UNIFORM AND SEMI-UNIFORM TOPOLOGY ON GENERAL FUZZY AUTOMATA

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Abstract. In this paper, we define the concepts of compatibility between two fuzzy subsets on Q, the set of states of a max- min general fuzzy automaton and transitivity in a max-min general fuzzy automaton. We then construct a uniform structure on Q, and define a topology on it. We also define the concept of semi-uniform structures on a nonempty set X and construct a semi-uniform structure on the set of states of a general fuzzy automaton. We then construct a semi-uniform structure on Σ^*, the set of all finite words on Σ, the set of input symbols of a general fuzzy automaton and, finally, using these semi-uniform structures, we construct two topologies on Q and Σ^* and discuss their properties.

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by Zadeh [12] and Wee [11] introduced the idea of fuzzy automata.

A fuzzy finite-state automaton (FFA) is a six-tuple $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the initial state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition, there is a membership value in $[0, 1]$ called the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use $\delta(q_i, a_k, q_j)$ to refer both to a transition and its Weight in the sense that whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, and otherwise, specifies the transition itself. The set of all transitions of $\tilde{F}$ will be denoted by Δ. The above definition is generally accepted as a formal definition of an FFA [6], [7], [8], [9].

The question of assignment of membership values to the next states is an important problem which should be clarified in the definition of FFA.When assigning membership values to states, there are two issues which must be dealt with: the assignment of a membership value to a state upon the completion of a transition and the cases where a state is forced to take several membership values simultaneously via overlapping transitions. In 2004, M. Doostfatemeh and S.C. Kremer

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extended the notion of fuzzy automata and introduced the notion of general fuzzy automata to deal with this issues [1]. We follow [1] and introduce some new notions and obtain related results.

Let Σ be a set. A word in Σ is the product of a finite sequence of elements in Σ. Λ will denote the empty word and Σ^* the set of all words on Σ. The length $\ell(x)$ of the word $x \in \Sigma^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ will denote the set of all fuzzy sets on X.

Definition 1.1. [1] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subset \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function,
(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,
(viii) $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters, μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1.2. [1] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t + 1$, the following algorithm will

assign a unified membership value to it:
(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i.

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.
(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

where
- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 1.3. [13] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

Also, if the input at time t_i be u_i, where $u_i \in \Sigma, \forall 1 \leq i \leq n$, then
$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,
$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$,
and recursively
$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$
$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\}$.

Definition 1.4. [13] Let $\tilde{F}^*$ be a max-min GFA. The response function $r^{\tilde{F}^*}$: $\Sigma^* \times Q \to [0, 1]$ of $\tilde{F}^*$, for any $x \in \Sigma^*$, $q \in Q$, is defined by

$$r^{\tilde{F}^*}(x, q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q).$$

Definition 1.5. [13] Let $\tilde{F}^*$ be a max-min GFA, λ be a fuzzy subset on Q, p, q belong to Q and $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^* \setminus \{\Lambda\}\}$. Also, let $\rho' : q_0 = p$, $q_1, \dots, q_n = q$ be a path from p to q and $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\}$. Then the degree of connectedness of states p and q with respect to λ is defined by

$$deg_\lambda(p, q) = \vee\{S_\lambda(\rho) : \rho \text{ is a path from } p \text{ to } q\}.$$

Theorem 1.6. [13] Let $\tilde{F}^*$ be a max-min GFA and λ be a fuzzy subset on Q. Then
(i) $deg_\lambda(q,q) = \overline{D}(\lambda)(q),\ \forall q \in Q$,
(ii) $deg_\lambda(p,q) = deg_\lambda(q,p),\ \forall p,q \in Q$.

Definition 1.7. [13] Let $\tilde{F}^*$ be a max-min GFA and λ be a fuzzy subset on Q. Then we say that p and q are connected with respect to λ if

$$deg_\lambda(p,q) = \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q).$$

Theorem 1.8. [13] Let $\tilde{F}^*$ be a max-min GFA and λ be a fuzzy subset on Q. Then p and q are connected with respect to λ if and only if there exists a path $\rho' : q_0 = p,\ q_1, \ldots, q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q), 0 \leq i \leq n$.

Definition 1.9. [3] Let X be a nonempty set and U, V be any subsets of $X \times X$. We now recall a definition and some notation, which will be used in the sequel.
(i) $\overline{\Delta} = \{(x,x) \in X \times X : x \in X\}$,
(ii) $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$,
(iii) $U \diamondsuit V = \{(x,y) \in X \times X : \exists z \in X \text{ s.t. } (x,z) \in U,\ (z,y) \in V\}$.

