IDEALS OF PSEUDO MV-ALGEBRAS BASED ON VAGUE SET THEORY

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Abstract. The notion of vague ideals in pseudo MV-algebras is introduced, and several properties are investigated. Conditions for a vague set to be a vague ideal are provided. Conditions for a vague ideal to be implicative are given. Characterizations of (implicative, prime) vague ideals are discussed. The smallest vague ideal containing a given vague set is established. Prime and implicative extension property for a vague ideal is discussed.

1. **Introduction**

In the real world there are vaguely specified data values in many applications, such as sensor information. Fuzzy set theory has been proposed to handle such vagueness by generalizing the notion of membership in a set. Essentially, in a fuzzy set, each element is associated with a point-value selected from the unit interval $[0, 1]$, which is termed the grade of membership in the set. A vague set, as well as an intuitionistic fuzzy set, is a further generalization of a fuzzy sets. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. The interval-based membership in vague sets is more expressive in capturing vagueness of data. In the literature, the notions of intuitionistic fuzzy sets and vague sets are regarded as an equivalent notion, in the sense that an intuitionistic fuzzy set is isomorphic to a vague set. Because of this view and intuitionistic fuzzy sets being earlier known as a tradition, the interesting features for handling vague data that are unique to vague sets are largely ignored. Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [7]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [1] studied vague groups. In this paper we introduce the notion of (implicative) vague ideals in pseudo MV-algebras, and investigate their properties. We provide conditions for a vague set to be a vague ideal. We also give conditions for a vague ideal to be implicative. We discuss characterizations of (implicative, prime) vague ideals. We establish the smallest vague ideal that contains a given vague set. We also discuss implicative and prime extension property for a vague ideal.

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2. **Basics**

In this section, we introduce some basic concepts related to pseudo MV-algebras and vague sets.

2.1. **Basic Results on Pseudo MV-algebras.** A *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim :$$

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(a2) $x \oplus 0 = 0 \oplus x = x$,
(a3) $x \oplus 1 = 1 \oplus x = 1$,
(a4) $1^\sim = 0$, $1^- = 0$,
(a5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(a6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$,
(a7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,
(a8) $(x^-)^\sim = x$.

If we define $x \leq y$ if and only if $x^- \oplus y = 1$, then $\leq$ is a partial order such that M is a bounded distributive lattice with the join $x \vee y$ and the meet $x \wedge y$ given by

$$x \vee y = x \oplus x^\sim \odot y = x \odot y^- \oplus y,$$

$$x \wedge y = x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y.$$

Let M be a pseudo MV-algebra and $x, y, z \in M$. Then the following properties are valid (see [4]).

(b1) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.
(b2) $(x \vee y)^- = x^- \wedge y^-$.
(b3) $x \leq y \Rightarrow z \odot x \leq z \odot y$, $x \odot z \leq y \odot z$.
(b4) $z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y)$.
(b5) $z \odot (x \oplus y) \leq z \odot x \oplus y$.
(b6) $(x^\sim)^- = x$.
(b7) $x \odot 1 = 1 \odot x = x$.
(b8) $x \oplus x^\sim = 1$, $x^- \oplus x = 1$.
(b9) $x \odot x^- = 0$, $x^\sim \odot x = 0$.
(b10) $x \odot (y \odot z) = (x \odot y) \odot z$.
(b11) $(x \oplus y)^- = y^- \odot x^-$, $(x \oplus y)^\sim = y^\sim \odot x^\sim$.
(b12) $x \leq y \Leftrightarrow x \odot y^- = 0 \Leftrightarrow y^\sim \odot x = 0$.
(b13) $x \odot y^- \wedge y \odot x^- = 0$, $x^\sim \odot y \wedge y^\sim \odot x = 0$.

A subset I of a pseudo MV-algebra M is called an *ideal* of M if it satisfies:

(i) $0 \in I$,
(ii) If $x, y \in I$, then $x \oplus y \in I$,
(iii) If $x \in I$, $y \in M$ and $y \leq x$, then $y \in I$.

For every subset $W \subseteq M$, we denote by $\langle W \rangle$ the ideal of M generated by W, that is, $\langle W \rangle$ is the smallest ideal containing W. By [4, Lemma 2.5],

$$\langle W \rangle = \{x \in M \mid x \leq y_1 \oplus \cdots \oplus y_k \text{ for some } y_1, \ldots, y_k \in W\}.$$

Let J be a proper ideal of a pseudo MV-algebra M (i.e., $J \neq M$). Then J is said to be *prime* if for every ideals J_1 and J_2 of M, $J = J_1 \cap J_2$ implies $J = J_1$ or $J = J_2$.

Proposition 2.1. [4] *For an ideal J of a pseudo MV-algebra M, the following are equivalent.*

(i) *J is prime.*
(ii) $(\forall x, y \in M)$ $(x \wedge y \in J \Rightarrow x \in J$ *or* $y \in J)$.

2.2. **Basic Results on Vague Sets.** Let U be a classical set of objects, called the universe of discourse, where an element of U is denoted by u.

A *fuzzy set* $A = \{\langle u, \mu_A(u)\rangle \mid u \in U\}$ in U is characterized by a membership function $\mu_A : U \to [0,1]$. An intuitionistic fuzzy set (IFS)

$$A = \{\langle u, \mu_A(u), \gamma_A(u)\rangle \mid u \in U\}$$

in U is characterized by a membership function, μ_A, and a non-membership function, γ_A, as follows:

$$\mu_A : U \to [0,1],\ \gamma_A : U \to [0,1], \text{ and } 0 \leq \mu_A + \gamma_A \leq 1.$$

A *vague set* (VS) A in U is characterized by two membership functions given by (see [1]):

(1) A truth membership function

$$t_A : U \to [0,1],$$

(2) A false membership function

$$f_A : U \to [0,1],$$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), 1 - f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A and is denoted by $V_A(u)$.

