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REMARKS ON COMPLETENESS OF LATTICE-VALUED CAUCHY SPACES

G. JÄGER

Abstract. We study different completeness definitions for two categories of lattice-valued Cauchy spaces and the relations between these definitions. We also show the equivalence of a so-called completion axiom and the existence of a completion.

1. Introduction

Lattice-valued Cauchy spaces and completions were first defined in [7]. A slightly more general definition was subsequently given in [1], where also regularity for such spaces was studied. Other papers dealing with the same subject are [13] and [15]. The major interest in (lattice-valued) Cauchy spaces is in defining completeness and constructing completions. A first attempt was already made in [7] and the construction was translated to the more general category in [1]. Also [15] constructs a completion. All lattice-valued generalizations of Cauchy spaces are at the same time generalizations of the probabilistic case [11, 12]. The completeness concept in [11] explicitely demands that a probabilistic Cauchy space is complete if and ony if each "level space" is complete. This means that each value $\alpha \in [0,1]$ and each Cauchy filter in that level there is a point such that the filter converges to this point with probability at least α. However, as it is stated, we may have for each "level" α a different convergence point. As the unit interval is linearly ordered, we then see that, for a Cauchy filter, a convergence point of a higher level is also a convergence point of the lower levels and it seems therefore that we can do with one convergence point overall. However, as is shown in this paper for the lattice-valued case, this is not always true. To ensure a single convergence point for a Cauchy filter we need to demand a so-called left-continuity condition. This observation is the starting point for this paper. After collecting the necessary notions and notations in a preliminary section, we introduce the two categories of lattice-valued Cauchy spaces that we consider in this paper in the next section. Section 3 then defines and discusses, for each of the considered categories, two different completeness notions. The final section is devoted to the construction of a completion for a non-complete lattice-valued Cauchy space. This construction shows that the existence of a completion is equivalent to a completion axiom. This had been observed and stated before, however the construction in [7] only leads to a weak completion and not to a completion as claimed. We finally draw some conclusions in the last section.

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2. Preliminaries

We consider in this paper frames, i.e. complete lattices L, where finite meets distribute over arbitrary joins, i.e. where $\alpha \wedge \bigvee_{i\in I} \beta_i = \bigvee_{i\in I}(\alpha \wedge \beta_i)$ for all $\alpha, \beta_i \in L (i \in I)$ and all index sets I. The bottom element of L is denoted by $\perp$ and the top element by $\top$.

The lattice operations are extended pointwise from L to L^X, the set of all L-sets $a, b, c, ...$ on X. In particular, we define, for $A \subseteq X$ and $\alpha \in L$, the L-set $\alpha_A \in L^X$ by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = \perp$ for $x \notin A$.

A *stratified L-filter* on X [3] is a mapping $\mathcal{F} : L^X \longrightarrow L$ with the properties (F1) $\mathcal{F}(\top_X) = \top$, $\mathcal{F}(\perp_X) = \perp$, (F2) $\mathcal{F}(a) \le \mathcal{F}(b)$ whenever $a \le b$, (F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \le \mathcal{F}(a \wedge b)$ for all $a, b \in L^X$ and (Fs) $\alpha \le \mathcal{F}(\alpha_X)$ for all $\alpha \in L$. The set of all stratified L-filters on X is denoted by $\mathcal{F}^s_L(X)$. An example of a stratified L-filter is the *principal L-filter* $[x]$ defined by $[x](a) = a(x)$ for all $a \in L^X$. The set $\mathcal{F}^s_L(X)$ is ordered pointwise, i.e. for $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$ we denote $\mathcal{F} \le \mathcal{G}$ if for all L-sets $a \in L^X$ we have $\mathcal{F}(a) \le \mathcal{G}(a)$. Maximal elelements in this order are called *stratified L-ultrafilters*. For a family of stratified L-filters $(\mathcal{F}_j)_{j\in J}$, the meet is then given by $\bigwedge_{j\in J} \mathcal{F}_j(a) = \bigwedge_{j\in J}(\mathcal{F}_j(a))$ for $a \in L^X$. For $A \subseteq X$ we define $[A] = \bigwedge_{x\in A}[x]$. Two stratified L-filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$ have an upper bound if and only if $\mathcal{F}(a) \wedge \mathcal{G}(b) = \perp$ whenever $a \wedge b = \perp_X$, see [3]. In this case we say that $\mathcal{F} \vee \mathcal{G}$ exists. For a mapping $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}^s_L(X)$ we define $\varphi(\mathcal{F}) \in \mathcal{F}^s_L(Y)$ by $\varphi(\mathcal{F})(b) = \mathcal{F}(\varphi^{\leftarrow}(b))$, where for $b \in L^Y$ it is defined $\varphi^{\leftarrow}(b)(x) = b(\varphi(x))$. For $\mathcal{G} \in \mathcal{F}^s_L(Y)$ we define $\varphi^{\leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ by $\varphi^{\leftarrow}(\mathcal{G})(a) = \bigvee\{\mathcal{G}(b) \,:\, \varphi^{\leftarrow}(b) \le a\}$. Then $\varphi^{\leftarrow}(\mathcal{G}) \in \mathcal{F}^s_L(X)$ if and only if $\mathcal{G}(b) = \perp$ whenever $\varphi^{\leftarrow}(b) = \perp_X$ (see [4]). If $Y \subseteq X$ and $\iota : Y \hookrightarrow X, y \longmapsto y$ is the inclusion mapping, then we denote for $\mathcal{F} \in \mathcal{F}^s_L(X)$ its *trace on* Y, $\mathcal{F}_Y = \iota^{\leftarrow}(\mathcal{F})$, in case this is a stratified L-filter on Y. Also, for $\mathcal{G} \in \mathcal{F}^s_L(Y)$ we denote $[\mathcal{G}] = \iota(\mathcal{G}) \in \mathcal{F}^s_L(X)$.

