FUZZY BOUNDED SETS AND TOTALLY FUZZY BOUNDED SETS IN I-TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, a new definition of fuzzy bounded sets and totally fuzzy bounded sets is introduced and properties of such sets are studied. Then a relation between totally fuzzy bounded sets and N-compactness is discussed. Finally, a geometric characterization for fuzzy totally bounded sets in I- topological vector spaces is derived.

1. Introduction

The notions of fuzzy vector spaces and fuzzy topological vector spaces were first introduced by Katsaras and Liu [8] and then modified by Katsaras [6] and in [7]. According to the standardized terminology in [5], fuzzy topological vector spaces should be called I -topological vector spaces, where $I = [0, 1]$. I-topological vector spaces have been extensively studied [4, 15, 16, 17, 18, 19]. Two different kind of neighborhood structures for I-topologies have been used; the first uses the neighborhood introduced by Warren [14] and the second uses the quasi-coincidence neighborhood introduced by Pu and Liu [11]. The relations between these two kinds of neighborhoods structures were studied by Fang [2]. It is well known that the notions of boundedness and totally boundedness play an important role in theory of classical topological vector spaces and hence it is important to study the concepts of boundedness and totally boundedness in the theory of I-topological vector spaces. The first attempt in this regard was made by Katsaras [7].On the other hand, there is a famous theorem in classical topological vector space which states that $A \subseteq X$ is compact if and only if A is complete and totally bounded. Unfortunately this famous theorem cannot be generalized to I-topological vector spaces. Maybe there exists an inextricable difficulty in the notion as defined by Wu [16]. To amend the situation, we first give a new definition of fuzzy bounded sets and study the elementary properties of such sets. We show that our definition differs from the definition introduced by Katsaras [7]. We also introduce a new definition for totally fuzzy bounded sets and study their properties. this definition also differs from the definition given by Wu [16]. Most important of all, we obtain a relation between N-compactness and totally boundedness in I-topological vector spaces. Finally, we

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derive a geometric characterization for fuzzy totally bounded sets in I- topological vector spaces.

2. **Preliminaries**

Throughout this paper, $I = [0, 1]$ and $\mathbb{N}$ is the set of all positive integers. Let X be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and θ denote the zero element of X. Also, let I^X denote the family of all fuzzy subsets of X and $2^{(X)}$ the family of all crisp subsets of X. For $r \in [0, 1]$, we denote a fuzzy set which takes the constant value r on X by $\underline{r}$. The symbol $f^{\rightarrow}$ shall mean that $f^{\rightarrow}$ is the fuzzy operator induced from an ordinary operator f and $f^{\leftarrow}$ will denote the reverse of $f^{\rightarrow}$ [9, Definition 2.1.12]. A fuzzy point x_λ on X $(0 < \lambda \leq 1)$ is a fuzzy set defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

where x is called the support of x_λ and λ is the value of x_λ $(0 < \lambda \leq 1)$, in symbols $V(x_\lambda) = \lambda$. Similarly, y_λ, $x^{(n)}_\lambda$, etc. are fuzzy points with respective values λ and supports y, $x^{(n)}$, etc. respectively. We use the symbols e, $S(n)$, etc. to denote fuzzy points when the support and value are trivial. The set of all fuzzy points on X is denoted by $Pt(I^X)$. A fuzzy net $S = \{S(n), n \in D\}$ or $S = \{x^{(n)}_{\lambda_n}\}_{n \in D}$ is a function $S : D \rightarrow Pt(I^X)$ where D is a directed set with order relation $\succ$. Let $A \in I^X$ and x_λ be a fuzzy point, we say that x_λ quasi-coincides with A $(x_\lambda \hat{q} A)$ if $A(x) > 1 - \lambda and that x_\lambda$ belongs to A $(x_\lambda \in A)$ if $A(x) \geq \lambda$. If x_λ does not quasi-coincide with A, we write $x_\lambda \neg \hat{q} A$.

Definition 2.1. (*Katsaras and Liu* [8]*, Yan and Fang* [20]). Let $A, B \in I^X$ and $k \in \mathbb{K}$. Then $A + B$ and kA are defined respectively by

$$(A + B)(x) = \sup_{s+t=x} (A(s) \wedge B(t)),$$

$$(kA)(x) = A(\tfrac{x}{k}) \text{ whenever } k \neq 0,$$

$$(0A)(x) = \begin{cases} \bigvee\limits_{z \in X} A(z) & \text{if } x = \theta, \\ 0 & \text{if } x \neq \theta. \end{cases}$$

In particular, for x_λ, $y_\mu \in Pt(I^X)$, we have

$$x_\lambda + y_\mu = (x + y)_{\lambda \wedge \mu}, \ \ kx_\lambda = (kx)_\lambda.$$

Definition 2.2. (*Katsaras* [6]). Let X be a vector space over $\mathbb{K}$. $A \in I^X$ is said to be balanced if $kA \subset A$ for each k with $|k| \leq 1$.

Definition 2.3. (*Wu and Fang* [15]*, Fang* [2]). Let X be a vector space over $\mathbb{K}$, $\lambda \in (0, 1]$. $A \in I^X$ is said to be Q$-\lambda-$absorbing if for each $x \in X$ there exists $\delta > 0$ such that $tx_\lambda \hat{q} A$ for all $|t| \leq \delta$. A is called absorbing, if it is Q$-\lambda-$absorbing for each $\lambda \in (0, 1]$.

Definition 2.4. (*Pu and Liu* [11]*, Liu and Luo* [9]). Let (X, δ) be an I-topological space and $x_\lambda \in Pt(I^X)$.

(1) A fuzzy set U on X is called a Q- neighborhood of x_λ if and only if there exists $G \in \delta$ such that $x_\lambda \hat{q} G \subset U$.
(2) A family U_{x_λ} of Q- neighborhood of x_λ is called a Q-neighborhood base of x_λ if and only if for every Q-neighborhood A of x_λ, there exists $U \in \mathrm{U}_{x_\lambda}$ such that $U \subset A$.

Definition 2.5. (*Liu and Luo* [9]). Let $S = \{S(n), n \in D\}$ be a fuzzy net in X. S is said to be quasi-coincident with A if, for each $n \in D$, $S(n)$ is quasi-coincident with A. S is said to be eventually quasi-coincident with A if there exists an element $n_0 \in D$ such that $S(n)$ is quasi-coincident with A whenever $n \in D$ and $n \succ n_0$. S is said to be frequently quasi-coincident with A if, for each element $n_0 \in D$, there exists an element $n \in D$ $(n \succ n_0)$ such that $S(n)$ is quasi-coincident with A.

Let (X, δ) be an I-topological space. A fuzzy net $S = \{S(n), n \in D\}$ is said to converge to a fuzzy point e($S \to e$) if, for each Q- neighborhood U of e, S is eventually quasi-coincident with U. A fuzzy point e is called a cluster point of a fuzzy net $S = \{S(n), n \in D\}$ ($S \infty e$) if, for each Q-neighborhood U of e, S is frequently quasi-coincident with U.

Definition 2.6. (*Liu and Luo* [9]). A fuzzy net $T = \{T(m), m \in E\}$ in X is called a fuzzy subnet of a fuzzy net $S = \{S(n), n \in D\}$ in X, if there exists a mapping $N : E \to D$ such that

(1) $T = S \circ N$, i.e. for each $m \in E$, $T(m) = S(N(m))$.

(2) for each $n \in D$, there exists $m_0 \in E$ such that $N(m) \succ n$ whenever $m \in E$, $m \succ m_0$.

Lemma 2.7. *(Liu and Luo* [9]*). Let (X, δ) be an I-topological space. A fuzzy point e is a cluster point of a fuzzy net $S = \{S(n), n \in D\}$ if and only if S has a fuzzy subnet T converging to e.*

Definition 2.8. (*Wang* [13]). Let (X, δ) be an I-topological space. For a fuzzy net $S = \{S(n), n \in D\}$, let λ_n be the value of $S(n)$. The crisp net $\{\lambda_n, n \in D\}$ in $(0, 1]$ will be called the value net of S and denoted $V(S)$. If $V(S)$ converges to a real number $\lambda \in (0, 1]$, then we say that S is a $\lambda-$net. In particular, if $\lambda_n = \lambda$ holds for all $n \in D$, then we say that S is a constant $\lambda-$net.

Remark 2.9. If $D = \mathbb{N}$, then we call S a $\lambda-$sequence.

Definition 2.10. (*Wang* [13]). Let (X, δ) be an I-topological space. $A \in I^X$ is called N-compact, if each $\lambda-$net$(\lambda \in (0, 1])$ contained in A has at least a cluster in A with value λ. In particular, if $A = \underline{\mathbf{1}}$ is N-compact, we call (X, δ) a N-compact I-topological space.

Definition 2.11. (*Katsaras* [6]). A stratified I-topology δ on a vector space X is said to be an I-vector topology, if the two mappings

$$f : X \times X \to X,\ (x, y) \mapsto x + y,$$

$$g : \mathbb{K} \times X \to X,\ (k, x) \mapsto kx,$$

are continuous, where $\mathbb{K}$ is equipped with the I-topology induced by the usual topology, and $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product I-topologies.