Definition 1.10. [3], [4], [10] By a uniformity on X we shall mean a nonempty collection κ of subsets of $X \times X$ which satisfies the following conditions:
(i) $\overline{\Delta} \subseteq U$ for any $U \in \kappa$,
(ii) If $U \in \kappa$, then $U^{-1} \in \kappa$,
(iii) If $U \in \kappa$, then there exists a $V \in \kappa$, such that $V \diamondsuit V \subseteq U$,
(iv) If $U, V \in \kappa$, then $U \cap V \in \kappa$,
(v) If $U \in \kappa$ and $U \subseteq V \subseteq X \times X$, then $V \in \kappa$.
The pair (X, κ) is called a uniform structure.

2. Uniform Topology on Max-Min General Fuzzy Automata

Definition 2.1. Let $\tilde{F}^*$ be a max-min GFA and λ_1, λ_2 be two fuzzy subsets on Q. We say that λ_1 is compatible with λ_2 if

$$\overline{D}(\lambda_1)(p) = \overline{D}(\lambda_2)(p),\ \forall p \in Q.$$

Example 2.2. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min GFA and λ_1, λ_2 be two fuzzy subsets on Q, where $Q = \{q_0\}$, $\Sigma = \{a\}$, $\tilde{R} = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$, $F_1(\mu, \delta) = Min(\mu, \delta)$, $Z = \emptyset$, ω and F_2 are not applicable, $\delta(q_0, a, q_0) = 0.4$, $\lambda_1(q_0) = 0.4$ and $\lambda_2(q_0) = 0.5$. Then we have

$$\mu^{t_1}(q_0) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_0) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_0)) = F_1(1, 0.4) = 0.4,$$

$$\begin{aligned} \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^2, q_0) &= \bigvee_{q \in Q_{act}(t_1)} [\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q) \wedge \tilde{\delta}((q, \mu^{t_1}(q)), a, q_0)] \\ &= \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_0) \wedge \tilde{\delta}((q_0, \mu^{t_1}(q_0)), a, q_0) \\ &= F_1(1, 0.4) \wedge F_1(0.4, 0.4) = 0.4 \wedge 0.4 = 0.4, \end{aligned}$$

$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^n, q_0) = 0.4$, $\forall n = 1, 2, 3, \ldots$,
$r^{\tilde{F}^*}(a, q_0) = \bigvee_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), a, q_0) = \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_0) = 0.4$,
$r^{\tilde{F}^*}(a^2, q_0) = \bigvee_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), a^2, q_0) = \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^2, q_0) = 0.4$,
$r^{\tilde{F}^*}(a^n, q_0) = 0.4$, $\forall n = 1, 2, 3, \ldots$.

$$\begin{aligned}
\overline{D}(\lambda_1)(q_0) &= \vee\{\lambda_1(q_0) \wedge r^{\tilde{F}^*}(x, q_0) : x \in \Sigma^* \backslash \{\Lambda\}\} \\
&= [\lambda_1(q_0) \wedge r^{\tilde{F}^*}(a, q_0)] \vee [\lambda_1(q_0) \wedge r^{\tilde{F}^*}(a^2, q_0)] \vee \ldots \\
&= (0.4 \wedge 0.4) \vee (0.4 \wedge 0.4) \vee \cdots = 0.4, \\
\overline{D}(\lambda_2)(q_0) &= \vee\{\lambda_2(q_0) \wedge r^{\tilde{F}^*}(x, q_0) : x \in \Sigma^* \backslash \{\Lambda\}\} \\
&= [\lambda_2(q_0) \wedge r^{\tilde{F}^*}(a, q_0)] \vee [\lambda_2(q_0) \wedge r^{\tilde{F}^*}(a^2, q_0)] \vee \ldots \\
&= (0.5 \wedge 0.4) \vee (0.5 \wedge 0.4) \vee \cdots = 0.4.
\end{aligned}$$

Since $\overline{D}(\lambda_1)(q_0) = \overline{D}(\lambda_2)(q_0)$, therefore λ_1 is compatible with λ_2.

Definition 2.3. Let $\tilde{F}^*$ be a max-min GFA and λ be a fuzzy subset on Q. Then we say that λ is transitive if when p and q are connected with respect to λ and q and r are connected with respect to λ, then p and r are connected with respect to λ.

Theorem 2.4. Let $\tilde{F}^*$ be a max-min GFA and λ be a transitive fuzzy subset on Q. Define

$U_\lambda = \{(p, q) \in Q \times Q : p, q$ are connected with respect to $\lambda\}$,
$\kappa^* = \{U_\lambda : \lambda$ is compatible with the all fuzzy subsets on $Q\}$.