As we can see that the difference between vague sets and intuitionistic fuzzy sets is due to the definition of membership intervals. We have $[t_A(u), 1 - f_A(u)]$ for u in a vague set A, but $\langle u, \mu_A(u), \gamma_A(u)\rangle$ for u in an intuitionistic fuzzy set A. Here the semantics of μ_A and γ_A are the same as $t_A(u)$ and $f_A(u)$, respectively. However, the boundary $1 - f_A(u)$ is able to indicate the possible existence of a data value, as already mentioned in [2]. This subtle difference gives rise to a simpler but meaningful graphical view of data sets. Consider a vague set in Fig. 1 ([5]) and an intuitionistic fuzzy set in Fig. 2 ([5]) respectively. It can be seen that, the shaded part formed by the boundary in a given VS in Fig. 1([5]) naturally represents the possible existence of data. Thus, the "hesitation region" corresponds to the intuition of representing vague data.

For our discussion, we shall use the following notations, which are given in [1], on interval arithmetic.

Notion. Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two elements of $I[0,1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly we understand the relations $I_1 \leq I_2$ and $I_1 = I_2$. Clearly the relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \supseteq I_2$ and conversely. We define the term "*imax*" to mean the maximum of two intervals as

$$imax(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)].$$

Similarly we define "*imin*". The concept of "*imax*" and "*imin*" could be extended to define "*isup*" and "*iinf*" of infinite number of elements of $I[0,1]$.

It is obvious that $L = \{I[0,1], isup, iinf, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [1]).

For $\alpha, \beta \in [0,1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 2.2. [1] Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α, β)*-cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts are also called *vague-cuts* of the vague set A.

Definition 2.3. [1] The α*-cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\alpha,\beta)} = A_\alpha$.

Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Vague Ideals

In what follows let M be a pseudo MV-algebra unless otherwise specified.

Definition 3.1. A vague set A of M is called a *vague ideal* of M if the following conditions are true:

(c1) $(\forall x, y \in M)$ $(V_A(x \oplus y) \geq imin\{V_A(x), V_A(y)\})$,
(c2) $(\forall x, y \in M)$ $(y \leq x \Rightarrow V_A(y) \geq V_A(x))$.

that is,

(1) for all $x, y \in M$,

$$\begin{aligned} t_A(x \oplus y) &\geq \min\{t_A(x), t_A(y)\}, \\ 1 - f_A(x \oplus y) &\geq \min\{1 - f_A(x), 1 - f_A(y)\}, \end{aligned} \tag{3.1}$$

(2) for all $x, y \in M$, $y \leq x$ implies

$$t_A(y) \geq t_A(x),\ 1 - f_A(y) \geq 1 - f_A(x) \tag{3.2}$$

It is easily seen that (c2) forces

(c3) $(\forall x \in M)\ (V_A(0) \geq V_A(x))$,

that is, for every $x \in M$,

$$t_A(0) \geq t_A(x),\ 1 - f_A(0) \geq 1 - f_A(x). \tag{3.3}$$

Example 3.2. Let I be an ideal of M and let A be a vague set of M defined by

$$V_A(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in I, \\ [\beta_1, \beta_2], & \text{otherwise,} \end{cases} \tag{3.4}$$

where $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in I[0, 1]$ with $[\alpha_1, \alpha_2] > [\beta_1, \beta_2]$. Let $x, y \in M$. If $x, y \in I$, then $x \oplus y \in I$ and so

$$V_A(x \oplus y) = [\alpha_1, \alpha_2] = imin\{V_A(x), V_A(y)\}.$$

If $x \notin I$ or $y \notin I$, then $V_A(x) = [\beta_1, \beta_2]$ or $V_A(y) = [\beta_1, \beta_2]$. Thus

$$V_A(x \oplus y) \geq [\beta_1, \beta_2] = imin\{V_A(x), V_A(y)\}.$$

Let $x, y \in M$ be such that $y \leq x$. If $y \in I$, then $V_A(y) = [\alpha_1, \alpha_2] \geq V_A(x)$. Assume that $y \notin I$. Then $x \notin I$, and thus $V_A(y) = [\beta_1, \beta_2] = V_A(x)$. Therefore A is a vague ideal of M.

Proposition 3.3. *Let A be a vague ideal of M. Then*

(i) $(\forall x, y \in M)\ (V_A(x \odot y) \geq imin\{V_A(x), V_A(y)\})$.
(ii) $(\forall x, y \in M)\ (V_A(x \wedge y) \geq imin\{V_A(x), V_A(y)\})$.
(iii) $(\forall x, y \in M)\ (V_A(x \vee y) = imin\{V_A(x), V_A(y)\})$.
(iv) $(\forall x, y \in M)\ (V_A(x \oplus y) = imin\{V_A(x), V_A(y)\})$.