3. Stratified L-Cauchy Tower Spaces and Stratified L-Cauchy Spaces

A *stratified L-Cauchy tower space* [1, 15] is a pair $(X, \overline{C})$, where X is a set, $\overline{C} = (C_\alpha)_{\alpha\in L}$ and $C_\alpha \subseteq \mathcal{F}^s_L(X)$ for all $\alpha \in L$, which satisfies the following axioms.

(LCT1) $[x] \in C_\alpha$ for each $\alpha \in L$ and $x \in X$;
(LCT2) $\mathcal{G} \in C_\alpha$ whenever $\mathcal{F} \in C_\alpha$ and $\mathcal{G} \ge \mathcal{F}$;
(LCT3) $\mathcal{F} \wedge \mathcal{G} \in C_\alpha$ whenever $\mathcal{F}, \mathcal{G} \in C_\alpha$ and $\mathcal{F} \vee \mathcal{G}$ exists;
(LCT4) $C_\alpha \subseteq C_\beta$ whenever $\beta \le \alpha$;
(LCT5) $C_\perp = \mathcal{F}^s_L(X)$.

A mapping $f : (X, \overline{C}) \longrightarrow (Y, \overline{D})$ between two stratified L-Cauchy tower spaces, $(X, \overline{C}), (Y, \overline{D})$, is called *Cauchy-continuous* if for all $\alpha \in L$, $f(\mathcal{F}) \in D_\alpha$ whenever $\mathcal{F} \in C_\alpha$. The category with objects the stratified L-Cauchy tower spaces and morphisms the Cauchy continuous mappings is denoted by $SL\text{-}CTS$. A space $(X, \overline{C}) \in |SL\text{-}CTS|$ is called *left-continuous* if $\mathcal{F} \in C_\alpha$ whenever there is $A \subseteq L$, $\bigvee A = \alpha$ such that $\mathcal{F} \in C_\beta$ for all $\beta \in A$. The subcategory of $SL\text{-}CTS$ with all left-continuous spaces as objects is denoted by $SL\text{-}LC\text{-}CTS$. The category $SL\text{-}CTS$ is topological and Cartesian closed and $SL\text{-}LC\text{-}CTS$ is bireflective in $SL\text{-}CTS$, see [1]. If $L = \{0,1\}$, then SL-CTS can be identified with the category CHY of Cauchy spaces [10, 14].

A stratified L-Cauchy tower space induces a *stratified L-limit tower space* $(X, \overline{q^C})$ by defining $x \in q^C_\alpha(\mathcal{F})$ if $\mathcal{F} \wedge [x] \in C_\alpha$. Such a stratified L-limit tower space [2] is a pair $(X, \overline{q})$ where X is a set, $\overline{q} = (q_\alpha)_{\alpha \in L}$ and $q_\alpha : \mathcal{F}^s_L(X) \longrightarrow X$ which satisfies the following axioms.

(LLT1) $x \in q_\alpha([x])$ for all $x \in X$ and $\alpha \in L$;
(LLT2) $x \in q_\alpha(\mathcal{G})$, whenever $x \in q_\alpha(\mathcal{F})$ and $\mathcal{G} \geq \mathcal{F}$;
(LLT3) $x \in q_\alpha(\mathcal{F} \wedge \mathcal{G})$ whenever $x \in q_\alpha(\mathcal{F})$ and $x \in q_\alpha(\mathcal{G})$;
(LLT4) $x \in q_\beta(\mathcal{F})$ whenever $x \in q_\alpha(\mathcal{F})$ and $\beta \leq \alpha$;
(LLT5) $q_0([X]) = X$.