Then κ^* satisfies the conditions (i)-(iv) of Definition 1.10.

Proof. (i) Let $U_k \in \kappa^*$ and $(q, q) \in \overline{\Delta}$. By Theorem 1.6, since $deg_\lambda(q, q) = \overline{D}(\lambda)(q)$, hence q and q are connected with respect to λ. Thus $(q, q) \in U_\lambda$. Therefore $\overline{\Delta} \subseteq U_\lambda$ for any $U_\lambda \in \kappa^*$.
(ii) Let $U_\lambda \in \kappa^*$. Then we have

$$\begin{aligned}
(p, q) \in U_\lambda^{-1} \Longleftrightarrow (q, p) \in U_\lambda &\Longleftrightarrow q \text{ and } p \text{ are connected with respect to } \lambda \\
&\Longleftrightarrow p \text{ and } q \text{ are connected with respect to } \lambda \\
&\Longleftrightarrow (p, q) \in U_\lambda.
\end{aligned}$$

Thus $U_\lambda^{-1} = U_\lambda \in \kappa^*$.
(iii) Let $U_\lambda \in \kappa^*$. We claim $U_\lambda \Diamond U_\lambda \subseteq U_\lambda$. Let $(p, q) \in U_\lambda \Diamond U_\lambda$. Then there exists $r \in U_\lambda$ such that $(p, r) \in U_\lambda$ and $(r, q) \in U_\lambda$. Thus p and r are connected with respect to λ, r and q are connected with respect to λ. Since λ is transitive, then p and q are connected with respect to λ. So $(p, q) \in U_\lambda$ and $U_\lambda \Diamond U_\lambda \subseteq U_\lambda$.
(iv) If $U_{\lambda_1} \in \kappa^*$, $U_{\lambda_2} \in \kappa^*$, then $U_{\lambda_1} \cap U_{\lambda_2} \subseteq U_{\lambda_1}$. We show that $U_{\lambda_1} \subseteq U_{\lambda_1} \cap U_{\lambda_2}$. Let $(p, q) \in U_{\lambda_1}$. Then p and q are connected with respect to λ_1. Thus, by Theorem 1.8, there exists a path $\rho' : q_0 = p, q_1, \ldots, q_n = q$ such that $\overline{D}(\lambda_1)(q_i) \geq \overline{D}(\lambda_1)(p) \wedge$

$\overline{D}(\lambda_1)(q)$, $0 \leq i \leq n$. Since λ_1 is compatible with λ_2, hence $\overline{D}(\lambda_1)(p) = \overline{D}(\lambda_2)(p)$, $\forall p \in Q$. Thus $\overline{D}(\lambda_2)(q_i) \geq \overline{D}(\lambda_2)(p) \wedge \overline{D}(\lambda_2)(q)$, $0 \leq i \leq n$. So by Theorem 1.8, p and q are connected with respect to λ_2. Therefore $(p, q) \in U_{\lambda_2}$. Thus $U_{\lambda_1} \subseteq U_{\lambda_1} \cap U_{\lambda_2}$. Consequently, $U_{\lambda_1} \cap U_{\lambda_2} = U_{\lambda_1} \in \kappa^*$. □

Theorem 2.5. Let $\tilde{F}^*$ be a max-min GFA, λ be a transitive fuzzy subset on Q and

$$U_\lambda = \{(p, q) \in Q \times Q : p, q \text{ are connected with respect to } \lambda\},$$
$$\kappa^* = \{U_\lambda : \lambda \text{ is compatible with the all fuzzy subsets on } Q\},$$
$$\kappa = \{U \subseteq Q \times Q : \text{there exists } U_\lambda \in \kappa^* \text{ and } U_\lambda \subseteq U\}.$$

Then κ satisfies a uniformity on Q and the pair (Q, κ) is a uniform structure.

Proof. (i) Let $U \in \kappa$. Then there exists $U_\lambda \in \kappa^*$ such that $U_\lambda \subseteq U$. By Theorem 2.4, $\overline{\Delta} \subseteq U_\lambda$. So $\overline{\Delta} \subseteq U$.
(ii) Let $U \in \kappa$. So there exists $U_\lambda \in \kappa^*$ such that $U_\lambda \subseteq U$. By Theorem 2.4, $U_\lambda^{-1} \in \kappa^*$. Now, let $(p, q) \in U_\lambda^{-1}$. Then we have

$$(q, p) \in U_\lambda \subseteq U \Rightarrow (q, p) \in U \Rightarrow (p, q) \in U^{-1}.$$