Proof. Note that $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ for all $x, y \in M$. Then

$$t_A(x \odot y) \geq t_A(x \wedge y) \geq t_A(x \vee y) \geq t_A(x \oplus y) \geq \min\{t_A(x), t_A(y)\},$$

$$\begin{aligned} 1 - f_A(x \odot y) \quad &\geq 1 - f_A(x \wedge y) \geq 1 - f_A(x \vee y) \\ &\geq 1 - f_A(x \oplus y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

Since $x \oplus y \geq x \vee y \geq x, y$ for all $x, y \in M$, we have

$$\begin{aligned} &t_A(x \oplus y) \leq t_A(x \vee y) \leq t_A(x), t_A(y), \\ &t_A(x \oplus y) \leq t_A(x \vee y) \leq \min\{t_A(x), t_A(y)\}, \\ &1 - f_A(x \oplus y) \leq 1 - t_A(x \vee y) \leq 1 - f_A(x), 1 - f_A(y), \\ &1 - f_A(x \oplus y) \leq 1 - f_A(x \vee y) \leq \min\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let A be a vague set of M. Then A is a vague ideal of M if and only if it satisfies* (c1) *and*

(c4) $(\forall x, y \in M)\ (V_A(x \wedge y) \geq V_A(x))$, *that is, for all* $x, y \in M$,

$$t_A(x \wedge y) \geq t_A(x),\ 1 - f_A(x \wedge y) \geq 1 - f_A(x). \tag{3.5}$$

Proof. Assume that A is a vague ideal of M and let $x, y \in M$. Since $x \wedge y \leq x$, it follows from (c2) that $V_A(x \wedge y) \geq V_A(x)$. Suppose that A satisfies (c1) and (c4). Let $x, y \in M$ be such that $y \leq x$. Then $x \wedge y = y$ and so $V_A(y) = V_A(x \wedge y) \geq V_A(x)$ by (c4). Hence A is a vague ideal of M. □

Proposition 3.5. *Every vague ideal A of M satisfies the following inequality*

$$(\forall x, y \in M)\,(V_A(y) \geq imin\{V_A(x), V_A(x^{\sim} \odot y)\}). \tag{3.6}$$

Proof. Let A be a vague ideal of M. Since $y \leq x \vee y = x \oplus x^{\sim} \odot y$ for all $x, y \in M$, it follows from (c1) and (c2) that

$$V_A(y) \geq V_A(x \oplus x^{\sim} \odot y) \geq imin\{V_A(x), V_A(x^{\sim} \odot y)\}.$$

This completes the proof. □

Proposition 3.6. *Let A be a vague set of M that satisfies* (c3) *and* (3.6)*. Then A satisfies the condition* (c2) *and*

$$(\forall x, y \in M)\,(V_A(y) \geq imin\{V_A(x), V_A(y \odot x^{-})\}). \tag{3.7}$$

Proof. Assume that A satisfies (c3) and (3.6). Let $x, y \in M$ be such that $y \leq x$. Using (b3) and (b9), we have $x^{\sim} \odot y \leq x^{\sim} \odot x = 0$ and so $x^{\sim} \odot y = 0$. It follows from (c3) and (3.6) that

$$V_A(y) \geq imin\{V_A(x), V_A(x^{\sim} \odot y)\} = imin\{V_A(x), V_A(0)\} = V_A(x)$$

so that (c2) is valid. Note that

$$(y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x) \leq (y \odot x^{-})^{\sim} \odot (y \odot x^{-}) \oplus x = 0 \oplus x = x$$

so from (c2) that $V_A(x) \leq V_A((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))$. Now since

$$x^{\sim} \odot y \leq x \oplus x^{\sim} \odot y = y \odot x^{-} \oplus x,$$

it follows from (c2) that $V_A(x^{\sim} \odot y) \geq V_A(y \odot x^{-} \oplus x)$ so that

$$\begin{aligned} V_A(y) &\geq imin\{V_A(x), V_A(x^{\sim} \odot y)\} \geq imin\{V_A(x), V_A(y \odot x^{-} \oplus x)\} \\ &\geq imin\{V_A(x), imin\{V_A(y \odot x^{-}), V_A((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))\}\} \\ &\geq imin\{V_A(x), imin\{V_A(y \odot x^{-}), V_A(x)\}\} \\ &= imin\{V_A(x), V_A(y \odot x^{-})\}. \end{aligned}$$

This completes the proof. □

Proposition 3.7. *If a vague set A of M satisfies conditions* (c3) *and* (3.7)*, then A is a vague ideal of M.*

Proof. Let $x, y \in M$ be such that $y \leq x$. Then $y \odot x^{-} \leq x \odot x^{-} = 0$ by (b3) and (b9), and thus $y \odot x^{-} = 0$. Using (c3) and (3.7), we have

$$V_A(y) \geq imin\{V_A(x), V_A(y \odot x^{-})\} = imin\{V_A(x), V_A(0)\} = V_A(x).$$

Thus (c2) is valid. Note that

$$(x \oplus y) \odot y^- = (x \oplus (y^-)^\sim) \odot y^- = x \wedge y^- \leq x$$

for all $x, y \in M$ so from (3.7) and (c2) that

$$V_A(x \oplus y) \geq imin\{V_A(y), V_A((x \oplus y) \odot y^-)\} \geq imin\{V_A(y), V_A(x)\}.$$

Hence (c1) is valid, and A is a vague ideal of M. □

Combining Propositions 3.5, 3.6 and 3.7, we have the following characterization of a vague ideal in a pseudo MV-algebra.

Theorem 3.8. *For a vague set A of M, the following are equivalent:*

(i) *A is a vague ideal of M.*
(ii) *A satisfies the conditions* (c3) *and* (3.6)*.*
(iii) *A satisfies the conditions* (c3) *and* (3.7)*.*

Proposition 3.9. *Let A be a vague set of M. If A satisfies conditions* (c3) *and*

$$(3.8) \qquad (\forall x, y, z \in M)\,(V_A(x \odot y) \geq imin\{V_A(x \odot y \odot z), V_A(z^\sim \odot y)\}),$$

then A is a vague ideal of M. Moreover, A satisfies:

(i) $(\forall x, y \in M)$ $(V_A(x \odot y) = V_A(x \odot y \odot y))$,
(ii) $(\forall x \in M)$ $(\forall n \in \mathbb{N})$ $(V_A(x) = V_A(x^n))$,

where $x^n = x^{n-1} \odot x = x \odot x^{n-1}$ and $x^0 = 1$.

