof. Let the tip and the tail of η be a_0 and t_0, respectively, and

$$\eta = \eta_0 \supseteq \eta_1 \supseteq ... \supseteq \eta_n = \eta_{t_0}^{a_0},$$

be a central series for η. As, $\eta_i \subseteq \eta$, by Proposition 2.15,

$$[\eta_i, \eta_i] \subseteq [\eta_i, \eta].$$

Moreover, by the definition of a central series

$$[\eta_i, \eta] \subseteq \eta_{i+1},$$

which implies $[\eta_i, \eta_i] \subseteq \eta_{i+1}$. This proves that every central series is, also, a solvable series. $\square$

Theorem 3.14. *Let $\eta \in L(\mu)$ be a proper L-subgroup. If η is nilpotent, then η is solvable.*

Proof. Let η be nilpotent. Then, by Theorem 2.20, η has a central series and by Lemma 3.13, every central series is a solvable series. Hence, by Theorem 3.11, the result follows. $\square$

In order to discuss the level subset characterization of a solvable L-subgroup, firstly we recall the following well known notion of an upper well ordered subset from the literature:

Definition 3.15. A non empty subset X of a lattice L is said to be upper well ordered if every non empty subset A of X contains its supremum i.e. if $sup\ A = a_0$, then $a_0 \in A$.

The following proposition is immediate:

Proposition 3.16. *Let $\eta \in L^{\mu}$. Then, η has sup-property if and only if $Im\ \eta$ is an upper well ordered subset of L.*

The above characterization allows a generalization of the concept of sup-property to an arbitrary family of L-subsets. This generalization widens the scope of application of sup-property.

Definition 3.17. Let $\{\eta_i\}_{i\in I} \subseteq L^{\mu}$. Then, $\{\eta_i\}_{i\in I}$ is said to be a supstar family if $\underset{i\in I}{\cup} Im\ \eta_i$ is an upper well ordered subset of L.

It is clear, from the above definition and Proposition 3.16, that each member of a supstar family of L-subsets satisfies sup-property. As a particular case, we say that two L-subsets μ and η are jointly supstar if $Im\ \mu \cup Im\ \eta$ is an upper well ordered subset of L.

Proposition 3.18. *Let $\eta, \theta \in L^{\mu}$. If η and θ are jointly supstar, then η and θ possess sup-property. Converse is true if the lattice L is a chain.*

Lemma 3.19. *Let $\eta \in L(\mu)$ and possesses sup-property. Then,*

$$Im\ \eta^{(i)} \subseteq Im\ \eta \bigcup \{inf\ \eta\}\,.$$

Proof. Let $x \in G$. By Theorem 3.3,

$$[\eta, \eta](x) = \underset{a\in Im(\eta,\eta)}{\vee} \{a : x \in \langle(\eta, \eta)_a\rangle\}.$$

Since η has sup-property, by Proposition 3.16, $Im\ \eta$ is upper well ordered. Thus, it follows that, $Im\ \eta \bigcup \{inf\ \eta\}$ is also upper well ordered. We now show that

$$Im\ (\eta, \eta) \subseteq Im\ \eta \bigcup \{Inf\ \eta\}.$$

For this purpose, let $x \in G$. If x is not a commutator, then

$$(\eta, \eta)(x) = inf\ \eta \in Im\ \eta \bigcup \{Inf\ \eta\}.$$

If x is a commutator, then

$$(\eta, \eta)(x) = \underset{x=[y,z]}{\vee} \{\eta(y \wedge \eta(z))\}.$$

Since $Im\ \eta$ is upper well ordered, we have

$$\eta(y) \vee \eta(z) = \eta(y) \text{ or } \eta(z).$$

Thus,

$$\eta(y) \wedge \eta(z) = \eta(z) \text{ or } \eta(y),$$

which implies $\bigvee \{\eta(y) \wedge \eta(z) : x = [y, z]\} \in Im\ \eta$. Therefore,