A mapping $f : (X, \overline{q}) \longrightarrow (Y, \overline{p})$ between two stratified L-limit tower spaces is called *continuous* if $f(x) \in p_\alpha(f(\mathcal{F}))$ whenever $x \in q_\alpha(\mathcal{F})$. The category with objects the stratified L-limit tower spaces and morphisms the continuous mappings is denoted by SL-LTS.

A stratified L-Cauchy space [7] is a pair (X, C) of a set X and a mapping $C : \mathcal{F}^s_L(X) \longrightarrow L$ which satisfies the following axioms.

(LC1) $\forall\, x \in X, C([x]) = \top$
(LC2) $\mathcal{F} \leq \mathcal{G} \quad \Longrightarrow \quad C(\mathcal{F}) \leq C(\mathcal{G})$
(LC3) If $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}^s_L(X)$, then $C(\mathcal{F}) \wedge C(\mathcal{G}) \leq C(\mathcal{F} \wedge \mathcal{G})$.

A mapping $\varphi : (X, C) \longrightarrow (X', C')$ between two stratified L-Cauchy spaces (X, C) and (X', C') is called *Cauchy-continuous* if for all $\mathcal{F} \in \mathcal{F}^s_L(X)$ we have $C(\mathcal{F}) \leq C'(\varphi(\mathcal{F}))$. The category which has as objects the stratified L-Cauchy spaces and as morphisms the Cauchy-continuous mappings is denoted by SL-CHY. The category SL-CHY is topological and Cartesian closed [7]. In case $L = \{0,1\}$ we can identify SL-CHY with the category CHY of Cauchy spaces [10, 14].

A stratified L-Cauchy space induces a stratified L-limit space by defining $\lim_C \mathcal{F}(x) = C(\mathcal{F} \wedge [x])$. Such a *stratified L-limit space* [5] is a set X together with a limit map $\lim : \mathcal{F}^s_L(X) \longrightarrow L^X$ which satisfies the axioms

(LL1) $\lim[x](x) = \top$ for all $x \in X$,
(LL2) $\lim \mathcal{F} \leq \lim \mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$,
(LL3) $\lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$.

A mapping $\varphi : (X, \lim) \longrightarrow (X', \lim')$ between two stratified L-limit spaces is called *continuous* if for all $\mathcal{F} \in \mathcal{F}^s_L(X)$ and all $x \in X$ we have $\lim \mathcal{F}(x) \leq \lim{}' \varphi(\mathcal{F})((\varphi(x))$. The category with objects all stratified L-limit spaces and the continuous mappings as morphisms is denoted by SL-LIM. Also this category is topological and Cartesian closed [5]. A stratified L-limit space is called a *T1-space* if $\lim[x](y) = \top$ implies $x = y$ and it is called a *T2-space* if $\lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top$ implies $x = y$ [6]. A subset $A \subseteq X$ is called *dense in* $(X, \lim)$ if for every $x \in X$, the set $H^\top_A(x) = \{\mathcal{F} \in \mathcal{F}^s_L(X) \,:\, \mathcal{F}_A \in \mathcal{F}^s_L(A),\ \lim \mathcal{F}(x) = \top\}$ is non-empty [6].

Boustique and Richardson [1] introduce the following functors.

$$\Theta : \begin{cases} SL\text{-}CHY & \longrightarrow & SL\text{-}CTS \\ (X, C) & \longmapsto & (X, \Theta C) \\ f & \longmapsto & f \end{cases}$$

where $\mathcal{F} \in (\Theta C)_\alpha$ if $C(\mathcal{F}) \geq \alpha$, and

$$\Psi : \begin{cases} SL\text{-}CTS & \longrightarrow & SL\text{-}CHY \\ (X, \overline{C}) & \longmapsto & (X, \Psi\overline{C}) \\ f & \longmapsto & f \end{cases}$$

where $\Psi(\overline{C})(\mathcal{F}) = \bigvee\{\alpha \in L \;:\; \mathcal{F} \in C_\alpha\}$. In fact, the functor Θ maps onto SL-LC-CTS and if we restrict Ψ on SL-LC-CTS, these functors provide isomorphism functors between SL-CHY and SL-LC-CTS, see [1].

4. **Completeness Notions**

We first consider two completeness notions in SL-CTS.

A space $(X, \overline{C}) \in |SL\text{-}CTS|$ is called *weakly complete* if for all $\alpha \in L$, for $\mathcal{F} \in C_\alpha$ there is $x = x(\alpha, \mathcal{F}) \in X$ such that $\mathcal{F} \wedge [x] \in C_\alpha$. It is called *complete* if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ there is $x = x(\mathcal{F}) \in X$ such that, for $\alpha \in L$ we have $\mathcal{F} \wedge [x] \in C_\alpha$ whenever $\mathcal{F} \in C_\alpha$.