Proof. Taking $x = y$, $y = 1$ and $z = x^-$ in (3.8) and using (a8) and (b7), we have

$$\begin{aligned} V_A(y) &= V_A(y \odot 1) \geq imin\{V_A(y \odot 1 \odot x^-), V_A((x^-)^\sim \odot 1)\} \\ &= imin\{V_A(y \odot x^-), V_A(x)\}. \end{aligned}$$

It follows from Theorem 3.8 that A is a vague ideal of M. Now taking $z = y$ in (3.8) and using (b9) and (c3), we get

$$\begin{aligned} V_A(x \odot y) &\geq imin\{V_A(x \odot y \odot y), V_A(y^\sim \odot y)\} \\ &= imin\{V_A(x \odot y \odot y), V_A(0)\} \\ &= V_A(x \odot y \odot y). \end{aligned}$$

On the other hand, since $x \odot y \odot y \leq x \odot y$, we see that $V_A(x \odot y \odot y) \geq V_A(x \odot y)$. Then (i) holds.

The proof of (ii) is by induction on n. If $n = 1$, then (ii) is obviously true. If we put $x = 1$ and $y = x$ in (i), then

$$V_A(x) = V_A(1 \odot x) = V_A(1 \odot x \odot x) = V_A(x^2).$$

Now assume that (ii) is valid for every positive integer $k > 2$. Then

$$V_A(x^{k+1}) = V_A(x^{k-1} \odot x \odot x) = V_A(x^{k-1} \odot x) = V_A(x^k) = V_A(x).$$

Therefore (ii) is true. □

Theorem 3.10. *Let A be a vague set of M. Then the following assertions are equivalent.*

(i) *A is a vague ideal of M.*

(ii) $(\forall x, y, z \in M)$ $(z \odot x^- \odot y^- = 0 \Rightarrow V_A(z) \geq imin\{V_A(x), V_A(y)\}.)$
(iii) $(\forall x, y, z \in M)$ $(x^\sim \odot y^\sim \odot z = 0 \Rightarrow V_A(z) \geq imin\{V_A(x), V_A(y)\}.)$

Proof. (i) $\Rightarrow$ (ii). Assume that A is a vague ideal of M. Then A satisfies (3.7). Hence

$$V_A(z) \geq imin\{V_A(x), V_A(z \odot x^-)\}$$

and

$$V_A(z \odot x^-) \geq imin\{V_A(y), V_A(z \odot x^- \odot y^-)\}$$

for all $x, y, z \in M$. It follows that

$$\begin{aligned} V_A(z) &\geq imin\{V_A(x), V_A(y), V_A(z \odot x^- \odot y^-)\} \\ &= imin\{V_A(x), V_A(y), V_A(0)\} \\ &= imin\{V_A(x), V_A(y)\}, \end{aligned}$$

which proves (ii).

(ii) $\Rightarrow$ (iii). Let $x, y, z \in M$ be such that $x^\sim \odot y^\sim \odot z = 0$. Then

$$(y \oplus x)^\sim \odot z = 0$$

by (b11), and so $z \odot (y \oplus x)^- = 0$ by (b12). It follows from (b11) that $z \odot x^- \odot y^- = 0$. Using (ii), we have

$$V_A(z) \geq imin\{V_A(x), V_A(y)\}.$$

(iii) $\Rightarrow$ (i). Suppose that (iii) is valid. Since $x^\sim \odot x^\sim \odot 0 = 0$ for all $x \in M$, we have

$$V_A(0) \geq imin\{V_A(x), V_A(x)\} = V_A(x).$$

Using (b9), we get $(x^\sim \odot y)^\sim \odot x^\sim \odot y = 0$ for all $x, y \in M$. It follows from (iii) that

$$V_A(y) \geq imin\{V_A(x^\sim \odot y), V_A(x)\}.$$

Using Theorem 3.8, we conclude that A is a vague ideal of M. □

Corollary 3.11. *A vague set A of M is a vague ideal of M if and only if it satisfies:*

$$(\forall x, y, z \in M)\,(z \leq x \oplus y \Rightarrow V_A(z) \geq imin\{V_A(x), V_A(y)\}.$$

Using induction on n, we have the following corollary.

Corollary 3.12. *A vague set A of M is a vague ideal of M if and only if it satisfies:*

$$x \leq y_1 \oplus y_2 \oplus \cdots \oplus y_n \Rightarrow V_A(x) \geq imin\{V_A(y_1), V_A(y_2), \cdots, V_A(y_n)\}$$

for all $x, y_1, y_2, \cdots, y_n \in M$.

Proposition 3.13. *For any vague set A of M, the condition* (3.8) *is equivalent to the following condition:*

(3.9) $(\forall x, y, z \in M)\,(V_A(x \odot y) \geq imin\{V_A(x \odot y \odot z^-), V_A(z \odot y)\}.$

Proof. (3.8) $\Rightarrow$ (3.9): Let $x, y, z \in M$. Using (3.8) and (a8), we have

$$\begin{aligned} V_A(x \odot y) &\geq imin\{V_A(x \odot y \odot z^-), V_A((z^-)^\sim \odot y)\} \\ &= imin\{V_A(x \odot y \odot z^-), V_A(z \odot y)\}. \end{aligned}$$

(3.9) $\Rightarrow$ (3.8): Applying (3.9) we see that

$$V_A(x \odot y) \geq imin\{V_A(x \odot y \odot (z^\sim)^-), V_A(z^\sim \odot y)\}.$$

From this we obtain (3.8), because $(z^{\sim})^{-} = z$ by (b6). □

In [6], Walendziak introduced the notion of implicative ideals in pseudo MV-algebras. An ideal I of M is said to be *implicative* if it satisfies the following implication:

$$(\forall x, y, z \in M)\,(x \odot y \odot z \in I \ \&\ z^{\sim} \odot y \in I \ \Rightarrow\ x \odot y \in I).$$

Definition 3.14. Let A be a vague ideal of M. We say that A is *implicative* if it satisfies the condition (3.8) (or (3.9)).