$$(\eta, \eta)(x) \in Im\ \eta.$$

Hence,

$$Im\ (\eta, \eta) \subseteq Im\ \eta \cup \{inf\ \eta\}.$$

Thus, $Im\ (\eta, \eta)$ being subset of an upper well ordered subset, is also upper well ordered. So, by Proposition 3.16, (η, η) possesses sup-property. Therefore, the Theorem 3.3 applies. Consequently,

$$Im\ [\eta, \eta] \subseteq Im\ (\eta, \eta) \subseteq Im\ \eta \cup \{inf\ \eta\},$$

i.e. $Im\ \eta^{(1)} \subseteq Im\ \eta \cup \{inf\ \eta\}$. Now suppose that for some positive integer k,

$$Im\ \eta^{(k)} \subseteq Im\ \eta \cup \{inf\ \eta\}\,. \tag{1}$$

Then, $Im\ \eta^{(k)}$ being a subset of an upper well ordered subset is also upper well ordered. Hence, by Proposition 3.16, $\eta^{(k)}$ possesses sup-property. Again, it can be verified that

$$Im\ \eta^{(k+1)} \subseteq Im\ \eta^{(k)} \cup \left\{inf\ \eta^{(k)}\right\}. \tag{2}$$

By Proposition 3.4, $Inf\eta^{(k)} = Inf\eta$. Consequently, by (1) and (2),

$$Im\ \eta^{(k+1)} \subseteq Im\ \eta \cup \{inf\ \eta\}\,.$$

Hence, the result follows from the Principle of Mathematical Induction. □

Corollary 3.20. *Let $\eta \in L(\mu)$ and possesses sup-property. Then,*
(i) $\eta^{(i)}$ possesses sup-property,
(ii) $\eta^{(i)}$ and η are jointly supstar.

Here, we recall the following results from [3].

Lemma 3.21. *Let $\eta \in L(\mu)$ and possesses sup-property. If $a_0 = \bigvee_{x \in G} \{\eta(x)\}$, then $\langle \eta \rangle_a = \langle \eta_a \rangle$ for each $a \leq a_0$.*

Theorem 3.22. *Let $\eta, \theta \in L(\mu)$ be jointly supstar and $a_0 = \eta(e) \wedge \theta(e)$. Then, for each $a \nleq inf\eta \bigwedge inf\theta$ and $a \leq a_0$,*

$$[\eta, \theta]_a = [\eta_a, \theta_a].$$

Lemma 3.23. *Let $\eta \in L(\mu)$ and possesses sup-property. Then, for each $a \nleq inf\eta$ and $a \leq \eta(e)$,*

$$\left(\eta^{(i)}\right)_a = \left(\eta_a\right)^{(i)}.$$

Proof. Let $a \nleq inf\eta$ and $a \leq \eta(e)$. We prove the result by applying induction on i. As η possesses sup-property, by Theorem 3.22,

$$\left(\eta^{(1)}\right)_a = [\eta_a, \eta_a] = (\eta_a)^{(1)}.$$

Now suppose that, for some positive integer k, $\left(\eta^{(k-1)}\right)_a = (\eta_a)^{(k-1)}$. By Lemma 3.19,

$$Im\ \eta^{(k-1)} \subseteq Im\ \eta \cup \{inf\ \eta\}.$$

Moreover, by Proposition 3.16, $Im\ \eta$ is upper well ordered. Thus, it follows that $Im\ \eta \cup \{inf\ \eta\}$ is also upper well ordered. Hence, $Im\ \eta^{(k-1)}$ being a subset of an upper well ordered subset is also upper well ordered. So, by Proposition 3.16, $\eta^{(k-1)}$ possesses sup-property. Moreover, by Proposition 3.4, $Inf\ \eta^{(k-1)} = Inf\ \eta$ so that $a \nleq Inf\ \eta$. Hence, by Theorem 3.22, we have

$$\begin{aligned}\left(\eta^{(k)}\right)_a &= \left[\eta^{(k-1)}, \eta^{(k-1)}\right]_a \\ &= \left[(\eta^{(k-1)})_a, (\eta^{(k-1)})_a\right] \\ &= \left[(\eta_a)^{(k-1)}, (\eta_a)^{(k-1)}\right] \\ &= (\eta_a)^{(k)}.\end{aligned}$$