Weak completeness is the completeness notion that was introduced in [1] under the name "completeness". In this case, the "point of convergence", x, may vary from "level to level", i.e. it may depend on α. Example 4.2 and Example 4.3 below show that this will become noticable in the non-left-continuous case. The concept of completeness introduced here is stronger and we believe that it is more adequate, as the convergence point only depends on the L-filter $\mathcal{F}$ and not on the level α. For left-continuous spaces both definitions coincide.

Lemma 4.1. *Let $(X, \overline{C}) \in |SL\text{-}CTS|$.*
(i) If $(X, \overline{C})$ is complete, it is weakly complete.
(ii) If $(X, \overline{C})$ is left-continuous and weakly complete, then it is complete.

Proof. (i) is obvious. In order to prove (ii) let $\mathcal{F} \in \mathcal{F}_L^s(X)$. We denote $A = \{\alpha \in L \;:\; \mathcal{F} \in C_\alpha\}$ and $\delta = \bigvee A$. By left-continuity then $\mathcal{F} \in C_\delta$ and hence there is $x = x(\delta, \mathcal{F}) \in X$ such that $\mathcal{F} \wedge [x] \in C_\delta$. Let now $\mathcal{F} \in C_\alpha$. Then $\alpha \in A$ and hence $\alpha \leq \delta$ which implies $C_\delta \subseteq C_\alpha$. Hence $\mathcal{F} \wedge [x] \in C_\alpha$ and the point x does not depend on α but only on $\mathcal{F}$. □

The following examples show that we cannot omit the assumption of left-continuity in (ii).

Example 4.2. We consider $L = \{\perp, \alpha, \beta, \top\}$ with $\alpha \wedge \beta = \perp$, $\alpha \vee \beta = \top$ and α, β uncomparable. Then L is a complete Boolean algebra. Let further X be an infinite set and let $\mathcal{U} \in \mathcal{F}_L^s(X)$ be a stratified L-ultrafilter such that $\mathcal{U} \not\geq [x]$ for all $x \in X$ (cf. [8]). Fix $x_0, y_0 \in X$ with $x_0 \neq y_0$. We define the following Cauchy tower structure on X.

- $C_\perp = \mathcal{F}_L^s(X)$.

- $\mathcal{F} \in C_\alpha$ if $\mathcal{F} \geq \mathcal{U} \wedge [x_0]$ or if $\mathcal{F} \geq [x]$ for some $x \in X$.
- $\mathcal{F} \in C_\beta$ if $\mathcal{F} \geq \mathcal{U} \wedge [y_0]$ or if $\mathcal{F} \geq [x]$ for some $x \in X$.
- $\mathcal{F} \in C_\top$ if $\mathcal{F} \geq [x]$ for some $x \in X$.

Then $(X, \overline{C})$ is a stratified L-Cauchy tower space. All axioms except (LCT3) are clear. For (LCT3), let $\mathcal{F}, \mathcal{G} \in C_\alpha$ and let $\mathcal{F} \vee \mathcal{G}$ exist. If both $\mathcal{F}, \mathcal{G} \geq \mathcal{U} \wedge [x_0]$, then also $\mathcal{F} \wedge \mathcal{G} \geq \mathcal{U} \wedge [x_0]$ and hence $\mathcal{F} \wedge \mathcal{G} \in C_\alpha$. Similarly, if $\mathcal{F} \geq \mathcal{U} \wedge [x_0]$ and $\mathcal{G} \geq [x_0]$, then also $\mathcal{F} \wedge \mathcal{G} \geq \mathcal{U} \wedge [x_0]$ and hence $\mathcal{F} \wedge \mathcal{G} \in C_\alpha$. If $\mathcal{F} \geq \mathcal{U} \wedge [x_0]$ and $\mathcal{G} \geq [x]$ with $x \neq x_0$, then $(\mathcal{U} \wedge [x_0]) \vee [x]$ exists. We choose $F = X \setminus \{x\}$. Then $\top_F \wedge \top_{X \setminus F} = \bot_X$ but $(\mathcal{U} \wedge [x_0])(\top_F) = \top$ and $[x](\top_{X \setminus F}) = \top$, a contradiction. Hence this case cannot occur. Similarly $\mathcal{F} \geq [x]$, $\mathcal{G} \geq [y]$ with $x \neq y$ is impossible. The same arguments can be used for $\mathcal{F}, \mathcal{G} \in C_\beta$ and hence (LCT3) is true. We note, that $(X, \overline{C})$ is not left-continuous, because $\mathcal{U} \in C_\alpha$ and $\mathcal{U} \in C_\beta$ but $\mathcal{U} \notin C_\top = C_{\alpha \vee \beta}$.