Thus $U_\lambda^{-1} \subseteq U^{-1}$ and $U_\lambda^{-1} \in \kappa^*$. Therefore $U^{-1} \in \kappa$.
(iii) Let $U \in \kappa$. Then there exists $U_\lambda \in \kappa^*$ such that $U_\lambda \subseteq U$. By Theorem 2.4, $U_\lambda \Diamond U_\lambda \subseteq U_\lambda$. On the other hand, since $\kappa^* \subseteq \kappa$, then $U_\lambda \in \kappa$. So $U_\lambda \Diamond U_\lambda \subseteq U$ and $U_\lambda \in \kappa$.
(iv) Let $U_1, U_2 \in \kappa$. Then there exist $U_{\lambda_1}, U_{\lambda_2} \in \kappa^*$ such that $U_{\lambda_1} \subseteq U_1$, $U_{\lambda_2} \subseteq U_2$. Thus $U_{\lambda_1} \cap U_{\lambda_2} \subseteq U_1 \cap U_2$. By Theorem 2.4, $U_{\lambda_1} \cap U_{\lambda_2} = U_{\lambda_1}$. So we get that $U_{\lambda_1} \subseteq U_1 \cap U_2$ and $U_{\lambda_1} \in \kappa^*$. Therefore $U_1 \cap U_2 \in \kappa$.
(v) Let $U \in \kappa$, $U \subseteq V \subseteq Q \times Q$. Then there exists $U_\lambda \in \kappa^*$ such that $U_\lambda \subseteq U$. Thus $U_\lambda \subseteq V$ and $U_\lambda \in \kappa^*$. Therefore $V \in \kappa$. □

Theorem 2.6. Let $\tilde{F}^*$ be a max-min GFA, $p \in Q$, $U \in \kappa$, $U[p] = \{q \in Q : (p, q) \in U\}$ and $\tau = \{Q' \subseteq Q : \forall p \in Q', \exists U \in \kappa, U[p] \subseteq Q'\}$. Then τ is a topology on Q.

Proof. (i) It is clear that $\emptyset$ and Q are in τ.
(ii) Let $Q_1, Q_2 \in \tau$ and $q \in Q_1 \cap Q_2$. Then there exist $U, V \in \kappa$ such that $U[q] \subseteq Q_1$ and $V[q] \subseteq Q_2$. Let $W = U \cap V$. By Theorem 2.5, $W \in \kappa$ and also $W[q] \subseteq U[q] \cap V[q] \subseteq Q_1 \cap Q_2$. Thus $Q_1 \cap Q_2 \in \tau$.
(iii) Let $Q_i \in \tau$, $\forall i \in I$. Then $\forall p \in Q_i$, $\exists U_i \in \kappa : U_i[p] \subseteq Q_i$. Let $U = \bigcup_{i \in I} U_i$. By Theorem 2.5, since we have $U_1 \subseteq \bigcup_{i \in I} U_i \subseteq Q \times Q$, $U_1 \in \kappa$, hence $U = \bigcup_{i \in I} U_i \in \kappa$. Also, $U[p] = \bigcup_{i \in I} U_i[p] \subseteq \bigcup_{i \in I} Q_i$, $\forall p \in \bigcup_{i \in I} Q_i$. Therefore $\bigcup_{i \in I} Q_i \in \tau$. Consequently τ is a topology on Q. □

Remark. τ is called the uniform topology on Q induced by κ.

Theorem 2.7. Let $\tilde{F}^*$ be a max-min GFA and λ be a transitive fuzzy subset on Q. Then $U_\lambda[q]$ is clopen in (Q, τ), for any $q \in Q$.

Proof. (i) We first show that $U_\lambda[q]$ is open. Let $p \in U_\lambda[q]$. If $r \in U_\lambda[p]$, then $(q,p) \in U_\lambda$, $(p,r) \in U_\lambda$. Thus q and p are connected with respect to λ and p and r are also connected with respect to λ. Since λ is transitive, hence q and r are connected with respect to λ. So $r \in U_\lambda[p]$. Therefore $U_\lambda[p] \subseteq U_\lambda[q]$, $\forall p \in U_\lambda[q]$ and $U_\lambda \in \kappa$. So $U_\lambda[q] \in \tau$. Consequently $U_\lambda[q]$ is an open set in (Q,τ).
(ii) We now show that $U_\lambda[q]$ is closed. Let $p \in (U_\lambda[q])^C$. Then q and p are not connected with respect to λ. If $r \in U_\lambda[p]$, then p and r are connected with respect to λ. Also, if $r \in U_\lambda[q]$, then q and r are connected with respect to λ. Since λ is transitive, q and p are connected with respect to λ, which is a contradiction. So $r \in (U_\lambda[q])^C$. Therefore $U_\lambda[p] \subseteq (U_\lambda[q])^C$, $\forall p \in (U_\lambda[q])^C$ and $U_\lambda \in \kappa$. Thus $(U_\lambda[q])^C \in \tau$. Consequently $U_\lambda[q]$ is a closed set in (Q,τ). □