A vector space X with an I-vector topology δ, denoted by (X, δ), is called an I-topology vector space (I-tvs).

Lemma 2.12. *(Fang* [2]*). Each I-tvs (X, δ) has a Q-neighborhood base of θ_λ consisting of a balanced and absorbing Q-neighborhood.*

Definition 2.13. Let (X, δ) be an I-tvs. The $\lambda-$net $S = \{S(n), n \in D\}$ is a called $\lambda-$Cauchy net if, for every Q-neighborhood U of θ_λ, there exists $n_0 \in D$ such that

$$S(n') - S(n'')\hat{q}U \text{ whenever } n', n'' \in D,\ n', n'' \succ n_0.$$

Example 2.14. Let $X = \mathbb{R}$, $\delta = \{\underline{r} \mid r \in I\}$. Then clearly, (X, δ) is an I-tvs. In particular, if we let $D = \mathbb{N}$, then the $\frac{1}{4}$-sequence $\{\frac{1}{n}{}_{\frac{1}{4}+\frac{1}{n}}\}$ is a $\frac{1}{4}$-Cauchy net.

Definition 2.15. Let (X, δ) be an I-tvs. $A \in I^X$ is said to be complete, if each $\lambda-$Cauchy net in A converges to a fuzzy point in A with value λ.

Example 2.16. Let $X = \mathbb{R}$, $\delta = \{\underline{r} \mid r \in I\}$. By example 2.14, we know that (X, δ) is an I-tvs. Define the fuzzy set $A \in I^{\mathbb{R}}$ is as follows:

$$A(x) = \begin{cases} \frac{1}{2}, & x = 1 \\ \frac{1}{3}, & x = 2 \\ \frac{1}{4}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

It is easy to verify that A is a complete fuzzy set in (X, δ).

Definition 2.17. (*Katsaras* [7]) Let (X, δ) be an I-tvs. The fuzzy set $A \in I^X$ is said to be fuzzy bounded, if for each Warren neighborhood U of θ and every $r \in (0, U(\theta))$, there is a $t > 0$ such that $A \bigcap \underline{r} \subseteq tU$.

Definition 2.18. (*Wu and Fang* [15]). Let (X, δ) be an I-tvs. A fuzzy set $A \in I^X$ is said to be $\lambda-$bounded($\lambda \in (0, 1]$) if, for each Q- neighborhood U of θ_λ, there exists $k > 0$ and $r \in (1 - \lambda, 1]$ such that $A \bigcap \underline{r} \subset kU$. A is said to be bounded if A is $\lambda-$bounded for all $\lambda \in (0, 1]$.

3. **Fuzzy Boundedness**

In this section, we introduce a new definition of fuzzy bounded sets in I-tvs which is quite different from definitions 2.17 or definition 2.18. Then we discuss elementary properties of fuzzy bounded sets in the sense of this new definition.

We first prove that definitions 2.17 and 2.18 are equivalent.

Proposition 3.1. *A fuzzy set A in I-tvs (X, δ) is fuzzy bounded in the sense of Katsaras iff A is fuzzy bounded in the sense of Wu and Fang.*

Proof. Suppose that $A \in I^X$ is a fuzzy bounded set in the sense of Katsaras and B is a W- neighborhood base of θ. By [2, Theorem 3.2], for each $\lambda \in (0, 1]$, the family of fuzzy sets $\mathrm{B}_\lambda = \{B \mid B \in \mathrm{B}, B(\theta) > 1 - \lambda\}$ is a Q- neighborhood base of θ_λ. So, for each Q-neighborhood U of θ_λ, there exists $B \in \mathrm{B}$ such that $B(\theta) > 1 - \lambda$ and $B \subseteq U$. Since A is fuzzy bounded in the sense of Katsaras, for each $r \in (1-\lambda, B(\theta))$, there exists $t > 0$ such that $A \bigcap \underline{r} \subseteq tB \subseteq tU$. This implies that A is $\lambda-$bounded in the sense of the definition 2.17. Since $\lambda \in (0,1]$ is arbitrary, A is fuzzy bounded in the sense of Wu and Fang.

On the other hand, for each W-neighborhood V of θ and each $r \in (0, V(\theta))$, let $\lambda = 1 - r$. Then we have $V(\theta) > 1 - \lambda$. By [2, Theorem 3.2], V is a Q-neighborhood of θ_λ. Since A is λ-bounded, there exist $t > 0$ and $s \in (1-\lambda, 1]$ such that $A \bigcap \underline{s} \subseteq tV$. Since $s > 1 - \lambda = r$, the relation $A \bigcap \underline{r} \subseteq tV$ holds. Hence A is fuzzy bounded in the sense of Katsaras. □

Definition 3.2. Let (X, δ) be an I-tvs, $A \in I^X$ is said to be fuzzy bounded if, for each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in A, we have $\frac{1}{n}x^{(n)}_{\lambda_n} \to \theta_\lambda$.

Remark 3.3. By definition 3.2, we know that each fuzzy point $x_\lambda \in Pt(I^X)$ is fuzzy bounded.

Example 3.4. Let $X = \mathbb{R}$. For all $\lambda \in (0,1]$, we define the family of fuzzy sets U_λ of θ_λ as follows:

$$\mathrm{U}_\lambda = \begin{cases} \{\underline{r} \mid r > 1-\lambda\} & \text{if } 0 < \lambda \leq \frac{1}{2}, \\ \{U^t_a \mid 1-\lambda < t \leq 1,\ a > 0\} & \text{if } \frac{1}{2} < \lambda \leq 1. \end{cases}$$

where

$$U^t_a(x) = \begin{cases} t & \text{if } x \in (1-a, a), \\ 0 & \text{if } x \notin (1-a, a). \end{cases}$$

It is easy to verify that the family of fuzzy sets $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ satisfies the conditions (1)-(5) in [1, Theorem 4.1]. Thus there exists a unique fuzzy topology δ such that (X, δ) is an I-tvs and, for each $\lambda \in (0,1]$, U_λ is a Q-neighborhood base of θ_λ. $\delta = \{A \in I^X \mid \forall\ x_\mu \hat{q} A,\ \exists U \in \mathrm{U}_\mu, x + U \subseteq A\}$.

Let $A = \{n_{\frac{1}{3}}\}_{n\in\mathbb{N}}$, then it is easy to show that A is fuzzy bounded in the sense of definition 3.2. On the other hand, since A is not $\dfrac{2}{3}-$bounded in the sense of Wu and Fang, it is not fuzzy bounded in the sense of Wu and Fang or Katsaras.

Henceforth we shall use definition 3.2 for the notion of fuzzy bounded, we always

Theorem 3.5. *Let (X, δ) be an I-tvs. If A, B are fuzzy bounded sets, then*
(1) $A \cap B$;
(2) $A \cup B$;
(3) $A + B$;
(4) aA $(a \in \mathbb{K}, a \neq 0)$,
are fuzzy bounded sets.

Proof. (1) The proof is trivial.

(2) For each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in $A\cup B$, there exist a $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}_1}$ in A and a $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}_2}$ in B, such that $\mathbb{N}_1\cup\mathbb{N}_2=\mathbb{N}$. Since A, B are fuzzy bounded sets, hence

$$\tfrac{1}{n}x^{(n)}_{\lambda_n}\to\theta_\lambda\ (n\in\mathbb{N}_1)\text{ and }\tfrac{1}{n}x^{(n)}_{\lambda_n}\to\theta_\lambda\ (n\in\mathbb{N}_2).$$

So, for each Q-neighborhood U of θ_λ, there exists $n_i\in\mathbb{N}_i$ such that $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}U$ whenever $n\in\mathbb{N}_i,\ n\geq n_i,\ (i=1,2)$. For $n_i\in\mathbb{N}_i\subset\mathbb{N}$, there exists $n_0\in\mathbb{N}$ such that $n_0\geq n_1,n_2$. So for all $n\in\mathbb{N},\ n\geq n_0$, we also have $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}U$. Hence $A\cup B$ is fuzzy bounded.

(3) For each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in $A+B$ and a decreasing sequence of real numbers sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0+$, there exist $y^{(n)} and z^{(n)}$ such that $y^{(n)}+z^{(n)}=x^{(n)}$, $\{y^{(n)}_{\lambda_n-\varepsilon_n}\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in A and $\{z^{(n)}_{\lambda_n-\varepsilon_n}\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in B. Since A and B are fuzzy bounded, we have

$$\tfrac{1}{n}y^{(n)}_{\lambda_n-\varepsilon_n}\to\theta_\lambda,\ \tfrac{1}{n}z^{(n)}_{\lambda_n-\varepsilon_n}\to\theta_\lambda. \tag{3.1}$$

For each Q-neighborhood U of θ_λ, there exists a Q-neighborhood V of θ_λ, such that $V+V\subset U$. By (3.1), there exist $n_1,n_2\in\mathbb{N}$, such that

$$\tfrac{1}{n}y^{(n)}_{\lambda_n-\varepsilon_n}\hat{q}V\text{ whenever }n\in\mathbb{N},\ n\geq n_1$$

and

$$\tfrac{1}{n}z^{(n)}_{\lambda_n-\varepsilon_n}\hat{q}V\text{ whenever }n\in\mathbb{N},\ n\geq n_2.$$

Now if $n_0=\max\{n_1,\ n_2\}$, then for all $n\in\mathbb{N},\ n\geq n_0$, we have

$$\begin{aligned}nU(x^{(n)})&\geq n(V+V)(x^{(n)})=(nV+nV)(x^{(n)})\\&\geq(nV)(y^{(n)})\wedge(nV)(z^{(n)})>1-\lambda_n+\varepsilon_n>1-\lambda_n.\end{aligned}$$

i.e. $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}U$, so $\frac{1}{n}x^{(n)}_{\lambda_n}\to\theta_\lambda$. Therefore $A\cup B$ is fuzzy bounded.