The space $(X, \overline{C})$ is weakly complete. For $\delta \in L$, whenever $\mathcal{F} \in C_\delta$, then $\mathcal{F} \geq \mathcal{U} \wedge [x_0]$ and hence also $\mathcal{F} \wedge [x_0] \in C_\delta$, or $\mathcal{F} \geq \mathcal{U} \wedge [y_0]$ and hence also $\mathcal{F} \wedge [y_0] \in C_\delta$, or $\mathcal{F} \geq [x]$ and then also $\mathcal{F} \wedge [x] = [x] \in C_\delta$. $(X, \overline{C})$ is, however, not complete. For the stratified L-filter $\mathcal{U}$ we have $\mathcal{U} \wedge [x_0] \in C_\alpha$ and $\mathcal{U} \wedge [y_0] \in C_\beta$ but $\mathcal{U} \wedge [y_0] \notin C_\alpha$. For, if we assume $\mathcal{U} \wedge [y_0] \geq [x]$ for some $x \in X$, this implies $\mathcal{U} \geq [x]$, a contradiction, and if we assume $\mathcal{U} \wedge [y_0] \geq \mathcal{U} \wedge [x_0]$, then, defining $F = X \setminus \{y_0\}$, we obtain $\mathcal{U}(\top_F) = \top$, $[y_0](\top_F) = \bot$ and $[x_0](\top_F) = \top$. This implies $(\mathcal{U} \wedge [x_0])(\top_F) = \top$ but $(\mathcal{U} \wedge [y_0])(\top_F) = \bot$, a contradiction.

Example 4.3. Let $L = [0, 1]$ and let $X = [0, 1]$. Again fix a stratified L-ultrafilter $\mathcal{U} \in \mathcal{F}_L^s(X)$ with $\mathcal{U} \not\geq [x]$ for all $x \in X$. We define $\mathcal{F} \in C_\alpha$ if $\mathcal{F} \geq [x]$ for some $x \in X$ or if $\mathcal{F} \geq \mathcal{U} \wedge [x]$ with some $\alpha < x < 1$. Note that $\mathcal{F} \in C_1$ if $\mathcal{F} \geq [x]$ for some $x \in X$. We show that $(X, \overline{C}) \in |SL\text{-}CTS|$. The axioms (LCT1), (LCT2) and (LCT4) are clear from the definition. For the axiom (LCT3) we only discuss the case that $\mathcal{F}, \mathcal{G} \in C_\alpha$, $\mathcal{F} \vee \mathcal{G}$ exists and $\mathcal{F} \geq \mathcal{U} \wedge [x]$, $\mathcal{G} \geq \mathcal{U} \wedge [y]$ where $x \neq y$. Then $(\mathcal{U} \wedge [x]) \vee (\mathcal{U} \wedge [y])$ exists. We choose two complement-finite sets $F_1, F_2 \subseteq X$ with $x \in F_2, y \in F_1$, and $F_1 \cap F_2 = \emptyset$. Then $\mathcal{U} \wedge [x](1_{F_2}) = 1$ and $\mathcal{U} \wedge [y](1_{F_1}) = 1$ and $1_{F_2} \wedge 1_{F_1} = 0_X$, a contradiction. The other cases are easy. Clearly, for all $\alpha < 1$ we have $\mathcal{U} \in C_\alpha$ but $\mathcal{U} \notin C_1$. Hence $(X, \overline{C})$ is not left-continuous. By construction, $(X, \overline{C})$ is weakly complete. However, the space is not complete. To see this we choose for $x < 1$ a value $x < \alpha < 1$. Then $\mathcal{U} \in C_\alpha$ but $\mathcal{U} \wedge [x] \notin C_\alpha$.

A space $(X, C) \in |SL\text{-}CHY|$ is called *weakly complete* [13] if for every $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $C(\mathcal{F}) \leq \bigvee_{x \in X} C(\mathcal{F} \wedge [x])$. It is called *complete* [7] if for every $\mathcal{F} \in \mathcal{F}_L^s(X)$ there is $x \in X$ such that $C(\mathcal{F}) \leq C(\mathcal{F} \wedge [x])$.

Weak completenenss is the completeness concept introduced in [13]. Clearly, if $(X, C) \in |SL\text{-}CHY|$ is complete, then it is weakly complete. However, the converse is not true as the following example shows.

Example 4.4. We consider the space $(X, \overline{C})$ from Example 4.2 and consider the space $(X, \Psi\overline{C}) \in |SL\text{-}CHY|$. Then $\Psi\overline{C}(\mathcal{U}) = \bigvee\{\delta \in L : \mathcal{U} \in C_\delta\} = \alpha \vee \beta = \top$. However, $\Psi\overline{C}(\mathcal{U} \wedge [x_0]) = \alpha$, $\Psi\overline{C}(\mathcal{U} \wedge [y_0]) = \beta$ and $\Psi\overline{C}(\mathcal{U} \wedge [x]) = \bot$ for all $x \notin \{x_0, y_0\}$. Hence $(X, \Psi\overline{C})$ is not complete. Note that $\bigvee_{x \in X} \Psi\overline{C}(\mathcal{U} \wedge [x]) = \alpha \vee \beta = \top$ and that $(X, \Psi\overline{C})$ is weakly complete by Lemma 4.6 below.