Thus, the result is established by the Principle of Mathematical Induction. □

Theorem 3.24. *Let $\eta \in L(\mu)$ be a proper L-subgroup and possesses sup-property. Then, η is a solvable L-subgroup of μ of solvable length at most n if and only if η_a is a solvable subgroup of μ_a of solvable length at most n for each $a \nleq inf\ \eta$ and $a \leq \eta(e)$.*

Proof. $\Rightarrow$ Let $a \nleq inf\ \eta$ and $a \leq \eta(e)$. As η is a solvable L-subgroup of μ of solvable length at most n, there exists a positive integer m such that

$$\eta = \eta^{(0)} \supseteq \eta^{(1)} \supseteq ... \supseteq \eta^{(m)} = \eta_{t_0}^{a_0},$$

where $\eta_{t_0}^{a_0}$ is the trivial L-subgroup of η and $m \leq n$. Moreover, as $\eta^{(i)} \supseteq \eta^{(i-1)}$, by Proposition 2.1, $(\eta^{(i)})_a \supseteq (\eta^{(i-1)})_a$ for each i. Thus,

$$\eta_a = (\eta^{(0)})_a \supseteq (\eta^{(1)})_a \supseteq ... \supseteq (\eta^{(m)})_a = (\eta_{t_0}^{a_0})_a,$$

As $a \nleq t_0$, $(\eta_{t_0}^{a_0})_a = \{e\}$. Thus,

$$\eta_a = (\eta^{(0)})_a \supseteq (\eta^{(1)})_a \supseteq ... \supseteq (\eta^{(m)})_a = (\eta_{t_0}^{a_0})_a = \{e\}.$$

Since η possesses sup-property, by Lemma 3.23, $(\eta^{(i)})_a = (\eta_a)^{(i)}$. Hence,

$$\eta_a = (\eta_a)^{(0)} \supseteq (\eta_a)^{(1)} \supseteq ... \supseteq (\eta_a)^{(m)} = \{e\}.$$

Thus, η_a is a solvable subgroup of μ_a of solvable length m, $m \leq n$.
$\Leftarrow$ As η_a is a solvable subgroup of μ_a for each choice of $a \nleq inf\ \eta$ and $a \leq \eta(e)$, there exists a positive integer m_a such that

$$\eta_a = (\eta_a)^{(0)} \supseteq (\eta_a)^{(1)} \supseteq ... \supseteq (\eta_a)^{(m_a)} = \{e\}.$$

Let $m = sup\ m_a$, where $a \in Im\ \eta \sim \{t_0\}$. Then, $m \leq n$ and

$$\eta_a = (\eta_a)^{(0)} \supseteq (\eta_a)^{(1)} \supseteq ... \supseteq (\eta_a)^{(m)} = \{e\}.$$

Also, for each $a \leq a_0$ and $a \not\leq t_0$, $(\eta_a)^{(i)} = (\eta^{(i)})_a$. Hence, for each $a \in Im\ \eta \sim \{t_0\}$, we have

$$\eta_a = (\eta^{(0)})_a \supseteq (\eta^{(1)})_a \supseteq ... \supseteq (\eta^{(m)})_a = \{e\} = (\eta_{t_0}^{a_0})_a.$$

Also, by Proposition 3.4, $Inf\ \eta^{(i)} = Inf\ \eta = t_0$. So, we have

$$(\eta^{(i)})_a = G = (\eta_a)^{(i)} \text{ for } a = t_0.$$

Therefore, for each $a \in Im\ \eta \cup \{t_0\}$

$$\eta_a = (\eta^{(0)})_a \supseteq (\eta^{(1)})_a \supseteq ... \supseteq (\eta^{(m)})_a = (\eta_{t_0}^{a_0})_a.$$