(4) If $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in aA, then $\{\frac{1}{a}x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in A. Since A is fuzzy bounded, we have

$$\tfrac{1}{n}\tfrac{1}{a}x^{(n)}_{\lambda_n}\to\theta_\lambda. \tag{3.2}$$

For each Q-neighborhood U of θ_λ, there exist $\delta>0$ and a Q-neighborhood V of θ_λ such that $tV\subset U$ for all $t\in\mathbb{K}$ with $|t-a|<\delta$. In particular, for $t=a$, we have $aV\subset U$. By (3.2), there exists $n_0\in\mathbb{N}$ such that

$$\tfrac{1}{n}\tfrac{1}{a}x^{(n)}_{\lambda_n}\hat{q}V\text{ whenever }n\in\mathbb{N},\ n\geq n_0.$$

So

$$\tfrac{1}{n}x^{(n)}_{\lambda_n}\hat{q}aV\subset U\text{ whenever }n\in\mathbb{N},\ n\geq n_0,$$

i.e. $\frac{1}{n}x^{(n)}_{\lambda_n}\to\theta_\lambda$. Hence aA is fuzzy bounded. □

Remark 3.6. By Remark 3.3 and property (2) in Theorem 3.5, every finite set in I-tvs (X,δ) is fuzzy bounded.

Theorem 3.7. *Suppose that (X, δ) is an I- Tvs. Then $A \in I^X$ is fuzzy bounded in (X, δ) if and only if for each $\lambda \in [0, 1)$, $\sigma_{1-\lambda}(A)$ is bounded in the classical topological vector space $(X, \sigma_\lambda(\delta))$, where $\sigma_\lambda(A) = \{x \in X \mid A(x) > \lambda\}$.*

Proof. Necessity. Let $\lambda \in [0, 1)$, and $\{x^{(n)}\}_{n\in\mathbb{N}}$ be a sequence in $\sigma_{1-\lambda}(A)$. We shall prove that $\frac{1}{n}x^{(n)} \to \theta$.

For any open neighborhood U of θ in $(X, \sigma_\lambda(\delta))$, there is $V \in \delta$ such that $\theta \in U = \sigma_\lambda(V)$. Obviously, V is a Q -neighborhood of $\theta_{1-\lambda}$. Because $\{x^{(n)}\}_{n\in\mathbb{N}}$ is a sequence in $\sigma_{1-\lambda}(A)$, i.e. $A(x^{(n)}) > 1 - \lambda$, so $\{x^{(n)}_{1-\lambda}\}_{n\in\mathbb{N}}$ is a $(1-\lambda)$-sequence in A. Since A is fuzzy bounded, $\frac{1}{n}x^{(n)}_{1-\lambda} \to \theta_{1-\lambda}$, which implies that for the Q-neighborhood V of $\theta_{1-\lambda}$, there exists $n_0 \in \mathbb{N}$ such that

$$\tfrac{1}{n}x^{(n)}_{1-\lambda}\hat{q}V \text{ whenever } n \in \mathbb{N},\ n \geq n_0.$$

So for all $n \in \mathbb{N}$ such that $n \geq n_0$ and $nV(x^{(n)}) > \lambda$, we have $\frac{1}{n}x^{(n)} \in \sigma_\lambda(V) = U$. Hence $\frac{1}{n}x^{(n)} \to \theta$. Therefore $\sigma_{1-\lambda}(A)$ is bounded.

Sufficiency. For each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in A and any Q-neighborhood W of θ_λ, there is $U \in \delta$ with $\theta_\lambda\hat{q}U$ such that $U \subseteq W$. Since $U(\theta) > 1-\lambda$, there is $\varepsilon > 0$ such that $U(\theta) > 1 - \lambda + \varepsilon$. This means $\theta \in \sigma_{1-\lambda+\varepsilon}(U) \in \sigma_{1-\lambda+\varepsilon}(\delta)$. Moreover, $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in A. Hence there exists $n_0 \in \mathbb{N}$ such that $\lambda_n > \lambda - \varepsilon$ for all $n \geq n_0$. Thus $x^{(n)} \in \sigma_{\lambda-\varepsilon}(A)$ for all $n \geq n_0$. By

hypothesis, there is $n_1 \geq n_0$ such that $\frac{1}{n}x^{(n)} \in \sigma_{1-\lambda+\varepsilon}(U)$ for all $n \geq n_1$. This implies that $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}\chi_{\sigma_{1-\lambda+\varepsilon}(U)}$. It is also easy to verify that $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}\underline{(1-\lambda+\varepsilon)}$. Thus we have $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}\underline{(1-\lambda+\varepsilon)} \bigwedge \chi_{\sigma_{1-\lambda+\varepsilon}(U)}$. Since $U = \bigvee_{r\in[0,1]} \underline{r} \bigwedge \chi_{\sigma_r(U)} \geq \underline{(1-\lambda+\varepsilon)} \bigwedge \chi_{\sigma_{1-\lambda+\varepsilon}(U)}$, so $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}U \subseteq W$ for all $n \geq n_1$. Hence A is a fuzzy bounded set. □

Lemma 3.8. *([17]) Let (X, T) be a crisp topological vector space,* U *an open neighborhood base of θ and $(X, \omega(\mathrm{T}))$ the induced I-topological vector space. Then for each $\lambda \in (0, 1]$, the family of fuzzy sets $\mathrm{U}_\lambda = \{\mathcal{X}_U \bigcap \underline{r} \mid U \in \mathrm{U},\ r > 1 - \lambda\}$ is a Q-neighborhood base of θ_λ.*

Theorem 3.9. *Let $(X, \omega(\mathrm{T}))$ be the I-tvs induced by crisp topological vector space (X, T). Then $A \subseteq X$ is bounded in (X, T) if and only if $\mathcal{X}_A$ is fuzzy bounded in $(X, \omega(\mathrm{T}))$.*

Proof. Necessity. Clearly, for each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in $\mathcal{X}_A$, $\{x^{(n)}\}_{n\in\mathbb{N}}$ is a sequence in A. Since A is bounded, we have

$$\tfrac{1}{n}x^{(n)} \to \theta. \tag{3.3}$$

By Lemma 3.8, for any Q-neighborhood U of θ_λ, there are $V \in \mathrm{T}$ and $\alpha \in (1-\lambda, 1]$ such that $\mathcal{X}_V \bigcap \underline{\alpha} \subset U$. Hence, by (3.3), it there exists $n_0 \in \mathbb{N}$ such that

$$\tfrac{1}{n}x^{(n)} \in V \text{ whenever } n \in \mathbb{N},\ n \geq n_0.$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$, there exists $n_1 \in \mathbb{N}$ such that $\lambda_n + \alpha > 1$ for all $n \geq n_1$. Let $n_2 = \max\{n_0,\ n_1\}$, then $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}\mathcal{X}_V \bigcap \underline{\alpha}$ for all $n \geq n_2$. i.e., $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}U$. So $\frac{1}{n}x^{(n)}_{\lambda_n} \to \theta_\lambda$ and therefore $\mathcal{X}_A$ is fuzzy bounded.