3. Semi-Uniform Topology on General Fuzzy Automata

Definition 3.1. By a semi-uniformity on X we shall mean a nonempty collection κ of subsets of $X \times X$ which satisfies the following conditions:
(i) $\overline{\Delta} \cap U \neq \emptyset$ for some $U \in \kappa$,
(ii) If $U \in \kappa$, then $U^{-1} \in \kappa$,
(iii) If $U \in \kappa$, then there exists a $V \in \kappa$, such that $V \diamondsuit V \subseteq U$,
(iv) If $U, V \in \kappa$, then $U \cap V \in \kappa$,
(v) If $U \in \kappa$ and $U \subseteq V \subseteq X \times X$, then $V \in \kappa$.
The pair (X,κ) is called a semi-uniform structure.

Definition 3.2. Let $\tilde{F}$ be a GFA and $p \in Q$. If $n = \bigwedge_{q_0 \in \tilde{R}} (\wedge\{|\rho| : \rho \text{ is a path of } q_0 \text{ to } p\})$, then we say that the order of p is $n+1$.
In fact, if ρ is the path $q_0, q_1, \dots, q_m = p$ and $q_0 \in \tilde{R} = Q_{act}(t_0)$, then $m = |\rho|$. We denote the order of p by $ord(p)$.

Theorem 3.3. Let $\tilde{F}$ be a GFA and

$$U_n = \{(p,q) \in Q \times Q : ord(p) = ord(q) = n\}, \forall n \geq 1, U_0 = \emptyset,$$
$$V_n = \{(x,y) \in \Sigma^* \times \Sigma^* : \ell(x) = \ell(y) = n-1\}, \forall n \geq 1, V_0 = \emptyset,$$
$$\kappa_1^* = \{U_n : n = 0,1,2,3,\dots\},$$
$$\kappa_2^* = \{V_n : n = 0,1,2,3,\dots\}.$$

Then κ_1^*, κ_2^* satisfy the conditions (i)-(iv) of Definition 3.1.

Proof. (i) Let $q \in Q$ and $ord(q) = n$. Since $(q,q) \in \overline{\Delta}$ and $(q,q) \in U_n$, hence $\overline{\Delta} \cap U_n \neq \emptyset$ for some $U_n \in \kappa_1^*$.
(ii) Let $U_n \in \kappa_1^*$. Then

$$(p,q) \in U_n^{-1} \Longleftrightarrow (q,p) \in U_n \Longleftrightarrow ord(q) = ord(p) = n \Longleftrightarrow (p,q) \in U_n.$$

Thus $U_n^{-1} = U_n \in \kappa_1^*$.
(iii) If $U_n \in \kappa_1^*$, then it is clear that $U_n \diamondsuit U_n \subseteq U_n$.

(iv) If $U_n \in \kappa_1^*$, $U_m \in \kappa_1^*$, then $U_n \cap U_m = \begin{cases} U_0, n \neq m \\ U_n, n = m \end{cases}$. Thus $U_n \cap U_m \in \kappa_1^*$. So κ_1^* satisfies the conditions (i)-(iv) of Definition 3.1. Also,

(a) If $x \in \Sigma^*$ and $\ell(x) = n$, since $(x, x) \in V_{n+1}$ and $(x, x) \in \overline{\Delta}$, hence $\overline{\Delta} \cap V_{n+1} \neq \emptyset$ for some $V_{n+1} \in \kappa_2^*$.

(b) If $V_n \in \kappa_2^*$, then

$$(x, y) \in V_n^{-1} \Longleftrightarrow (y, x) \in V_n \Longleftrightarrow \ell(x) = \ell(y) = n - 1 \Longleftrightarrow (x, y) \in V_n.$$

Thus $V_n^{-1} = V_n \in \kappa_2^*$.

(c) If $V_n \in \kappa_2^*$, then it is clear that $V_n \Diamond V_n \subseteq V_n$.

(d) If $V_n \in \kappa_2^*$, $V_m \in \kappa_2^*$, then $V_n \cap V_m = \begin{cases} V_0, n \neq m \\ V_n, n = m \end{cases}$. Thus $V_n \cap V_m \in \kappa_2^*$. Therefore κ_2^* satisfies the conditions (i)-(iv) of Definition 3.1. □

Example 3.4. Consider the general fuzzy automaton in Fig. 1, where $Q = \{q_0, q_1, q_2, q_3, q_4\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols and $\tilde{R} = Q_{act}(t_0) = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$.