Proposition 3.15. *Let I be an ideal of M. Then I is implicative if and only if the vague set A which is described in Example 3.2 is an implicative vague ideal of M.*

Proof. Straightforward. □

Lemma 3.16. *Let A be a vague ideal of M. Then*

$$(\forall x, y \in M)\,(V_A(x \odot y) \geq imin\{V_A(x \odot y \odot y), V_A(y \wedge y^{\sim})\}). \tag{3.10}$$

Proof. Applying (b7), (b8) and (b5), we have

$$x \odot y = (x \odot y) \odot 1 = (x \odot y) \odot (y \oplus y^{\sim}) \leq (x \odot y) \odot y \oplus y^{\sim}.$$

Using (b4), we obtain

$$\begin{aligned} x \odot y &\leq y \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &\leq (x \odot y \odot y \oplus y) \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &= x \odot y \odot y \oplus (y \wedge y^{\sim}). \end{aligned}$$

It follows from (3.2) and (3.1) that

$$\begin{aligned} t_A(x \odot y) &\geq t_A(x \odot y \odot y \oplus (y \wedge y^{\sim})) \\ &\geq \min\{t_A(x \odot y \odot y), t_A(y \wedge y^{\sim})\} \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(x \odot y) &\geq 1 - f_A(x \odot y \odot y \oplus (y \wedge y^{\sim})) \\ &\geq \min\{1 - f_A(x \odot y \odot y), 1 - f_A(y \wedge y^{\sim})\}, \end{aligned}$$

that is, $V_A(x \odot y) \geq V_A(x \odot y \odot y \oplus (y \wedge y^{\sim}) \geq imin\{V_A(x \odot y \odot y), V_A(y \wedge y^{\sim})\}$. This completes the proof. □

Theorem 3.17. *Let A be a vague ideal of M. Then the following statements are equivalent:*

(i) *A is implicative.*
(ii) $(\forall x, y \in M)$ $(V_A(x \odot y) = V_A(x \odot y \odot y))$.
(iii) $(\forall x \in M)$ $(x^2 = 0 \Rightarrow V_A(x) = V_A(0))$.
(iv) $(\forall x \in M)$ $(V_A(x \wedge x^{-}) = V_A(0))$.
(v) $(\forall x \in M)$ $(V_A(x \wedge x^{\sim}) = V_A(0))$.

Proof. (i) ⇒ (ii): This is by Proposition 3.9.

(ii) ⇒ (iii): Taking $x = 1$ and $y = x$ in (ii), we get

$$V_A(x) = V_A(1 \odot x) = V_A(1 \odot x \odot x) = V_A(x^2).$$

Then (iii) is obviously true.

(iii) $\Rightarrow$ (iv): Using (b3) and (b9) we have

$$(x \wedge x^-)^2 = (x \wedge x^-) \odot (x \wedge x^-) \leq x \odot x^- = 0.$$

Consequently, $(x \wedge x^-)^2 = 0$. Hence $V_A(x \wedge x^-) = V_A(0)$.

(iv) $\Rightarrow$ (v): Since $x \wedge x^\sim = x^\sim \wedge x = x^\sim \wedge (x^\sim)^-$, it follows from (iv) that $V_A(x \wedge x^\sim) = V_A(0)$.

(v) $\Rightarrow$ (i): Note that

$$V_A(x \odot y) \geq imin\{V_A(x \odot y \odot y), V_A(y \wedge y^\sim)\}$$

by Lemma 3.16. Therefore

$$V_A(x \odot y) \geq \min\{V_A(x \odot y \odot y), V_A(0)\} = V_A(x \odot y \odot y).$$

Applying (b3) and (b5) we get

$$x \odot y \odot y \leq x \odot y \odot (z \vee y) = x \odot y \odot (z \oplus z^\sim \odot y) \leq x \odot y \odot z \oplus z^\sim \odot y.$$

Since A is a vague ideal, we have

$$\begin{aligned} V_A(x \odot y \odot y) &\geq V_A(x \odot y \odot z \oplus z^\sim \odot y) \\ &\geq \min\{V_A(x \odot y \odot z), V_A(z^\sim \odot y)\}. \end{aligned}$$

Thus A satisfies the condition (3.8), i.e., A is implicative. □

Theorem 3.18. (Implicative extension property for vague ideals) *Let A be an implicative vague ideal of M and B any vague ideal of M such that A is contained in B and $V_A(0) = V_B(0)$. Then B is an implicative vague ideal of M.*

Proof. Since A is an implicative vague ideal of M, we get $V_A(x \wedge x^\sim) = V_A(0)$ for all $x \in M$ by Theorem 3.17. It follows from hypothesis that

$$V_B(0) = V_A(0) = V_A(x \wedge x^\sim) \leq V_B(x \wedge x^\sim)$$

so that $V_B(x \wedge x^\sim) = V_B(0)$. Using Theorem 3.17, we know that B is an implicative vague ideal of M. □

Theorem 3.19. *If A is a vague ideal of M, then its nonempty (α, β)-cuts*

$$A_{(\alpha,\beta)} := \{x \in M \mid V_A(x) \geq [\alpha, \beta]\}$$

is a crisp ideal of M for all $\alpha, \beta \in [0, 1]$.