Sufficiency. It is easy to see that for each sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in A and each $\lambda \in (0, 1]$, $\{x^{(n)}_\lambda\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in $\mathcal{X}_A$. Since $\mathcal{X}_A$ is fuzzy bounded, we have

$$\frac{1}{n}x^{(n)}_\lambda \to \theta_\lambda. \tag{3.4}$$

Clearly, for any neighborhood U of θ in (X, T), $\mathcal{X}_U$ is a Q-neighborhood of θ_λ in $(X, \omega(\mathrm{T}))$. By (3.4), it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n}x^{(n)}_\lambda \hat{q}\mathcal{X}_U \text{ whenever } n \in \mathbb{N},\ n \geq n_0.$$

So for all $n \in \mathbb{N},\ n \geq n_0$, $(nU)(x^{(n)}) > 1 - \lambda$, for each $\lambda \in (0, 1]$. Because $\lambda \in (0, 1]$ is arbitrary, we have $x^{(n)} \in nU$, i.e., $\frac{1}{n}x^{(n)} \in U$. Hence $\frac{1}{n}x^{(n)} \to \theta$ and therefore A is bounded. □

Theorem 3.10. *Let $\{(X_i, \delta_i)\}_{i\in\Lambda}$ be a family of I- topological vector spaces, and (X, δ) be the product space of $\{(X_i, \delta_i)\}_{i\in\Lambda}$. If A_i is fuzzy bounded in X_i $(i \in \Lambda)$, then $A = \prod_{i\in\Lambda} A_i$ is fuzzy bounded in product space X.*

Proof. For each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in A, $\{P_i^{\rightarrow}(x^{(n)}_{\lambda_n})\}_{n\in\mathbb{N}}$ is a $\lambda-$sequence in A_i $(\forall i \in \Lambda)$. Since A_i is $\lambda-$bounded, we have

$$\frac{1}{n}P_i^{\rightarrow}(x^{(n)}_{\lambda_n}) \to \theta^i_\lambda\ (i \in \Lambda). \tag{3.5}$$

For every Q-neighborhood U of θ_λ, there exists a finite set Λ_0 and Q-neighborhood U_i of θ^i_λ in X_i such that $\bigcap_{i\in\Lambda_0} P_i^{\leftarrow}(U_i) \subset U$. By (3.5), it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n}P_i^{\rightarrow}(x^{(n)}_{\lambda_n})\hat{q}U_i \text{ whenever } n \in \mathbb{N},\ n \geq n_0.$$

By continuity of P_i, we have $\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q}P_i^{\leftarrow}(U_i)$ $(i \in \Lambda_0)$. So

$$\frac{1}{n}x^{(n)}_{\lambda_n}\hat{q} \bigcap_{i\in\Lambda_0} P_i^{\leftarrow}(U_i) \subset U.$$

This implies that $\frac{1}{n}x^{(n)}_{\lambda_n} \to \theta_\lambda$. Therefore A is fuzzy bounded. □

4. Totally Fuzzy Boundedness

Definition 4.1. Let (X, δ) be an I-tvs, $A \in I^X$ is said to be totally fuzzy bounded if each $\lambda-$net contained in A has a $\lambda-$Cauchy subnet.

Remark 4.2. In [16], Wu introduced the notion of totally fuzzy bounded sets as follows: $A \in I^X$ is called λ -totally fuzzy bounded if, for each Q-neighborhood U of θ_λ, there exists $r \in (1 - \lambda, 1]$ and a finite set $F = \{z_1, \cdots, z_n\}$ such that $A \bigcap \underline{r} \subseteq \bigcup_{i=1}^{n}(z_i + U) = F + U$; A is called totally fuzzy bounded if A is λ-totally fuzzy bounded for each $\lambda \in (0, 1]$. In example 3.4, it is easily seen that if $A = \frac{1}{3}$, then A is totally fuzzy bounded in the sense of the definition 4.1, and it is clear that A is not $\dfrac{2}{3}$−totally fuzzy bounded in the sense of Wu [14], i.e., A is not totally fuzzy bounded in the sense of Wu [16]. Henceforth, we shall only consider totally bounded sets in the sense of definition 4.1 .

Theorem 4.3. *Let (X, δ) be an I-tvs. If A, B are totally fuzzy bounded, then the sets* (1) $A \cap B$, (2) $A \cup B$ *and* (3) aA $(a \in \mathbb{K}, a \neq 0)$ *are totally fuzzy bounded.*

Proof. (1) The proof is trivial.

(2) For each $\lambda-$net $S = \{S(n), n \in D\}$ in $A \cup B$, if D is a finite direct set, then there exists $n_0 \in D$ such that $S(n) = S(n_0)$ whenever $n \in D,\ n \succ n_0$. It is easy to see that $S_1 = \{S(n) \mid n \succ n_0\}$ is a $\lambda-$Cauchy net, hence S has a $\lambda-$Cauchy subnet S_1.

If D is an infinite direct set, then there exists an infinite $\lambda-$subnet T of S in A or B. We assume that T is a subnet of S in A. Since A is totally fuzzy bounded, T has a $\lambda-$Cauchy subnet T_1. So T_1 is a $\lambda-$Cauchy subnet of S. Hence $A \cup B$ is fuzzy totally bounded.

(3) For each $\lambda-$net $S = \{S(n), n \in D\}$ in aA, $S_1 = \{\frac{1}{a}S(n), n \in D\}$ is a $\lambda-$net in A. Since A is totally fuzzy bounded, S_1 has a $\lambda-$Cauchy subnet $T_1 = \{T_1(m), m \in E\}$. Let $T = \{T(m)$, where, for $m \in E\}$, $T(m) = aT_1(m)$. We may prove that T is a $\lambda-$Cauchy subnet of S.

In fact, it is easy to see that T is a $\lambda-$net. Next we shall prove that T is a subnet of S. In fact, since T_1 is a subnet of S_1, there exists a mapping $N : E \to D$ such that $T_1 = S_1 \circ N$, and, for all $n \in D$, there exists $m_0 \in E$ such that $N(m) \succ n$ whenever $m \in E,\ m \succ m_0$. Thus

$$T(m) = aT_1(m) = aS_1(N(m)) = S(N(m)),\ m \in E, \text{ i.e. } T = S \circ N.$$

Finally, we prove that T is a $\lambda-$Cauchy subnet. In fact, for every Q-neighborhood U of θ_λ, there exists $\delta > 0$ and a Q-neighborhood V of θ_λ such that $tV \subset U$ for all $t \in \mathbb{K}$ with $|t - a| < \delta$. In particular, if $t = a$, we have $aV \subset U$. Since T_1 is a $\lambda-$Cauchy net, there exists $m_0 \in E$ such that

$$T_1(m') - T_1(m'')\hat{q}V \text{ whenever } m', m'' \in E,\ m', m'' \succ m_0.$$

It follows that for all $m', m'' \in E,\ m', m'' \succ m_0$,

$$T(m') - T(m'') = aT_1(m') - aT_1(m'') = a(T_1(m') - T_1(m''))\hat{q}aV \subset U.$$

So T is a $\lambda-$Cauchy net. Hence aA is totally fuzzy bounded. □

Theorem 4.4. *Let (X, δ) be an I-tvs and $A \in I^X$ be totally fuzzy bounded. Then A is fuzzy bounded.*

Proof. For each $\lambda-$sequence $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ in A, we shall prove that $\frac{1}{n}x^{(n)}_{\lambda_n} \to \theta_\lambda$. Suppose that $\frac{1}{n}x^{(n)}_{\lambda_n} \nrightarrow \theta_\lambda$. Then there exists a Q-neighborhood U of θ_λ and for every $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N},\ n_0 \geq n$ such that

$$\tfrac{1}{n_0}x^{(n_0)}_{\lambda_{n_0}}\neg\hat{q}U. \tag{4.1}$$

Since A is totally fuzzy bounded, $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$ has a $\lambda-$Cauchy subsequence $\{x^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$, where $N: E\to\mathbb{N}$ and since $\{x^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is a subsequence of $\{x^{(n)}_{\lambda_n}\}_{n\in\mathbb{N}}$, hence for all $n\in\mathbb{N}$, there exists $m_0^{(1)}\in E$ such that $N(m)\geq n$ whenever $m\in E,\ m\geq m_0^{(1)}$. By (4.1), it follows that

$$\tfrac{1}{N(m)}x^{(N(m))}_{\lambda_{N(m)}}\hat{q}U. \tag{4.2}$$

For each Q-neighborhood U of θ_λ, there exists a Q-neighborhood V of θ_λ such that $V+V\subset U$. By Lemma 2.12, we can assume that V is balanced and Q$-\lambda-$absorbed. Since $\{x^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is a $\lambda-$Cauchy net, there exists $m_0^{(2)}\in E$ such that

$$x^{(N(m'))}_{\lambda_{N(m')}}-x^{N(m'')}_{\lambda_{N(m'')}}\hat{q}V \text{ whenever } m',m''\in E,\ m',m''\geq m_0. \tag{4.3}$$

Since V is Q$-\lambda-$absorbed, for every $x_\lambda\in Pt(I^X)$, there exists $\delta>0$ such that

$$tx_\lambda\hat{q}V \text{ for all } |t|\leq\delta. \tag{4.4}$$

Now, $\frac{1}{N(m)}\to 0$. Hence there exists $m_0^{(3)}$ such that $\frac{1}{N(m)}<\delta$ whenever $m\in E,\ m\geq m_0^{(3)}$. By (4.4), it follows that

$$\tfrac{1}{N(m)}x_\lambda\hat{q}V \text{ whenever } m\in E,\ m\geq m_0^{(3)}. \tag{4.5}$$

For $m_0^{(1)},\ m_0^{(2)},\ m_0^{(3)}\in E$, there exists $m_0\in E$ such that $m_0\geq m_0^{(1)},\ m_0^{(2)},\ m_0^{(3)}$ and $\lambda_{N(m)}<\lambda$. Thus, for every $m\in E,\ m\geq m_0$, by (4.3) we have

$$x^{(N(m))}_{\lambda_{N(m)}}-x^{(N(m_0))}_{\lambda_{N(m_0)}}\hat{q}V, \text{ i.e } x^{(N(m))}_{\lambda_{N(m)}}\hat{q}x^{(N(m_0))}_{\lambda_{N(m_0)}}+V$$

and by (4.5), it follows that

$$\tfrac{1}{N(m)}x^{(N(m))}_{\lambda_{N(m)}}\hat{q}\tfrac{1}{N(m)}x^{(N(m_0))}_{\lambda_{N(m_0)}}+\tfrac{1}{N(m)}V\hat{q}V+\tfrac{1}{N(m)}V\subset V+V\subset U,$$

which contradicts (4.2). Hence $\frac{1}{n}x^{(n)}_{\lambda_n}\to\theta_\lambda$. Therefore A is fuzzy bounded. □

Theorem 4.5. *Suppose that (X,δ) is an I-tvs, then A is totally fuzzy bounded in (X,δ) if and only if for each $\lambda\in[0,1)$, $\sigma_{1-\lambda}(A)$ is totally bounded in the classical topological vector space $(X,\sigma_\lambda(\delta))$, where $\sigma_\lambda(A)=\{x\in X\mid A(x)>\lambda\}$.*

Proof. Necessity. It suffices to prove that for any $\lambda\in[0,1)$, every net $\{x^{(n)}\}_{n\in D}$ contained in $\sigma_{1-\lambda}(A)$ has a Cauchy subnet $\{y^{(m)}\}_{m\in E}$.