Lemma 4.5. *Let $(X, C) \in |SL\text{-}CHY|$. Then (X, C) is complete if and only if $(X, \Theta C)$ is weakly complete if and only if $(X, \Theta C)$ is complete.*

Proof. Let first (X, C) be complete and let $\mathcal{F} \in (\Theta C)_\alpha$. Then $C(\mathcal{F}) \geq \alpha$ and hence there is $x = x(\mathcal{F}) \in X$ such that $C(\mathcal{F} \wedge [x]) \geq C(\mathcal{F}) \geq \alpha$. But this means that $\mathcal{F} \wedge [x] \in (\Theta C)_\alpha$ and $(X, \Theta C)$ is complete, and hence also weakly complete.
Let now $(X, \Theta C)$ be weakly complete. Because $(X, \Theta C)$ is left-continuous, then also $(X, \Theta C)$ is complete. If $C(\mathcal{F}) = \alpha$, then $\mathcal{F} \in (\Theta C)_\alpha$ and hence there is $x = x(\mathcal{F}) \in X$ such that $\mathcal{F} \wedge [x] \in (\Theta C)_\alpha$, i.e. $C(\mathcal{F} \wedge [x]) \geq \alpha = C(\mathcal{F})$. □

Lemma 4.6. *Let $(X, \overline{C}) \in |SL\text{-}CTS|$.*
(i) If $(X, \overline{C})$ is weakly complete, then $(X, \Psi\overline{C})$ is weakly complete.
(ii) If $(X, \overline{C})$ is complete, then $(X, \Psi\overline{C})$ is complete.

Proof. (i) Let $(X, \overline{C})$ be weakly complete and let $\Psi\overline{C}(\mathcal{F}) = \bigvee\{\beta \in L \ : \ \mathcal{F} \in C_\beta\} = \alpha$. For $\mathcal{F} \in C_\beta$ there is $x = x(\mathcal{F}, \beta) \in X$ such that $\mathcal{F} \wedge [x] \in C_\beta$. Then $\Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \beta$ and hence $\bigvee_{x \in X} \Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \beta$. It follows that $\bigvee_{x \in X} \Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \bigvee\{\beta \in L \ : \ \mathcal{F} \in C_\beta\} = \Psi\overline{C}(\mathcal{F})$.
(ii) If $(X, \overline{C})$ is complete, we can argue as before, however $x = x(\mathcal{F})$ does then not depend on β but only on $\mathcal{F}$. For $\mathcal{F} \in C_\beta$ then $\Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \beta$ and hence $\Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \bigvee\{\beta \in L \ : \ \mathcal{F} \in C_\beta\} = \Psi\overline{C}(\mathcal{F})$. □

In the left-continuous case, we can say more.

Lemma 4.7. *Let $(X, \overline{C}) \in |SL\text{-}CTS|$ be left-continuous. Then $(X, \overline{C})$ is complete if and only of $(X, \Psi\overline{C})$ is complete.*

Proof. We only need to show that $(X, \Psi\overline{C})$ complete implies the completeness of $(X, \overline{C})$. Let $\mathcal{F} \in C_\alpha$. Then $\Psi\overline{C}(\mathcal{F}) \geq \alpha$. Hence there is $x = x(\mathcal{F}) \in X$ such that $\Psi\overline{C}(\mathcal{F} \wedge [x]) \geq \alpha$. Define $A = \{\beta \in L \ : \mathcal{F} \wedge [x] \in C_\beta\}$. Then $\delta = \bigvee A \geq \alpha$. For $\beta \in A$ we have $\mathcal{F} \wedge [x] \in C_\beta$ and hence by left-continuity, $\mathcal{F} \wedge [x] \in C_\delta$. From (CLT4) we conclude that $\mathcal{F} \wedge [x] \in C_\alpha$. □

We conclude this section by collecting some known results from the literature and prove some simple results about the productivity of the completeness notions.

For $(X_\lambda, \overline{C}^\lambda) \in |SL\text{-}CTS|$ for all $\lambda \in \Lambda$, with the projection mappings $pr_\mu : \prod_{\lambda \in \Lambda} X_\lambda \longrightarrow X_\mu$, the *product space* $(\prod_{\lambda \in \Lambda} X_\lambda, \pi - \overline{C})$ is defined as follows. For $\mathcal{F} \in \mathcal{F}_L^s(\prod_{\lambda \in \Lambda} X_\lambda)$ we have $\mathcal{F} \in (\pi - \overline{C})_\alpha$ if $pr_\lambda(\mathcal{F}) \in C_\alpha^\lambda$ for all $\lambda \in \Lambda$, see [1].