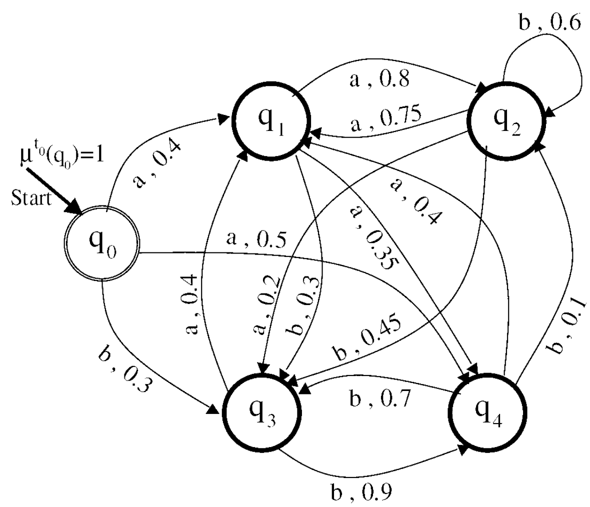

Fig. 1. The GFA of Example 3.4

Then we have

$ord(q_0) = 1, ord(q_1) = ord(q_3) = ord(q_4) = 2, ord(q_2) = 3,$

$U_0 = \emptyset, U_1 = \{(q_0, q_0)\}, U_3 = \{(q_2, q_2)\}, U_n = \emptyset, \forall n \geq 4,$

$U_2 = \{(q_1, q_1), (q_1, q_3), (q_1, q_4), (q_3, q_1), (q_3, q_3), (q_3, q_4), (q_4, q_1), (q_4, q_3), (q_4, q_4)\},$

$V_0 = \emptyset, V_1 = \{(\Lambda, \Lambda)\}, V_2 = \{(a, a), (a, b), (b, a), (b, b)\},$

$V_3 = \{(aa, aa), (aa, ab), (aa, ba), (aa, bb), \dots\}, \dots .$

Theorem 3.5. Let $\tilde{F}$ be a GFA and

$$\kappa_1 = \{U \subseteq Q \times Q : U_n \subseteq U, \quad \exists U_n \in \kappa_1^*\},$$
$$\kappa_2 = \{V \subseteq \Sigma^* \times \Sigma^* : V_n \subseteq V, \quad \exists V_n \in \kappa_2^*\}.$$

Then κ_1 satisfies a semi-uniformity on Q and the pair (Q, κ_1) is a semi-uniform structure. Also κ_2 satisfies a semi-uniformity on Σ^* and the pair (Σ^*, κ_2) is a semi-uniform structure.

Proof. By Theorem 3.3, κ_1 satisfies the conditions (i)-(iv) of Definition 3.1.

Now, let $U \in \kappa_1$, $U \subseteq U' \subseteq Q \times Q$. Then there exists $U_n \in \kappa_1^*$ such that $U_n \subseteq U$. Thus $U_n \subseteq U'$ and $U_n \in \kappa_1^*$. Therefore $U' \in \kappa_1$.

Again, by Theorem 3.3, κ_2 satisfies the conditions (i)-(iv) of Definition 3.1.

Now, let $V \in \kappa_2$, $V \subseteq V' \subseteq \Sigma^* \times \Sigma^*$. Then there exists $V_n \in \kappa_2^*$ such that $V_n \subseteq V$. Thus $V_n \subseteq V'$ and $V_n \in \kappa_2^*$. Therefore $V' \in \kappa_2$. □

Theorem 3.6. Let $\tilde{F}$ be a GFA, $p \in Q$, $x \in \Sigma^*$, $U \in \kappa_1$, $V \in \kappa_2$ and

$$\begin{aligned} U[p] &= \{q \in Q : (p,q) \in U\}, V[x] = \{y \in \Sigma^* : (x,y) \in V\}, \\ \tau_1 &= \{Q' \subseteq Q : \forall p \in Q', \ \ \exists U \in \kappa_1 \ \ , \ \ U[p] \subseteq Q'\}, \\ \tau_2 &= \{T \subseteq \Sigma^* : \forall x \in T, \ \ \exists V \in \kappa_2 \ \ , \ \ V[x] \subseteq T\}. \end{aligned}$$

Then τ_1 is a topology on Q and τ_2 is a topology on Σ^*.