For the net $\{x^{(n)}\}_{n\in D}$ in $\sigma_{1-\lambda}(A)$, it is easy to see that $\{x^{(n)}_{1-\lambda}\}_{n\in D}$ is a constant $(1-\lambda)-$net in A. Since A is totally fuzzy bounded, $\{x^{(n)}_{1-\lambda}\}_{n\in D}$ has a constant $(1-\lambda)-$Cauchy subnet $\{y^{(m)}_\lambda\}_{m\in E}$. Now we prove that $\{y^{(m)}\}_{m\in E}$ is a Cauchy subnet of $\{x^{(n)}\}_{n\in D}$.

It is easy to show that $\{y^{(m)}\}_{m\in E}$ is a subnet of $\{x^{(n)}\}_{n\in D}$ and hence we only need to prove that $\{y^{(m)}\}_{m\in E}$ is a Cauchy net. In fact, for every neighborhood U of θ in $(X,\sigma_\lambda(\delta))$, there exists $V\in\delta$ such that $\theta\in U=\sigma_\lambda(V)$. It is clear that V is a Q-neighborhood of $\theta_{1-\lambda}$. Since $\{y^{(m)}_{1-\lambda}\}_{m\in E}$ is a $(1-\lambda)-$Cauchy net, there exists $m_0\in E$ such that

$$y^{m'}_{1-\lambda}-y^{m''}_{1-\lambda}\hat{q}V \text{ whenever } m',m''\in E,\ m',m''\succ m_0,$$

i.e. $V(y^{m'}-y^{m''})>\lambda$. So $y^{m'}-y^{m''}\in\sigma_\lambda(V)=U$ whenever $m',\ m''\in E, m',m''\succ m_0$.
Hence $\{y^{(m)}\}_{m\in E}$ is a Cauchy net. Therefore $\sigma_{1-\lambda}(A)$ is totally bounded.

Sufficiency. For each $\lambda-$net $\{x^{(n)}_{\lambda_n}\}_{n\in D}$ in A and any Q- neighborhood W of θ_λ, there exists $U\in\delta$ with $\theta_\lambda\hat{q}U$ such that $U\subseteq W$. Since $U(\theta)>1-\lambda$, there is $\varepsilon>0$ such that $U(\theta)>1-\lambda+\varepsilon$. This means that $\theta\in\sigma_{1-\lambda+\varepsilon}(U)\in\sigma_{1-\lambda+\varepsilon}(\delta)$. Moreover, $\{x^{(n)}_{\lambda_n}\}_{n\in D}$ is a $\lambda-$net in A. Thus there exists $n_0\in D$ such that $\lambda_n>\lambda-\varepsilon$ for all $n\succ n_0, n\in D$. Hence $x^{(n)}\in\sigma_{\lambda-\varepsilon}(A)$ for all $n\succ n_0$. By hypothesis, the net $\{x^{(n)}\}_{n\succ n_0}$ has a Cauchy subnet $\{y^{(m)}\}_{m\in E}$, i.e., there is a mapping $N:E\to D$ with $N(m)\succ n_0$ for all $m\in E$ such that $y^{(m)}=x^{(N(m))}$, and for the open neighborhood $\sigma_{1-\lambda+\varepsilon}(U)$ of θ in $(X,\sigma_{1-\lambda+\varepsilon}(\delta))$, there exists $m_0\in E$ such that $y^{(m_1)}-y^{(m_2)}\in\sigma_{1-\lambda+\varepsilon}(U)$ for all $m_1,m_2\succ m_0, m_1,m_2\in E$. This implies that $y^{(m_1)}_{\lambda_{N(m_1)}}-y^{(m_2)}_{\lambda_{N(m_2)}}=(y^{(m_1)}-y^{(m_2)})_{\lambda_{N(m_1)}\bigwedge\lambda_{N(m_2)}}\hat{q}\chi_{\sigma_{1-\lambda+\varepsilon}(U)}$. In addition, it is easy to verify that $y^{(m_1)}_{\lambda_{N(m_1)}}-y^{(m_2)}_{\lambda_{N(m_2)}}\hat{q}\underline{(1-\lambda+\varepsilon)}$. Thus we have $y^{(m_1)}_{\lambda_{N(m_1)}}-y^{(m_2)}_{\lambda_{N(m_2)}}\hat{q}\underline{(1-\lambda+\varepsilon)}\bigwedge\chi_{\sigma_{1-\lambda+\varepsilon}(U)}$. Since

$$U=\bigvee_{r\in[0,1]}\underline{r}\bigwedge\chi_r(U)\geq\underline{(1-\lambda+\varepsilon)}\bigwedge\chi_{\sigma_{1-\lambda+\varepsilon}(U)},$$

we have $y^{(m_1)}_{\lambda_{N(m_1)}}-y^{(m_2)}_{\lambda_{N(m_2)}}\hat{q}U\subseteq W$ for all $m_1,m_2\succ m_0,\ m_1,m_2\in E$. Hence A is a totally fuzzy bounded set. □

Theorem 4.6. *Let $(X,\omega(\mathrm{T}))$ be the I-tvs induced by the crisp topological vector space (X,T). Then $A\subseteq X$ is totally bounded in (X,T) if and only if $\mathcal{X}_A$ is totally fuzzy bounded in $(X,\omega(\mathrm{T}))$.*

Proof. Necessity. For each $\lambda-$net $\{x^{(n)}_{\lambda_n}\}_{n\in D}$ in $\mathcal{X}_A$, it is easy to see that $\{x^{(n)}\}_{n\in D}$ is a net contained in A. Since A is totally bounded, $\{x^{(n)}\}_{n\in D}$ has a Cauchy subnet $\{y^{(N(m))}\}_{m\in E}$ where $N:E\to D$. Thus $\{y^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is the $\lambda-$Cauchy subnet of $\{x^{(n)}_{\lambda_n}\}_{n\in D}$.

In fact, it is easy to see that $\{y^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is a $\lambda-$net. Also, since $\{y^{(N(m))}\}_{m\in E}$ is a Cauchy subnet of $\{x^{(n)}\}_{n\in D}$ where $N:E\to D$, obviously $\{y^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is a subnet of $\{x^{(n)}_{\lambda_n}\}_{n\in D}$. Now we shall prove that $\{y^{N(m)}_{\lambda_{N(m)}}\}_{m\in E}$ is a $\lambda-$Cauchy net. By lemma 3.8, for every Q-neighborhood U of θ_λ, there exist $V\in\mathrm{T}$ with $\theta\in V$ and $\mu\in(1-\lambda,1]$ such that $V\bigcap\underline{\mu}\subseteq U$. Since $\{y^{(N(m))}\}_{m\in E}$ is a Cauchy net, there exists $m_0\in E$ such that

$$y^{(N(m'))} - y^{(N(m''))} \in V \text{ whenever } m', m'' \in E,\ m', m'' \succ m_0.$$

So $V(y^{(N(m'))} - y^{(N(m''))}) = 1$. Moreover, for $1-\mu < \lambda$ and a $\lambda-$net $\{x^{(n)}_{\lambda_n}\}_{n\in D}$, there exists $n_0 \in D$ such that $\lambda_n > 1-\mu$ for all $n \succ n_0$. Since $\{y^{(N(m))}_{\lambda_{N(m)}}\}_{m\in E}$ is the subnet of $\{x^{(n)}_{\lambda_n}\}_{n\in D}$, there exists $m_1 \in E$ such that $N(m) \succ n_0$ for all $m \succ m_1, m \in E$.