Lemma 4.8. *Let $(X_\lambda, \overline{C}^\lambda) \in |SL\text{-}CTS|$ be weakly complete (resp. complete) for all $\lambda \in \Lambda$. Then the product space $(\prod_{\lambda \in \Lambda} X_\lambda, \pi - \overline{C})$ is weakly complete (resp. complete).*

Proof. We only prove the case that all factors are complete. Let $\mathcal{F} \in (\pi - \overline{C})_\alpha$. Then for all $\lambda \in \Lambda$, $pr_\lambda(\mathcal{F}) \in C_\alpha^\lambda$. Hence, for all $\lambda \in \Lambda$ there is $x_\lambda \in X_\lambda$ such that $pr_\lambda(\mathcal{F}) \wedge [x_\lambda] \in C_\alpha^\lambda$. We define $x = (x_\lambda)_{\lambda \in \Lambda}$. Then $pr_\lambda(\mathcal{F} \wedge [x]) = pr_\lambda(\mathcal{F}) \wedge [x_\lambda] \in C_\alpha^\lambda$ for all $\lambda \in \Lambda$ and hence $\mathcal{F} \wedge [x] \in (\pi - \overline{C})_\alpha$. □

For $(X_\lambda, C_\lambda) \in |SL\text{-}CHY|$ for all $\lambda \in \Lambda$, the *product space* $(\prod_{\lambda\in\Lambda} X_\lambda, \pi - C)$ is defined as follows. For $\mathcal{F} \in \mathcal{F}_L^s(\prod_{\lambda\in\Lambda} X_\lambda)$ we have $\pi - C(\mathcal{F}) = \bigwedge_{\lambda\in\Lambda} C_\lambda(\mathcal{F})$, see [7].

Lemma 4.9. [7] *Let $(X_\lambda, C_\lambda) \in |SL\text{-}CHY|$ be complete for all $\lambda \in \Lambda$. Then the product space $(\prod_{\lambda\in\Lambda} X_\lambda, \pi - C)$ is complete.*

Lemma 4.10. [13] *(i) Let $(X_1, C_1), (X_2, C_2) \in |SL\text{-}CHY|$ be weakly complete. Then the product space $(X_1 \times X_2, C_1 \times C_2)$ is weakly complete.*
(ii) Let L be completely distributive and let $(X_\lambda, C_\lambda) \in |SL\text{-}CHY|$ be weakly complete for all $\lambda \in \Lambda$. Then the product space $(\prod_{\lambda\in\Lambda} X_\lambda, \pi - C)$ is weakly complete.

5. Completion

In this section we consider the category $SL\text{-}CHY$. For similar definitions and constructions in the category $SL\text{-}CTS$ see [1].

A completion $((X', C'), k)$ of (X, C) is a complete stratified L-Cauchy space (X', C') together with a dense embedding $k : (X, C) \longrightarrow (X', C')$. By this we mean that for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $C(\mathcal{F}) = C'(k(\mathcal{F}))$ and that $k(X)$ is dense in $(X', \lim_{C'})$.

In the case that the lattice L has a prime bottom element, i.e. that from $\alpha \wedge \beta = \bot$ it follows that $\alpha = \bot$ or $\beta = \bot$, for a non-complete space $(X, C) \in |SL\text{-}CHY|$, a completion was constructed in [7]. This construction generalizes similar constructions in probabilistic Cauchy spaces, see [11, 12]. A related, but different construction can be found in [15]. For lattice-valued convergence tower spaces, [1] provides a completion that can be considered as a "levelwise" version of the construction in [7]. We therefore describe the construction of [7] and point to a problem in that paper that is related to the different definitions of completeness. Already in [7] it is stated, generalizing the situation for probabilistic Cauchy spaces, that the existence of a completion is equivalent to the space satisfying a so-called completion axiom.

We denote, for $(X, C) \in |SL\text{-}CHY|$,

$$\mathcal{N}_C = \{\mathcal{V} \in \mathcal{F}_L^s(X) \ : \ C(\mathcal{V}) = \top, \lim_C \mathcal{V}(x) < \top, \forall x \in X\}.$$

We consider $\mathcal{N}_C$ as the set of non-convergent Cauchy L-filters.

The completion axiom for a space $(X, C) \in |SL\text{-}CHY|$ is stated as follows [7]. *For all $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\lim_C \mathcal{F}(x) \not\geq C(\mathcal{F})$ for all $x \in X$, there is $\mathcal{G} \in \mathcal{N}_C$ such that $C(\mathcal{F} \wedge \mathcal{G}) \geq C(\mathcal{F})$.*

Hence, for a non-complete lattice-valued Cauchy space that allows a completion the set $\mathcal{N}_C$ is non-empty. Because we are interested in constructing a completion, we will assume in the sequel that $\mathcal{N}_C \neq \emptyset$.