Moreover, in view of Lemma 3.19, we have

$$Im\ \eta^{(i)} \subseteq Im\ \eta \cup \{t_0\}.$$

Thus, $(\eta^{(i-1)})_a \supseteq (\eta^{(i)})_a$ for each $a \in Im\ \eta^{(i)}$. So, by Proposition 2.1, we have

$$\eta^{(i-1)} \supseteq \eta^{(i)} \text{ for each } i = 1, 2, ..., m.$$

Further,

$$(\eta^{(m)})_a = \{e\} = (\eta_{t_0}^{a_0})_a \quad \text{for each } a \in Im\ \eta \sim \{t_0\},$$
$$(\eta^{(m)})_a = \{G\} = (\eta_{t_0}^{a_0})_a \quad \text{for } a = \inf \eta = t_0.$$

This implies $(\eta^{(m)})_a = (\eta_{t_0}^{a_0})_a$ for each $a \in Im\ \eta \cup \{t_0\}$. Moreover, as $Im\ \eta^{(m)} \subseteq Im\ \eta \cup \{t_0\}$, it follows that $\eta^{(m)} = \eta_{t_0}^{a_0}$. Consequently,

$$\eta = \eta^{(0)} \supseteq \eta^{(1)} \supseteq ... \supseteq \eta^{(m)} = \eta_{t_0}^{a_0}.$$

This establishes that η is a solvable L-subgroup of μ of solvable length at most $m \leq n$. □

Corollary 3.25. *A subgroup H of a group of G is a solvable if and only if 1_H is a solvable L-subgroup of 1_G.*

Proof. Note that the L-subgroup 1_H possesses sup-property and has two level subgroups H and G. Thus, the result follows in view of Theorem 3.24. □

Here we provide an example of a non trivial solvable L-subgroup which is not nilpotent.

Example 3.26. Let

$$D_6 = \left\{ \langle x, y \rangle \ : \ x^2 = \ e \ = y^6, \ xy = y^{-1}x \right\}$$

be the dihedral group of degree 6. Then,

$$D_3 = \left\{ \langle x, y^2 \rangle \ : \ x^2 = \ e \ = (y^2)^3, \ xy^2 = y^{-2}x \right\}$$

is the dihedral subgroup of D_6. Also, the commutator subgroup $D_6^{'}$ of D_6 is given by

$$D_6 = \langle y^2 \rangle \ = \left\{ e, \ y^2, \ y^4 \right\}.$$

Now, define the following L-subsets of D_6:

$$\mu(z)=\begin{cases}\dfrac{1}{2} & \text{if } z\in D_3,\\ \\ \dfrac{1}{4} & \text{if } z\in D_6\sim D_3.\end{cases}$$

$$\eta(z)=\begin{cases}\dfrac{1}{3} & \text{if } z\in \langle x\rangle,\\ \\ \dfrac{1}{9} & \text{if } z\in D_3\sim \langle x\rangle,\\ \\ \dfrac{1}{27} & \text{if } z\in D_6\sim D_3.\end{cases}$$

Here $A\sim B$ means usual set difference and $\langle x\rangle$ denotes the subgroup of D_6 generated by 'x'. Clearly, $\eta\subseteq\mu$, $\eta\neq\mu$ and $\eta,\mu\in L(G)$. Further, the following is easy to verify:

$$(\eta,\eta)(e)\ =\tfrac{1}{3},\ (\eta,\eta)(D_6^{'}\sim\{e\})=\tfrac{1}{9} \text{ and } (\eta,\eta)(D_6\sim D_6^{'})=\tfrac{1}{27}.$$

This implies

$$\eta^{(1)}=[\eta,\eta]=\langle(\eta,\eta)\rangle.$$

Next, note that

$$\left(\eta^{(1)},\eta^{(1)}\right)(e)\ =\ \tfrac{1}{3} \text{ and } \left(\eta^{(1)},\eta^{(1)}\right)(D_6\sim\{e\})=\tfrac{1}{27}.$$

Thus,

$$\eta^{(2)}=\left[\eta^{(1)},\eta^{(1)}\right]=\left\langle(\eta^{(1)},\eta^{(1)})\right\rangle=\eta_{t_0}^{a_0},$$

where $a_0=\eta(e)$ and $t_0=inf\ \eta$. Consequently,

$$\eta=\eta^{(0)}\supseteq\eta^{(1)}\supseteq\eta^{(2)}=\eta_{t_0}^{a_0}$$

is the derived series for η. Hence, η is a solvable L-subgroup of μ. Further, one can verify easily that

$$Z_i(\eta)=\eta^{(1)} \text{ for each } i,$$

which implies that η is not a nilpotent L-subgroup of μ.