Let $m_2 \in E$ such that $m_2 \succ m_0,\ m_2 \succ m_1$. Then for each $m', m'' \succ m_2, m \in E$, we have $(y^{(N(m'))} - y^{(N(m''))})_{\lambda_{N(m')}\wedge\lambda_{N(m'')}}\hat{q}\underline{\mu}$. This implies that

$$(y^{(N(m'))} - y^{(N(m''))})_{\lambda_{N(m')}\wedge\lambda_{N(m'')}}\hat{q}V.$$

Note that $(y^{(N(m'))} - y^{(N(m''))})_{\lambda_{N(m')}\wedge\lambda_{N(m'')}} = y^{(N(m'))}_{\lambda_{N(m')}} - y^{(N(m''))}_{\lambda_{N(m'')}}$. So for all $m', m'' \in E,\ m', m'' \succ m_2$, we have

$$y^{(m')}_{\lambda_{N(m')}} - y^{(m'')}_{\lambda_{N(m'')}}\hat{q}U.$$

Thus $\{y^{m}_{\lambda_{N(m)}}\}_{m\in E}$ is a $\lambda-$Cauchy net and hence $\mathcal{X}_A$ is totally bounded.

Sufficiency. It is easy to see that for every net $\{x^{(n)}\}_{n\in D}$ in A and all $\lambda \in (0,1]$, $\{x^{(n)}_{\lambda}\}_{n\in D}$ is a constant $\lambda-$net in $\mathcal{X}_A$. Since $\mathcal{X}_A$ is totally fuzzy bounded, $\{x^{(n)}_{\lambda}\}_{n\in D}$ has a constant $\lambda-$Cauchy subnet $\{y^{(m)}_{\lambda}\}_{m\in E}$. Thus $\{y^{(m)}_{\lambda}\}_{m\in E}$ is a Cauchy subnet of $\{x^{(n)}\}_{n\in D}$ (The proof is similar to Theorem 3.9). Hence A is totally bounded. □

Theorem 4.7. *Let (X_1, δ_1), (X_2, δ_2) be two I- topological vector spaces and (X, δ) be the product space of (X_1, δ_1), (X_2, δ_2). If A_i is totally fuzzy bounded in X_i $(i = 1, 2)$, then $A = A_1 \times A_2$ is totally fuzzy bounded in the product space X.*

Proof. For each $\lambda-$net $S = \{S(n), n \in D\}$ in A, $S_i = \{P_i^{\rightarrow}(S(n)), n \in D\}$ is a $\lambda-$net in A_i, $(i = 1, 2)$. Since A_1 is totally fuzzy bounded, hence S_1 has a $\lambda-$Cauchy subnet $T_1 = \{P_1^{\rightarrow}(S(N_1(m))), m \in E\}$, where $N_1 : E \to D$.

Let $\widetilde{S_2} = \{P_2^{\rightarrow}(S(N_1(m))), m \in E\}$. It is easy to see that $\widetilde{S_2}$ is a $\lambda-$net in A_2. Since A_2 is also a totally fuzzy bounded set, hence $\widetilde{S_2}$ has a $\lambda-$Cauchy subnet $T_2 = \{P_2^{\rightarrow}(S(N_1(N_2(h)))), h \in H\}$ where $N_2 : H \to E$. We now prove that $T = \{S(N_1(N_2(h))), h \in H\}$ is a $\lambda-$Cauchy subnet of S.

It is easy to see that T is a $\lambda-$net. Now we show that T is the subnet of S. In fact, it is easy to show that $T = S \circ (N_1(N_2))$. Moreover, since T_1 is the subnet of S_1, for all $n \in D$, there exists $m_0 \in E$ such that

$$N_1(m) \succ n \text{ whenever } m \in E,\ m \succ m_0. \tag{4.6}$$

Since T_2 is the subnet of $\widetilde{S_2}$ and $m_0 \in E$, there exists $h_0 \in H$ such that

$$N_2(h) \succ m_0 \text{ whenever } h \in H,\ h \succ h_0. \tag{4.7}$$

By (4.6) and (4.7), it follows that for all $n \in D$, there exists $h_0 \in H$ such that

$$N_1(N_2(h)) \succ n \text{ whenever } h \in H,\ h \succ h_0.$$

We claim that T is a $\lambda-$Cauchy net. It suffices to prove that $\widetilde{T_1} = \{P_1^{\rightarrow}(S(N_1(N_2(h)))), h \in H\}$ is the $\lambda-$Cauchy subnet of T_1.

Clearly, $\widetilde{T_1}$ is a subnet of T_1. Since T_1 is a $\lambda-$Cauchy net, for every Q- neighborhood V_1 of θ^1_λ, there exists $m_0 \in E$ such that

$$P_1^{\rightarrow}(S(N_1(m'))) - P_1^{\rightarrow}(S(N_1(m'')))\hat{q}V_1 \text{ whenever } m', m'' \in E,\ m', m'' \succ m_0. \tag{4.8}$$

Since T_2 is a subnet of $\widetilde{S_2}$, for $m_0 \in E$, there exists $h_0 \in H$ such that $N_2(h) \succ m_0$ whenever $h \in H,\ h \succ h_0$. So for all $h', h'' \in H,\ h', h'' \succ h_0$, we have

$$N_2(h') \succ m_0, \quad N_2(h'') \succ m_0. \tag{4.9}$$

By (4.8) and (4.9), it follows that

$$P_1^{\rightarrow}(S(N_1(N_2(h')))) - P_1^{\rightarrow}(S(N_1(N_2(h'))))\hat{q}V_1 \text{ whenever } h', h'' \in H,\ h', h'' \succ h_0.$$

Hence $\widetilde{T_1}$ is a $\lambda-$Cauchy net.

For every Q-neighborhood U of θ_λ, there exists a Q-neighborhood U_i of θ^i_λ in (X, δ_i) $(i = 1, 2)$ such that $P_1^{\leftarrow}(U_1) \cap P_2^{\leftarrow}(U_2) \subset U$. Since $\widetilde{T_1},\ T_2$ are $\lambda-$Cauchy nets, for Q-neighborhoods U_i $(i = 1, 2)$ of θ^i_λ, there exists $h_0^{(i)} \in H$ such that for all $h', h'' \in H,\ h', h'' \succ h_0^{(i)}$, we have

$$P_1^{\rightarrow}(S(N_1(N_2(h')))) - P_1^{\rightarrow}(S(N_1(N_2(h'))))\hat{q}U_1, \tag{4.10}$$

$$P_2^{\rightarrow}(S(N_1(N_2(h')))) - P_2^{\rightarrow}(S(N_1(N_2(h'))))\hat{q}U_2, \tag{4.11}$$

By the continuity of P_i, we have

$$S(N_1(N_2(h'))) - S(N_1(N_2(h')))\hat{q}P_i^{\leftarrow}(U_1) \text{ whenever } h', h'' \in H,\ h', h'' \succ h_0^{(i)}\ (i = 1, 2).$$

For $h_0^{(1)},\ h_0^{(2)} \in H$, there exists $h_0 \in H$ such that $h_0 \succ h_0^{(1)}, h_0^{(2)}$. Also, for all $h', h'' \in H,\ h', h'' \succ h_0$, we have the relations (4.10) and (4.11). Thus

$$S(N_1(N_2(h'))) - S(N_1(N_2(h')))\hat{q}P_1^{\leftarrow}(U_1) \cap P_2^{\leftarrow}(U_2) \subset U.$$

Hence T is a $\lambda-$Cauchy net and therefore A is totally fuzzy bounded. □

Corollary 4.8. *Let (X_i, δ_i) be a family of I-topological vector spaces$(i = 1, 2, \cdots, n)$ and (X, δ) be the product space of (X_i, δ_i) $(i = 1, 2, \cdots, n)$. If A_i is totally fuzzy bounded in X_i $(i = 1, 2, \cdots, n)$, then $A = \prod_{i=1}^n A_i$ is totally fuzzy bounded in the product space X.*

Proof. The theorem follows easily by Theorem 4.7 and induction on n. □

Theorem 4.9. *Let (X, δ) be an I-tvs, then $A \in I^X$ is complete and totally fuzzy bounded if and only if A is N-compact.*

Proof. For each $\lambda-$net S contained in A, since A is totally fuzzy bounded, S has a $\lambda-$Cauchy subnet T. By the completeness of A, we have $T \rightarrow x_\lambda,\ x \in A$ and From Lemma 2.7, we can get that $S \infty x_\lambda$. Hence by definition 2.10, A is N-compact.

If A is N-compact, it is obvious that $A \in I^X$ is complete and totally fuzzy bounded. □

5. Geometric Characterization of Totally Fuzzy Boundedness and its Application

Definition 5.1. Let (X, δ) be an I-tvs, $A \in I^X$. We say that a quasi-neighborhood U of θ_λ forms a finite λ^--quasi-neighborhood cover of A, if there exist a finite set $F \subset X$ and $\nu \in (0, \lambda)$ such that for every point x_ν in A, we have $x_\nu \hat{q} F + U$.

Example 5.2. Let (X, δ) be the I-tvs of example 3.4. Then the fuzzy set $U = \underline{\frac{3}{4}}$ is a quasi-neighborhood of $\theta_{\frac{1}{3}}$. Define the set $A \in I^{\mathbb{R}}$ as follows:

$$A(x) = \begin{cases} \frac{1}{2} - \frac{1}{1+x}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

For any finite set F, clearly $F + U = \underline{\frac{3}{4}} = U$, and $\nu = \frac{7}{24} \in (0, \frac{1}{3})$, thus for each $x_{\frac{7}{24}} \in A$, we have $x_{\frac{7}{24}} \hat{q} F + U$. This means the quasi- neighborhood U of $\theta_{\frac{1}{3}}$ forms $\frac{1}{3}^{-}$-quasi -neighborhood cover of A.