Proof. Assume that A is a vague ideal of M and let $\alpha, \beta \in [0, 1]$ be such that $A_{(\alpha,\beta)} \neq \emptyset$. Obviously $0 \in A_{(\alpha,\beta)}$. Let $x, y \in M$ be such that $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x) \geq [\alpha, \beta]$ and $V_A(y) \geq [\alpha, \beta]$, that is, $t_A(x) \geq \alpha$, $1 - f_A(x) \geq \beta$, $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$. It follows that

$$t_A(x \oplus y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha,$$

$$1 - f_A(x \oplus y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta.$$

Hence $V_A(x \oplus y) \geq imin\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$, and so $x \oplus y \in A_{(\alpha,\beta)}$. Let $x, y \in M$ be such that $x \in A_{(\alpha,\beta)}$ and $y \leq x$. Then $V_A(y) \geq V_A(x) \geq [\alpha, \beta]$ by (c2), and so $y \in A_{(\alpha,\beta)}$. Therefore $A_{(\alpha,\beta)}$ is a crisp ideal of M. □

The ideals like $A_{(\alpha,\beta)}$ are also called *vague-cut ideals* of X.

Corollary 3.20. *If A is a vague ideal of M, then the set*

$$M_a := \{x \in M \mid V_A(x) \geq V_A(a)\}$$

is a crisp ideal of M for every $a \in M$.

Proof. Straightforward. □

Theorem 3.21. *Any ideal I of M is a vague-cut ideal of some vague ideal of M.*

Proof. Consider the vague set A of M given by

$$V_A(x) = \begin{cases} [\alpha, \alpha], & \text{if } x \in I, \\ [0, 0], & \text{if } x \notin I, \end{cases} \tag{3.11}$$

where $\alpha \in (0, 1)$. Since $0 \in I$, we have $V_A(0) = [\alpha, \alpha] \geq V_A(x)$ for all $x \in X$. Let $x, y \in M$. If $x, y \in I$, then $x \oplus y \in I$. Hence

$$V_A(x \oplus y) = [\alpha, \alpha] = imin\{V_A(x), V_A(y)\}.$$

If one of x and y does not belong to I, then one of $V_A(x)$ and $V_A(y)$ is equal to $[0, 0]$. Thus

$$V_A(x \oplus y) \geq [0, 0] = imin\{V_A(x), V_A(y)\}.$$

Assume that $y \leq x$. If $x \in I$, then $y \in I$. Thus $V_A(y) = V_A(x)$. If $x \notin I$, then $V_A(x) = [0, 0]$, and so $V_A(y) \geq V_A(x)$. Therefore A is a vague ideal of M. Obviously, $A_{(\alpha,\alpha)} = I$. □

Theorem 3.22. *Let A be a vague ideal of M. Then the set*

$$I_0 := \{x \in M \mid V_A(x) = V_A(0)\}$$

is a crisp ideal of M. Moreover, if A is implicative then I_0 is implicative.

Proof. Clearly $0 \in I_0$. Let $x, y \in M$ be such that $x, y \in I_0$. Then $V_A(x) = V_A(0) = V_A(y)$, and so

$$V_A(x \oplus y) \geq imin\{V_A(x), V_A(y)\} = V_A(0).$$

Since $V_A(0) \geq V_A(x)$ for all $x \in M$, it follows that $V_A(x \oplus y) = V_A(0)$. Hence $x \oplus y \in I_0$. Let $x \in I_0$ and $y \in M$ be such that $y \leq x$. Then $V_A(y) \geq V_A(x) = V_A(0)$, and hence $V_A(y) = V_A(0)$, i.e., $y \in I_0$. Consequently, I_0 is a crisp ideal of M. Now, suppose that A is implicative. Let $x, y, z \in M$. If $x \odot y \odot z \in I_0$ and $z^\sim \odot y \in I_0$, then $V_A(x \odot y \odot z) = V_A(0) = V_A(z^\sim \odot y)$. Since A is implicative, it follows from (3.8) that

$$V_A(x \odot y) \geq imin\{V_A(x \odot y \odot z), V_A(z^\sim \odot y)\} = V_A(0).$$

Hence $V_A(x \odot y) = V_A(0)$, which implies that $x \odot y \in I_0$. Therefore I_0 is implicative. □

Proposition 3.23. *If a vague set A of M satisfies the condition* (3.8)*, then*

$$(x \odot y \odot z \in A_{(\alpha,\beta)} \ \&\ z^\sim \odot y \in A_{(\alpha,\beta)} \ \Rightarrow\ x \odot y \in A_{(\alpha,\beta)}) \tag{3.12}$$

for all $x, y, z \in M$ and $\alpha, \beta \in [0, 1]$.

Proof. Assume that A satisfies the condition (3.8) and let $x, y, z \in M$ and $\alpha, \beta \in [0,1]$ be such that $x \odot y \odot z \in A_{(\alpha,\beta)}$ and $z^{\sim} \odot y \in A_{(\alpha,\beta)}$. Then $V_A(x \odot y \odot z) \geq [\alpha, \beta]$ and $V_A(z^{\sim} \odot y) \geq [\alpha, \beta]$. It follows from (3.8) that

$$V_A(x \odot y) \geq imin\{V_A(x \odot y \odot z), V_A(z^{\sim} \odot y)\} \geq [\alpha, \beta]$$

so that $x \odot y \in A_{(\alpha,\beta)}$. □

Applying Theorem 3.19 and Proposition 3.23, we have the following theorem.