Proof. (i) It is clear that $\emptyset$ and Q are in τ_1.
(ii) Let $Q_1, Q_2 \in \tau_1$ and $q \in Q_1 \cap Q_2$. Then there exist $U_1, U_2 \in \kappa_1$ such that $U_1[q] \subseteq Q_1$ and $U_2[q] \subseteq Q_2$. Let $W = U_1 \cap U_2$. By Theorem 3.5, $W \in \kappa_1$ and also $W[q] \subseteq U_1[q] \cap U_2[q] \subseteq Q_1 \cap Q_2$. Thus $Q_1 \cap Q_2 \in \tau_1$.
(iii) Let $Q_i \in \tau_1$, $\forall i \in I$. Then $\forall p \in Q_i$, $\exists U_i \in \kappa_1 : U_i[p] \subseteq Q_i$. Let $U = \bigcup_{i \in I} U_i$. By Theorem 3.5, since $U_1 \subseteq \bigcup_{i \in I} U_i \subseteq Q \times Q$ and $U_1 \in \kappa_1$, hence $U = \bigcup_{i \in I} U_i \in \kappa_1$. So $U[p] = \bigcup_{i \in I} U_i[p] \subseteq \bigcup_{i \in I} Q_i$, $\forall p \in \bigcup_{i \in I} Q_i$. Therefore $\bigcup_{i \in I} Q_i \in \tau_1$. Consequently, τ_1 is a topology on Q.
Also ,
(i) $\emptyset$ and Σ^* are in τ_2.
(ii) Let $T_1, T_2 \in \tau_2$ and $x \in T_1 \cap T_2$. Then there exist $V_1, V_2 \in \kappa_2$ such that $V_1[x] \subseteq T_1$ and $V_2[x] \subseteq T_2$. Let $T = V_1 \cap V_2$. By Theorem 3.5, $T \in \kappa_2$ and $T[x] \subseteq V_1[x] \cap V_2[x] \subseteq T_1 \cap T_2$. Thus $T_1 \cap T_2 \in \tau_2$.
(iii) Let $T_i \in \tau_2$, $\forall i \in I$. Then $\forall x \in T_i$, $\exists V_i \in \kappa_2 : V_i[x] \subseteq T_i$. Let $V = \bigcup_{i \in I} V_i$. By Theorem 3.5, since $V_1 \subseteq \bigcup_{i \in I} V_i \subseteq \Sigma^* \times \Sigma^*$, $V_1 \in \kappa_2$, hence $V = \bigcup_{i \in I} V_i \in \kappa_2$. So $V[x] = \bigcup_{i \in I} V_i[x] \subseteq \bigcup_{i \in I} T_i$, $\forall x \in \bigcup_{i \in I} T_i$. Therefore $\bigcup_{i \in I} T_i \in \tau_2$. Consequently τ_2 is a topology on Σ^*. □

Definition 3.7. Let $\tilde{F}$ be a GFA, $Q' \subseteq Q$, $\Sigma' \subseteq \Sigma^*$, $q, q' \in Q$, $x, y \in \Sigma^*$, $ord(q) = ord(q')$ and $\ell(x) = \ell(y)$. We say that $\tilde{F}$ is absorbing with respect to Q' and Σ' if

(i) $q \in Q' \Rightarrow q' \in Q'$,
(ii) $x \in \Sigma' \Rightarrow y \in \Sigma'$.

Also, $\tilde{F}$ is absorbing if $\tilde{F}$ is absorbing with respect to Q' and Σ' for every $Q' \subseteq Q$ and $\Sigma' \subseteq \Sigma^*$.

Theorem 3.8. Let $\tilde{F}$ be an absorbing GFA, $Q' \subseteq Q$ and $\Sigma' \subseteq \Sigma^*$. Then Q' is a clopen set in (Q, τ_1) and Σ' is a clopen set in (Σ^*, τ_2).

Proof. (i) We first show that $Q' = \bigcup_{q \in Q'} U_n[q]$, where $n = ord(q)$. Let $q \in Q'$. Since $q \in U_n[q]$, hence $Q' \subseteq \bigcup_{q \in Q'} U_n[q]$. Conversely, let $q' \in \bigcup_{q \in Q'} U_n[q]$. Then there exists $q \in Q'$ such that $q' \in U_n[q]$. Thus $q \in Q'$, $ord(q) = ord(q') = n$. Since $\tilde{F}$ is absorbing, hence $q' \in Q'$. So $\bigcup_{q \in Q'} U_n[q] \subseteq Q'$. Consequently, Q' is an open set in (Q, τ_1).