In order to study homomorphic images and homomorphic preimages of solvable L-subgroups, we first prove the following results:

Lemma 3.27. *Let $f:G\to G^{'}$ be a group homomorphism and $\eta\in L^{\mu}$. Then, $f(\langle\eta\rangle)=\langle f(\eta)\rangle$.*

Proof. Since $\eta\subseteq\langle\eta\rangle$, by Proposition 2.2, $f(\eta)\subseteq f(\langle\eta\rangle)$. By Theorem 2.8, $f(\langle\eta\rangle)$ being the homomorphic image of an L-subgroup, is an L-subgroup. Thus, $\langle f(\eta)\rangle\subseteq f(\langle\eta\rangle)$. To prove the reverse inclusion, let $a_0=\underset{x\in G}{\vee}\{\eta(x)\}$. Next, let $y\in G$. Then,

$$f\left(\langle\eta\rangle\right)(y)=\bigvee_{f(x)=y}\langle\eta\rangle(x)$$
$$=\bigvee_{f(x)=y}\{\bigvee_{a\leq a_0}\{a:x\in\langle\eta_a\rangle\}\}.$$

(by Theorem 3.1)

We show that for $a\leq a_0$
$$if\ x\in\langle\eta_a\rangle,\ \text{then}\ f(x)\in\left\langle(f(\eta))_a\right\rangle.$$
Let $a\leq a_0$ and $x\in\langle\eta_a\rangle$. Then,
$$x=x_1\ x_2\ ...\ x_n,\ \text{where either}\ x_i\ or\ x_i^{-1}\in\eta_a,$$
which implies $f(x)=f(x_1)\ f(x_2)\ ...\ f(x_n)$. Also,
$$f(\eta)(f(x_i))\geq\eta(x_i)\ \text{or}\ f(\eta)(f(x_i^{-1}))\geq\eta(x_i^{-1}).$$
Thus,
$$f(\eta)(f(x_i))\geq a\ \text{or}\ f(\eta)(f(x_i^{-1}))\geq a$$
accordingly as x_i or $x_i^{-1}\in\eta_a$. This implies
$$f(x_i)\in(f(\eta))_a\ \text{or}\ (f(x_i^{-1}))\in(f(\eta))_a$$
accordingly as x_i *or* $x_i^{-1}\in\eta_a$. Hence,
$$f\left(\langle\eta\rangle\right)(y)\leq\bigvee_{f(x)=y}\left\{\bigvee_{a\leq a_0}\{a\ :\ f(x)\in\left\langle(f(\eta))_a\right\rangle\}\right\}$$
$$=\bigvee_{f(x)=y}\{\langle f(\eta)\rangle(f(x))\}$$
$$=\langle f(\eta)\rangle(y).$$
This establishes the result. □

Lemma 3.28. *Let $f:G\to G^{'}$ be a group homomorphism and $\eta,\theta\in L^{\mu}$. Then,*
$$f([\eta,\theta])=\langle f\left((\eta,\theta)\right)\rangle.$$

Proof. Since $[\eta,\theta]=\langle(\eta,\theta)\rangle$, the result follows in view of Lemma 3.27. □

Lemma 3.29. *Let $f:G\to G^{'}$ be a group homomorphism and $\eta,\theta\in L^{\mu}$. Let $inf\ \eta=inf\ f(\eta)$ and $inf\ \theta=inf\ f(\theta)$. Then,*
$$f([\eta,\theta])=[f(\eta),f(\theta)].$$