Theorem 5.3. *Let (X, δ) be an I-tvs. Then A is totally fuzzy bounded if and only if each $\lambda \in (0, \bigvee_{x \in X} A(x)]$, and each Q- neighborhood U of θ_λ, U forms a finite λ^--quasi-neighborhood cover of A.*

Proof. Necessity. Suppose there exists $\lambda \in (0, \bigvee_{x \in X} A(x)]$ and Q-neighborhood U of θ_λ, such that for every finite set $F \in 2^{(X)}$ and every $\nu \in (0, \lambda)$, there exists a point x_ν in A such that $x_\nu \neg\hat{q} F + U$.

For $D = (0, \lambda) \times 2^{(X)}$, we define an order in D as follows: $(\nu_1, F_1) < (\nu_2, F_2)$ if and only if $\nu_1 < \nu_2,\ F_1 \subset F_2$, for all $(\nu_1, F_1),\ (\nu_2, F_2) \in D$. So D is a directed set.

Let $S = \{x_\nu^F \mid (\nu, F) \in D\}$, then S is a $\lambda-$net in A. In fact, for every $\nu \in (0, \lambda)$, if we choose $F_0 \in 2^{(X)}$, such that for each $(\mu, F) > (\nu, F_0)$, we have that $V(x_\mu^F) = \mu > \nu$. Thus S is a $\lambda-$net in A. So S has a $\lambda-$Cauchy subnet $T = \{T(m),\ m \in E\}$, for a mapping $N : E \to D$.

For Q-neighborhood U of θ_λ, there exists a Q-neighborhood V of θ_λ such that $V + V \subset U$. Since T is a $\lambda-$Cauchy net, there exists $m_0 \in E$ such that

$$T(m_1) - T(m_2) \hat{q} V \text{ whenever } m_1, m_2 \in E,\ m_1, m_2 > m_0.$$

Let $(\nu_0, F_0) = N(m_0),\ (\nu_1, F_1) = N(m_1)$, so

$$x_{\nu_0}^{F_0} - x_{\nu_1}^{F_1} \hat{q} V \text{ whenever } (\nu_0, F_0), (\nu_1, F_1) \in D,\ \nu_1 > \nu_0,\ F_1 \supset F_0.$$

i.e., $V(x^{F_0} - x^{F_1}) > 1 - \nu_0$. Thus $(V + V)(x^{F_0} - x^{F_1}) > 1 - \nu_0$ and hence there exists $z_0 \in X$ such that

$$V(x^{F_0} - z_0) > 1 - \nu_0 \text{ and } V(z_0 - x^{F_1}) > 1 - \nu_0.$$

Let $F_2 = F_0 \cup \{z_0\}$ and choose $\nu_2 \in (\nu_0, \lambda)$. There exists $m_3 \in E,\ m_3 > m_0$ such that $(\nu_3, F_3) > (\nu_2, F_2)$. So $\nu_3 > \nu_2 > \nu_0,\ F_3 \supset F_0$. Note that $x_{\nu_3}^{F_3} - z_0 = x_{\nu_3}^{F_3} - x_{\nu_0}^{F_0} + x_{\nu_0}^{F_0} - z_0$. Hence

$$\begin{aligned} U(x_{\nu_3}^{F_3} - z_0) &\geq (V + V)(x_{\nu_3}^{F_3} - z_0) \geq V(x_{\nu_3}^{F_3} - x_{\nu_0}^{F_0}) \wedge V(x_{\nu_0}^{F_0} - z_0) \\ &> (1 - \nu_0) \wedge (1 - \nu_0) = 1 - \nu_0. \end{aligned}$$

On the other hand, $x_{\nu_3}^{F_3} \neg\hat{q} F_3 + U$, i.e. $U(x^{F_3} - z) \leq 1 - \nu_3$, for all $z \in F_3$. In particular, if we choose $z = z_0$, it follows that $U(x^{F_3} - z_0) \leq 1 - \nu_3 < 1 - \nu_0$, which is a contradiction.

Sufficiency. For each $\lambda-$net $S = \{x^{(\alpha)}_{\lambda_\alpha},\ \alpha \in I\}$ contained in A, we shall prove that S has a $\lambda-$Cauchy subnet $T = \{y^{(\beta)}_{\mu_\beta},\ \beta \in \Lambda\}$.

For each subset B of A, we say S is frequently contained in B, if for each $\alpha \in I$, there exists $\alpha' \in I,\ \alpha' > \alpha$ such that $x^{(\alpha')}_{\lambda_{\alpha'}} \in B$.

Let $\mathcal{M} = \{\zeta :\ \zeta \subseteq \mathcal{P}(A)$, if $|\zeta| < \infty$, then S is frequently contained in $\bigcap \zeta\}$. Then $\mathcal{M} \neq \Phi$, since $\{A\} \in \mathcal{M}$. We define order $\succ$ as follows:

$$\zeta'' \succ \zeta' \text{ if and only if } \zeta'' \supset \zeta', \text{ for all } \zeta'', \zeta' \in \mathcal{M}.$$

Then the set $(\mathcal{M}, \succ)$ satisfies the condition of Zorns Lemma.

In fact, for every subset $\mathcal{M}_0$ of $\mathcal{M}$ with total order, there exists $\eta \in \mathcal{M}$ such that η is the maximal element of $\mathcal{M}_0$. Let $\eta = \bigcup\limits_{\zeta \in \mathcal{M}_0} \zeta$. Then it is easy to see that $\eta \subseteq \mathcal{P}(A)$. If $|\eta| < \infty$, then for each $\zeta \in \mathcal{M}_0,\ |\zeta| < \infty$, so S is frequently contained in $\bigcap \zeta$ and thus S is frequently contained in $\bigcap \bigcup\limits_{\zeta \in \mathcal{M}_0} \zeta = \bigcap \eta$. Hence $\eta \in \mathcal{M}$.

By Zorns Lemma, it follows that there exists a maximal element ζ_0 in $\mathcal{M}$ satisfying:

(i) $A \in \zeta_0$.

(ii) If $A_1, \cdots, A_n \in \zeta_0$, then $\bigcap_{i=1}^n A_i \in \zeta_0$.

(iii) If $\bigcup_{i=1}^n A_i \in \zeta_0$, where $A_i \subseteq A$ and S is frequently contained in A_i, then there exists $i_0,\ 1 \le i_0 \le n$ such that $A_{i_0} \in \zeta_0$.

To show (i), let $\widetilde{\zeta} = \zeta_0 \cup \{A\}$. Then $\widetilde{\zeta} \in \mathcal{M}$. In fact, it is easy to see that $\zeta \subseteq \mathcal{P}(A)$. If $|\widetilde{\zeta}| < \infty$, then $|\zeta_0| < \infty$. Since $\zeta_0 \in \mathcal{M}$, then S is frequently contained in $\bigcap \zeta_0$. Note that $S \subseteq A$. Hence S is frequently contained in $\bigcap(\zeta_0 \cup \{A\})$ and so $\widetilde{\zeta} \in \mathcal{M}$. By $\widetilde{\zeta} = \zeta_0 \cup \{A\}$, we have $\widetilde{\zeta} \succ \zeta_0$, Moreover, since ζ_0 is a maximal element, it follows that for every $\zeta \in \mathcal{M}$, $\zeta_0 \succ \zeta$, in particular, $\zeta_0 \succ \widetilde{\zeta}$, thus $\widetilde{\zeta} = \zeta_0$, therefore $A \in \zeta_0$.

(ii) Let $B = \bigcap_{i=1}^n A_i$, for each $A_j \in \zeta_0, (j = 1, 2, \cdots, m)$. Note that $A_1, \cdots, A_n \in \zeta_0$. Hence S is frequently contained in $B \bigcap (\bigcap_{j=1}^m A_j)$. By the maximality of ζ_0, we know that $B \in \zeta_0$.