Now, we show that ${Q'}^C = \bigcup_{q \notin Q'} U_n[q]$. Let $q \in {Q'}^C$. Since $q \in U_n[q]$, hence ${Q'}^C \subseteq \bigcup_{q \notin Q'} U_n[q]$.
Conversely, let $q' \in \bigcup_{q \notin Q'} U_n[q]$. Then there exists $q \in {Q'}^C$ such that $q' \in U_n[q]$. Thus $q \in {Q'}^C$ and $ord(q) = ord(q') = n$. Since $\tilde{F}$ is absorbing, hence $q' \in {Q'}^C$. So $\bigcup_{q \notin Q'} U_n[q] \subseteq {Q'}^C$. Consequently, Q' is a closed set in (Q, τ_1).
(ii) We first show that $\Sigma' = \bigcup_{x \in \Sigma'} V_n[x]$, where $n = \ell(x) + 1$. Let $x \in \Sigma'$. Since $x \in V_n[x]$, hence $\Sigma' \subseteq \bigcup_{x \in \Sigma'} V_n[x]$. Conversely, let $y \in \bigcup_{x \in \Sigma'} V_n[x]$. Then there exists $x \in \Sigma'$ such that $y \in V_n[x]$. Thus $x \in \Sigma'$, $\ell(x) = \ell(y) = n-1$. Since $\tilde{F}$ is absorbing, hence $y \in \Sigma'$. So $\bigcup_{x \in \Sigma'} V_n[x] \subseteq \Sigma'$. Consequently, Σ' is an open set in (Σ^*, τ_2).

Now, we prove ${\Sigma'}^C = \bigcup_{x \notin \Sigma'} V_n[x]$, where $n = \ell(x) + 1$. Let $x \in {\Sigma'}^C$. Since $x \in V_n[x]$, hence ${\Sigma'}^C \subseteq \bigcup_{x \notin \Sigma'} V_n[x]$. Conversely, let $y \in \bigcup_{x \notin \Sigma'} V_n[x]$. Then there exists $x \in {\Sigma'}^C$ such that $y \in V_n[x]$. Thus $x \in {\Sigma'}^C$ and $\ell(x) = \ell(y) = n - 1$. Since $\tilde{F}$ is absorbing, hence $y \in {\Sigma'}^C$. So $\bigcup_{x \notin \Sigma'} V_n[x] \subseteq {\Sigma'}^C$. Therefore Σ' is a closed set in (Σ^*, τ_2). □

Corollary 3.9. Let $\tilde{F}$ be an absorbing GFA. Then τ_1 is a discrete topology on Q and τ_2 is a discrete topology on Σ^*.

Proof. $\tilde{F}$ is an absorbing GFA. Hence, by Theorem 3.8, if $p \in Q$ and $x \in \Sigma^*$, then $\{p\}$ is a clopen set in (Q, τ_1) and $\{x\}$ is a clopen set in (Σ^*, τ_2). Thus τ_1 is a discrete topology on Q and τ_2 is a discrete topology on Σ^*. □

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References

[1] M. Doostfatemeh and S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175-214.
[2] I. M. Hanafy, A. M. Abd El-Aziz and T. M. Salman, *Semi θ-compactness in Intuitionistic Fuzzy Topological Spaces*, Iranian Journal of Fuzzy Systems, **3(2)** (2006), 53-62.
[3] K. D. Joshi, *Introduction to general topology*, New Age International Publisher, India, 1997.
[4] Y. B. Jun and H. S. Kim, *Uniform structure in positive implicative algebras*, International Mathematical Journal, **2** (2002), 215-218.
[5] S. P. Li, Z. Fang and J. Zhao, *P2-Connectedness in L-Topological Spaces*, Iranian Journal of Fuzzy Systems, **2(1)** (2005), 29-36.
[6] J. N. Mordeson and D. S. Malik, *Fuzzy automata and languages, theory and applications*, Cha-pman and Hall/CRC, London/Boca Raton, FL, 2002.
[7] D. S. Malik and J. N. Mordeson, *Fuzzy discrete structures*, Physica-Verlag, New York, 2000.
[8] W. Omlin, K. K. Giles and K. K. Thornber, *Equivalence in knowledge representation: automata, rnns, and dynamic fuzzy systems*, Proc. IEEE, **87(9)** (1999), 1623-1640.
[9] W. Omlin, K. K. Thornber and K. K. Giles, *Fuzzy finite-state automata can be deterministically encoded into recurrent neural networks*, IEEE Trans. Fuzzy Syst. **5(1)** (1998), 76-89.
[10] W. Page, *Topological uniform structures*, Dover Publication, Inc. New York, 1988.
[11] W. G. Wee, *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classif ication*, Ph.D. dissertation Purdue University, IN, 1967.
[12] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338-353.
[13] M. M. Zahedi, M. Horry and K. Abolpor, *Bifuzzy (General) topology on max-min general fuzzy automata*, Advanced in Fuzzy Mathematics, **3(1)** (2008), 51-68.

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