Theorem 3.24. *If A is a vague ideal of M that satisfies the condition* (3.8)*, then $A_{(\alpha,\beta)}$ is an implicative ideal of M for all $\alpha, \beta \in [0,1]$.*

Given a vague set A of M, we finally establish the smallest vague ideal of M that contains A. For two vague sets A and B of M, if $V_B(x) \geq V_A(x)$ for all $x \in M$ then we say that B contains A.

Theorem 3.25. *Let A be a vague set of M. Define a vague set B of M as follows:*

$$\begin{aligned} V_B(x) = \; & isup\{imin\{V_A(a_1), V_A(a_2), \cdots, V_A(a_n)\} \mid x \leq a_1 \oplus \\ & a_2 \oplus \cdots \oplus a_n \text{ for some } a_1, a_2, \cdots, a_n \in M\}. \end{aligned}$$

Then B is the smallest vague ideal of M that contains A.

Proof. Obviously, $V_B(0) \geq V_B(x)$ for all $x \in M$. Let $x, y \in M$ be such that

$$x \leq a_1 \oplus a_2 \oplus \cdots \oplus a_n$$

and

$$x^{\sim} \odot y \leq b_1 \oplus b_2 \oplus \cdots \oplus b_m$$

for some $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \in M$. Then

$$\begin{aligned} y \;\; & \leq \;\; x \vee y \;=\; x \oplus x^{\sim} \odot y \\ & \leq \;\; a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus b_1 \oplus b_2 \oplus \cdots \oplus b_m, \end{aligned}$$

and so

$$V_B(y) \geq imin\{V_A(a_1), V_A(a_2), \cdots, V_A(a_n), V_A(b_1), V_A(b_2), \cdots, V_A(b_m)\}.$$

Denote by

$$\begin{aligned} \Omega_1 := \; & \{imin\{V_A(a_1), V_A(a_2), \cdots, V_A(a_i)\} \mid x \leq a_1 \oplus \\ & a_2 \oplus \cdots \oplus a_i \text{ for some } a_1, a_2, \cdots, a_i \in M\} \end{aligned}$$

and

$$\begin{aligned} \Omega_2 := \; & \{imin\{V_A(b_1), V_A(b_2), \cdots, V_A(b_j)\} \mid x^{\sim} \odot y \leq b_1 \oplus \\ & b_2 \oplus \cdots \oplus b_j \text{ for some } b_1, b_2, \cdots, b_j \in M\}. \end{aligned}$$

Then

$$\begin{aligned} & imin\{V_B(x), V_B(x^{\sim} \odot y)\} = imin\{isup\Omega_1, isup\Omega_2\} \\ & = isup\{imin\{V_A(a_1, V_A(a_2), \cdots, V_A(a_i), V_A(b_1), \\ & \qquad V_A(b_2), \cdots, V_A(b_j)\} \mid x \leq a_1 \oplus a_2 \oplus \cdots \oplus a_i; \\ & \qquad x^{\sim} \odot y \leq b_1 \oplus b_2 \oplus \cdots \oplus b_j \\ & \qquad \text{for some } a_1, a_2, \cdots, a_i, b_1, b_2, \cdots, b_j \in M\}, \end{aligned}$$

and so $V_B(y) \geq imin\{V_B(x), V_B(x^{\sim} \odot y)\}$. By Theorem 3.8, we know that B is a vague ideal of M. Since $x \leq x \oplus x$ for all $x \in M$, we get

$$V_B(x) \geq imin\{V_A(x), V_A(x)\} = V_A(x)$$

for all $x \in M$, that is, B contains A. Now let C be a vague ideal of M that contains A. For any $x \in M$,

$$\begin{aligned} V_B(x) &= isup\{imin\{V_A(a_1), V_A(a_2), \cdots, V_A(a_n)\} \mid x \leq a_1 \oplus a_2 \oplus \cdots a_n \\ &\qquad \text{for some } a_1, a_2, \cdots, a_n \in M\} \\ &\leq isup\{imin\{V_C(a_1), V_C(a_2), \cdots, V_C(a_n)\} \mid x \leq a_1 \oplus a_2 \oplus \cdots a_n \\ &\qquad \text{for some } a_1, a_2, \cdots, a_n \in M\} \\ &\leq V_C(x). \end{aligned}$$

Therefore B is the smallest vague ideal of M that contains A. □

4. **Prime Vague Ideals**

In this section, we define the notion of a prime vague ideal of a pseudo MV-algebra and investigate its properties.

Definition 4.1. A vague ideal A of M is said to be *prime* if it is non-constant vague set and satisfies:

$$(\forall x, y \in M)\,(V_A(x \wedge y) = imax\{V_A(x), V_A(y)\}), \tag{4.1}$$

that is, for every $x, y \in M$,

$$\begin{aligned} t_A(x \wedge y) &= \max\{t_A(x), t_A(y)\}, \\ 1 - f_A(x \wedge y) &= \max\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned} \tag{4.2}$$

We provide characterizations of a prime vague ideal.

Theorem 4.2. *Let A be a non-constant vague ideal of M. Then the following are equivalent.*

(i) *A is a prime vague ideal of M.*
(ii) $(\forall x, y \in M)$ $(V_A(x \wedge y) = V_A(0) \Rightarrow V_A(x) = V_A(0)$ *or* $V_A(y) = V_A(0))$.
(iii) $(\forall x, y \in M)$ $(V_A(x \odot y^-) = V_A(0)$ *or* $V_A(y \odot x^-) = V_A(0))$.
(iv) $(\forall x, y \in M)$ $(V_A(x^{\sim} \odot y) = V_A(0)$ *or* $V_A(y^{\sim} \odot x) = V_A(0))$.