(iii) Suppose for every $i,\ 1 \le i \le n,\ A_i \notin \zeta_0$. By (ii), we have that there exists $B_i \in \zeta_0$ such that $A_i \bigcap B_i \notin \zeta_0$. So S is not frequently contained in $A_i \bigcap B_i$ and hence there exists $\alpha_i \in I$ such that $x^{(\alpha)}_{\lambda_\alpha} \notin A_i \bigcap B_i$ whenever $\alpha \succ \alpha_i$. So

$$(A_i \bigcap B_i)(x^{(\alpha)}) < \lambda_\alpha,\ (i = 1, 2, \cdots, n)$$

Choose $\alpha' \in I$ such that $\alpha' \succ \alpha_i$ $(i = 1, 2, \cdots, n)$. Since $B_i \in \zeta_0$ $(i = 1, 2, \cdots, n)$, we have by (ii) that $\bigcap_{i=1}^n B_i \in \zeta_0$. Note that $\bigcup_{i=1}^n A_i \in \zeta_0$, so $(\bigcup_{i=1}^n A_i) \bigcap (\bigcap_{i=1}^n B_i) \in \zeta_0$. Thus S is frequently contained in $(\bigcup_{i=1}^n A_i) \bigcap (\bigcap_{i=1}^n B_i)$, i.e. for $\alpha' \in I$, there exists $\alpha \in I,\ \alpha \succ \alpha'$ such that $x^{(\alpha)}_{\lambda_\alpha} \in (\bigcup_{i=1}^n A_i) \bigcap (\bigcap_{i=1}^n B_i)$. So

$$((\bigcup_{i=1}^n A_i) \bigcap (\bigcap_{i=1}^n B_i))(x^{(\alpha)}) \ge \lambda_\alpha. \tag{5.1}$$

On the other hand,

$$((\bigcup_{i=1}^n A_i) \bigcap (\bigcap_{i=1}^n B_i))(x^{(\alpha)}) \le (\bigcup_{i=1}^n (A_i \bigcap (\bigcap_{i=1}^n B_i)))(x^{(\alpha)})$$

$\leq \bigcup_{i=1}^{n}(A_i \bigcap B_i)(x^{(\alpha)}) < \lambda_\alpha$.

This contradicts (5.1).

Now, we prove that S has a λ-subnet T.

Let $\Lambda = \{(B, \alpha) \in \zeta_0 \times I,\ x_{\lambda_\alpha}^{(\alpha)} \in B\}$. We define the order $\prec^*$ as follows:

$(B_2, \alpha_2) \prec^* (B_1, \alpha_1)$ if and only if $B_1 \subset B_2,\ \alpha_2 \prec \alpha_1$, for all $(B_1, \alpha_1), (B_2, \alpha_2) \in \Lambda$

So $(\Lambda, \prec^*)$ is a directed set.

For every $(B, \alpha) \in \Lambda$, we let $y^{(B,\ \alpha)} = x^{(\alpha)},\ \mu_{(B,\ \alpha)} = \lambda_\alpha$, a mapping $N : \Lambda \to I,\ (B, \alpha) \mapsto \alpha$, then $T = \{y_{\mu_\beta}^{(\beta)},\ \beta \in \Lambda\}$ is a subnet of S. In fact, it is easy to see that $S = T \circ N$. Also, if for each $\alpha \in I$, we choose $(A, \alpha) \in \Lambda$, for every $(B, \alpha') \in \Lambda,\ (A, \alpha) \prec^* (B, \alpha')$, then $N(B, \alpha') = \alpha' \succ \alpha$.

By the definition of T, we can easily see that T is a $\lambda-$subnet.

Finally, we prove that T is a $\lambda-$Cauchy net.

For each $\lambda \in (0, \vee A(x)]$, and any Q- neighborhood V of θ_λ, there exists a balanced Q-neighborhood U of θ_λ such that $U + U \subset V$. Since A is totally fuzzy bounded, there exists a finite set $F = \{z_1, z_2, \cdots, z_n\}$ and $\nu \in (0, \lambda)$ such that for every fuzzy point x_ν in A, we have $x_\nu \hat{q} \bigcup_{i=1}^{n}(z_i + U)$. Since $S = \{\ x_{\lambda(\alpha)}^{\alpha} \mid \lambda \in \Lambda\}$ is a λ-net in A, for $\nu \in (0, \lambda)$, there exists $\alpha_0 \in \Lambda$ such that $\lambda(\alpha) > \nu$ for all $\alpha \succ \alpha_0$ with $\alpha \in \Lambda$. Then the set $A_\nu = \{\ x \mid A(x) \geq \nu\} \neq \emptyset$.

Let $A_\nu^i(x) = \begin{cases} 0, & x \notin A_\nu \\ A(x), & x_\nu \hat{q}(z_i + U),\ x_\nu \in A_\nu \end{cases}$. Thus $\bigcup_{i=1}^{n} A_\nu^i = \chi_{A_\nu} \bigcap A$. Since $S = \{\ x_{\lambda(\alpha)}^{\alpha} \mid \lambda \in \Lambda\}$ is again a λ-net in A, the set $\bigcup_{i=1}^{n} A_\nu^i \in \zeta_0$ and hence by (iii), there exists $1 \leq i_0 \leq n$ such that $A_\nu^{i_0} \in \zeta_0$. We choose $x_{\lambda_\alpha}^{(\alpha)} \in A_\nu^{i_0}$. So for every $(B_1, \alpha_1), (B_2, \alpha_2) \in \Lambda,\ (A_\nu^{i_0}, \alpha) \prec^* (B_1, \alpha_1),\ (A_\nu^{i_0}, \alpha) \prec^* (B_2, \alpha_2)$,

$$y_{\mu_{(B_2,\ \alpha_2)}}^{(B_2,\ \alpha_2)} - y_{\mu_{(B_1,\ \alpha_1)}}^{(B_1,\ \alpha_1)} = x_{\lambda_{\alpha_2}}^{(\alpha_2)} - x_{\lambda_{\alpha_1}}^{(\alpha_1)} \hat{q}(z_{i_0} + U) - (z_{i_0} + U) = U - U = U + U \subset V$$

So $T = \{y_{\mu_\beta}^{(\beta)},\ \beta \in \Lambda\}$ is a $\lambda-$Cauchy net. □

Remark 5.4. Theorem 5.3 is a geometric characterization of totally Fuzzy boundedness.

Theorem 5.5. *Let $f^{\to} : (X, \delta_1) \to (Y, \delta_2)$ be a continuous fuzzy linear operator. If A is totally fuzzy bounded in (X, δ_1), then $f^{\to}(A)$ is totally fuzzy bounded in (Y, δ_2).*

Proof. For each $\lambda \in (0, \bigvee_{y \in Y} f^{\to}(A)(y)]$ and every Q-neighborhood U of θ_λ^Y, since $f^{\to} : (X, \delta_1) \to (Y, \delta_2)$ is a continuous fuzzy linear operator, hence $f^{\leftarrow}(U)$ is a Q-neighborhood U of θ_λ^X. Clearly, $\lambda \in (0, \bigvee_{x \in X} A(x)]$. Since A is totally fuzzy bounded in (X, δ_1), there exist a finite set F and $\nu \in (0, \lambda)$ such that $x_\nu \hat{q} F + f^{\leftarrow}(U)$ for each $x_\nu \in A$.

It is obvious that $f^{\to}(F)$ is a finite set. Moreover, for each $y_\nu \in f^{\to}(A)$, there exists $x^* \in X$ such that $f(x^*) = y$. Since $x_\nu^* \hat{q} F + f^{\leftarrow}(U)$, it follows that

$$y_\nu = f(x_\nu^*) \hat{q} f^{\to}(F) + f^{\to}(f^{\leftarrow}(U)) \subseteq f^{\to}(F) + U.$$

By Theorem 5.3, $f^{\to}(A)$ is totally fuzzy bounded in (Y, δ_2). □

Theorem 5.6. *Let (X_i, δ_i) be a family of I-topological vector spaces $(i = 1, 2, \cdots, n)$ and (X, δ) be their product space. Then $B \in I^X$ is totally fuzzy bounded if and only if $B \subset \prod_{i=1}^n B_i$, where B_i is totally fuzzy bounded in (X_i, δ_i).*

Proof. Necessity. For each $i = 1, 2, \cdots, n$, let $B_i = P_i^{\rightarrow}(B)$. By the continuity of P_i and theorem 5.5, $P_i^{\rightarrow}(B)$ is totally fuzzy bounded, i.e., B_i is totally fuzzy bounded, and

$$B \subset \bigcap_{i=1}^n P_i^{\leftarrow}(P_i^{\rightarrow}(B)) = \prod_{i=1}^n P_i^{\rightarrow}(B) = \prod_{i=1}^n B_i.$$

Sufficiency. Since B_i is totally fuzzy bounded in (X_i, δ_i), by Corollary 4.8 it follows that $\prod_{i=1}^n B_i$ is totally fuzzy bounded in (X, δ). Now since $B \subset \prod_{i=1}^n B_i$, we can easily obtain that S is totally fuzzy bounded in (X, δ). □

Theorem 5.7. *Let (X, δ) be an I-tvs. If A, B are totally fuzzy bounded, then $A + B$ is totally fuzzy bounded.*

Proof. Consider a mapping $f : X \times X \to X,\ (x, y) \to x + y$. Then $f^{\rightarrow}(A \times B) = A + B$. By theorem 4.7, $A \times B$ is totally fuzzy bounded. Furthermore, It follows that $A + B$ is totally fuzzy bounded by theorem 5.5. □

6. **Conclusion**

After introducing a new definition of fuzzy bounded sets and totally fuzzy bounded sets, some interesting properties of these sets are studied. It is proved that $A \in I^X$ is a totally fuzzy bounded set in I-tvs (X, δ) if and only if A is N-compact in (X, δ). Finally, a geometric characterization for fuzzy totally bounded sets in I- topological vector spaces is obtained.

In the future we may generalize the notions of fuzzy bounded sets and totally fuzzy bounded sets to L-topological vector spaces [20], where L is a fuzzy lattice. as well as the relation between totally fuzzy bounded sets and SR-compactness introduced by Bai [1].

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