Lemma 3.30. *Let $f:G\to G^{'}$ be a group homomorphism and $\nu\in L(G^{'})$. Let $\lambda,\rho\in L^{\nu}$ and $inf\ \lambda=inf\ f^{-1}(\lambda)$, $inf\ \rho=inf\ f^{-1}(\rho)$. Then,*
$$f^{-1}([\lambda,\rho])\supseteq\left[f^{-1}(\lambda),f^{-1}(\rho)\right].$$

Theorem 3.31. *Let $f:G\to G^{'}$ be a group homomorphism and $\eta\in L^{\mu}$. Let $inf\ \eta=inf\ f(\eta)$. If η is solvable, then $f(\eta)$ is also solvable.*

Proof. By using Proposition 2.2 and Lemma 3.29, the derived series of $f(\eta)$ can be obtained by using the derived series of η. □

Theorem 3.32. *Let $f : G \to G^{'}$ be a group homomorphism and having solvable kernel $\nu \in Ł(G^{'})$. Let $\rho \in L(\nu)$ and $inf\ \rho = inf\ f^{-1}(\rho)$. If ρ is solvable, then $f^{-1}(\rho)$ is also solvable.*

Proof. Let ρ be a solvable subgroup with tip a_0 and tail t_0. Then, ρ has the following derived series:

$$\rho = \rho^{(0)} \supseteq \rho^{(1)} \supseteq \cdots \supseteq \rho^{(m)} = \rho_{t_0}^{a_0}.$$

As $\rho^{(i-1)} \supseteq \rho^{(i)}$, by Proposition 2.2,

$$f^{-1}(\rho^{(i-1)}) \supseteq f^{-1}(\rho^{(i)}).$$

We set $\delta_i = f^{-1}(\rho^{(i)})$ for $i = 0, 1, 2, ..., m$. Thus, by Theorem 2.9, $\delta_{i+1} \in L(\delta_i)$. Further,

$$\delta_m(x) = f^{-1}(\rho_{t_0}^{a_0}) = \begin{cases} a_0, & \text{if } \ x \in N, \\ t_0, & \text{if } \ x \notin N; \end{cases}$$

where N denotes the kernel of the homomorphism f. By Proposition 3.4, ρ and $\rho^{(i)}$ have identical tails and by the hypothesis, $inf\ \rho = inf\ f^{-1}(\rho)$. Thus, it follows that

$$inf\ \rho^{(i)} = inf\ f^{-1}\left(\rho^{(i)}\right) = t_0 \text{ for each } i = 1, 2, ..., m.$$

So, by Lemma 3.30, for each $i = 1, 2, ..., m$

$$[\delta_{i-1}, \delta_{i-1}] \subseteq \delta_i. \tag{3}$$

Thus, we have the following series of L-subgroups

$$\delta_0 = f^{-1}(\rho) \supseteq \delta_1 \supseteq \delta_2 \supseteq .. \supseteq \delta_m$$

satisfying (1). By the hypothesis the kernel N of the homomrphism f is solvable so that N has the following derived series:

$$N \supseteq N^{(1)} \supseteq N^{(2)} ... \supseteq N^{(n)} = \{e\}.$$

Further, for $i = 1, 2, ..., n$ define

$$\delta_{m+i}(x) = \begin{cases} a_0, & \text{if } \ x \in N^{(i)}, \\ t_0, & \text{if } \ x \notin N^{(i)}. \end{cases}$$

Then, clearly $\delta_{m+i} \in L(\rho)$ and $[\delta_{m+i-1}, \delta_{m+i-1}] = \delta_{m+i}$ for $i = 1, 2, ..., n$. Consequently,

$$f^{-1}(\rho) = \delta_0 \supseteq \delta_1 \supseteq \delta_2 \supseteq .. \supseteq \delta_{m+n} = (f^{-1}(\rho))_{t_0}^{a_0}.$$

is a solvable series for $f^{-1}(\rho)$. So, by Theorem 3.1, $f^{-1}(\rho)$ is a solvable subgroup. □

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