Proof. (i) ⇒ (ii). Assume that A is a prime vague ideal of M. Let $x, y \in M$ be such that $V_A(x \wedge y) = V_A(0)$, that is, $t_A(x \wedge y) = t_A(0)$ and $1 - f_A(x \wedge y) = 1 - f_A(0)$. Then

$$\max\{t_A(x), t_A(y)\} = t_A(x \wedge y) = t_A(0),$$
$$\max\{1 - f_A(x), 1 - f_A(y)\} = 1 - f_A(x \wedge y) = 1 - f_A(0),$$

and so $t_A(x) = t_A(0)$ or $t_A(y) = t_A(0)$, and $1 - f_A(x) = 1 - f_A(0)$ or $1 - f_A(y) = 1 - f_A(0)$. This shows that $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$.

(ii) ⇒ (iii). By (b13), we have $x \odot y^- \wedge y \odot x^- = 0$ for all $x, y \in M$. Hence $V_A(x \odot y^- \wedge y \odot x^-) = V_A(0)$. It follows from (ii) that $V_A(x \odot y^-) = V_A(0)$ or $V_A(y \odot x^-) = V_A(0)$.

(iii) ⇒ (iv). Replacing x and y by $x^{\sim}$ and $y^{\sim}$, respectively, in (iii) and using (b6), we have (iv).

(iv) $\Rightarrow$ (i). Assume that $V_A(x^{\sim} \odot y) = V_A(0)$ for all $x, y \in M$, that is, $t_A(x^{\sim} \odot y) = t_A(0)$ and $1 - f_A(x^{\sim} \odot y) = 1 - f_A(0)$. Note that

$$y \leq (x \vee y) \wedge (y \oplus x^{\sim} \odot y) = (x \oplus x^{\sim} \odot y) \wedge (y \oplus x^{\sim} \odot y) = (x \wedge y) \oplus (x^{\sim} \odot y)$$

for some $x, y \in M$. Since A is a vague ideal of M, it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} t_A(y) &\geq t_A((x \vee y) \wedge (y \oplus x^{\sim} \odot y)) \\ &\geq \min\{t_A(x \wedge y), t_A(x^{\sim} \odot y)\} \\ &= \min\{t_A(x \wedge y), t_A(0)\} = t_A(x \wedge y), \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(y) &\geq 1 - f_A((x \vee y) \wedge (y \oplus x^{\sim} \odot y)) \\ &\geq \min\{1 - f_A(x \wedge y), 1 - f_A(x^{\sim} \odot y)\} \\ &= \min\{1 - f_A(x \wedge y), 1 - f_A(0)\} = 1 - f_A(x \wedge y). \end{aligned}$$

Since $x \wedge y \leq y$, it follows from (3.2) that $t_A(x \wedge y) \geq t_A(y)$ and $1 - f_A(x \wedge y) \geq 1 - f_A(y)$. Hence $t_A(x \wedge y) = t_A(y)$ and $1 - f_A(x \wedge y) = 1 - f_A(y)$, that is, $V_A(x \wedge y) = V_A(y)$. By Theorem 3.8 and (c3), we know that

$$\begin{aligned} V_A(y) &\geq imin\{V_A(x), V_A(x^{\sim} \odot y)\} \\ &= imin\{V_A(x), V_A(0)\} \\ &= V_A(x). \end{aligned}$$

Consequently, $V_A(x \wedge y) = imax\{V_A(x), V_A(y)\}$, and so A is a prime vague ideal of M. Similarly, we can induce the implication (iv) $\Rightarrow$ (i) for the case of $V_A(y^{\sim} \odot x) = V_A(0)$. This completes the proof. $\square$

Theorem 4.3. *Let A be a vague ideal of M. Then A is prime if and only if*

$$I_0 = \{x \in M \mid V_A(x) = V_A(0)\}$$

is a prime ideal of M.

Proof. Assume that A is a prime vague ideal of M. Then I_0 is an ideal of M by Theorem 3.22. Since A is non-constant, I_0 is proper. Let $x, y \in M$ be such that $x \wedge y \in I_0$. Then

$$V_A(0) = V_A(x \wedge y) = imax\{V_A(x), V_A(y)\},$$

and so $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$. Therefore $x \in I_0$ or $y \in I_0$, and hence I_0 is a prime ideal of M by Proposition 2.1.

Conversely, suppose that I_0 is a prime ideal of M. Since I_0 is proper, A is non-constant. Let $x, y \in M$. Then $x \odot y^{-} \wedge y \odot x^{-} = 0 \in I_0$ by (b13). It follows from Proposition 2.1 that $x \odot y^{-} \in I_0$ or $y \odot x^{-} \in I_0$, that is, $V_A(x \odot y^{-}) = V_A(0)$ or $V_A(y \odot x^{-}) = V_A(0)$. Using Theorem 4.2, we conclude that A is a prime vague ideal of M. $\square$

Theorem 4.4. (Prime extension property for vague ideals) *Let A be a prime vague ideal of M and B any non-constant vague ideal of M such that B contains A and $V_A(0) = V_B(0)$. Then B is a prime vague ideal of M.*

Proof. Since A is a prime vague ideal of M, we have $V_A(x \odot y^-) = V_A(0)$ or $V_A(y \odot x^-) = V_A(0)$ for all $x, y \in M$ by Theorem 4.2. It follows from hypothesis that

$$V_B(0) = V_A(0) = V_A(x \odot y^-) \leq V_B(x \odot y^-)$$

or

$$V_B(0) = V_A(0) = V_A(y \odot x^-) \leq V_B(y \odot x^-)$$

so that $V_B(0) = V_B(x \odot y^-)$ or $V_B(0) = V_B(y \odot x^-)$ for all $x, y \in M$. Applying Theorem 4.2, we know that B is a prime vague ideal of M. □

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