Let $(X, C) \in |SL\text{-}CHY|$. We define $C_\top = \{\mathcal{F} \in \mathcal{F}_L^s(X) \ : \ C(\mathcal{F}) = \top\}$. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ we define the relation $\mathcal{F} \sim \mathcal{G}$ if $C(\mathcal{F} \wedge \mathcal{G}) = \top$. It is then not difficult to show that $\sim$ is an equivalence relation on $C_\top$. For $\mathcal{G} \in C_\top$ we denote the equivalence class by $\langle \mathcal{G} \rangle = \{\mathcal{F} \in C_\top \ : \ \mathcal{F} \sim \mathcal{G}\}$. With this, we define $X^* = X \cup \{\langle \mathcal{F} \rangle \ : \ \mathcal{F} \in \mathcal{N}_C\}$ and the embedding $j : X \longrightarrow X^*$, $j(x) = x$. For

$\Phi \in \mathcal{F}_L^s(X^*)$ we define $M_\Phi \subseteq L$ by

$$\alpha \in M_\Phi \iff \Phi \geq j(\mathcal{F}) \wedge \bigwedge_{i=1}^{n} [\langle \mathcal{F}_i \rangle]$$

with $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n \in \mathcal{N}_C$, $C(\mathcal{F} \wedge \bigwedge_{i=1}^n \mathcal{F}_i) \geq \alpha$. With this we define $C^*(\Phi) = \bigvee M_\Phi$. It is shown in [7] that $(X^*, C^*) \in |SL\text{-}CHY|$, $C^*(j(\mathcal{F})) = C(\mathcal{F})$ for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and that $j(X)$ is dense in $(X^*, \lim_{C^*})$. Further, the following result ([7], Lemma 7.6) is proved (but stated incorrectly).

Lemma 5.1. [7] *Let $\perp \in L$ be prime and let $(X, C) \in |SL\text{-}CHY|$ satisfy the completion axiom. If $\alpha \in M_\Phi$, then there is $x^* = x^*_\alpha \in X^*$ such that $\alpha \in M_{\Phi \wedge [x^*]}$.*

It is claimed in [7], that (X^*, C^*) is complete, but the proof of the lemma above in [7] shows that the point x^* may vary with α. So we only have the following result.

Lemma 5.2. *Let $\perp \in L$ be prime and let $(X, C) \in |SL\text{-}CHY|$ satisfy the completion axiom. Then (X^*, C^*) is weakly complete.*

Proof. If $\alpha \in M_\Phi$, then $\alpha \leq \bigvee M_{\Phi \wedge [x^*_\alpha]} = C^*(\Phi \wedge [x^*_\alpha]) \leq \bigvee_{x^* \in X^*} C^*(\Phi \wedge [x^*])$. Hence $C^*_\Phi = \bigvee M_\Phi \leq \bigvee_{x^* \in X^*} C^*(\Phi \wedge [x^*])$. □

In the sequel, we shall construct a completion of a non-complete space $(X, C) \in |SL\text{-}CHY|$ (without the assumption of a prime bottom element of the lattice) and show the following result.

Theorem 5.3. *Let $(X, C) \in |SL\text{-}CHY|$ be non-complete. Then the existence of a completion is equivalent to (X, C) satisfying the completion axiom.*

Proof. It was shown in [7], Lemma 7.7, that if a completion exists, the completion axiom must be satisfied. We prove the converse and assume that (X, C) satisfies the completion axiom. We define again the set $X^* = X \cup \{\langle \mathcal{F} \rangle \ : \ \mathcal{F} \in \mathcal{N}_C\}$ and, following a classical idea of Reed [14] (cf. also [9]), we consider a fixed map $\sigma : X^* \longrightarrow \mathcal{F}_L^s(X)$ with the properties $\sigma(x) = x$ for $x \in X$ and $\sigma(\langle \mathcal{F} \rangle) \in \langle \mathcal{F} \rangle$ for $\mathcal{F} \in \mathcal{N}_C$. For $a \in L^X$ we define the L-set a^σ on X^* by $a^\sigma(x) = a(x)$ and $a^\sigma(\langle \mathcal{F} \rangle) = \sigma(\langle \mathcal{F} \rangle)(a)$. The following properties are then easily established, where $a, b \in L^X$ and $\alpha \in L$: $(a \wedge b)^\sigma = a^\sigma \wedge b^\sigma$, $a \leq b$ implies $a^\sigma \leq b^\sigma$, $\alpha_{X^*} \leq (\alpha_X)^\sigma$ and $\perp_{X^*} = (\perp_X)^\sigma$ and $\top_{X^*} = (\top_X)^\sigma$. For $\Phi \in \mathcal{F}_L^s(X^*)$ we define $\Phi^\sigma : L^X \longrightarrow L$ by $\Phi^\sigma(a) = \Phi(a^\sigma)$. Then $\Phi^\sigma \in \mathcal{F}_L^s(X)$ and the following properties for $\Phi, \Psi \in \mathcal{F}_L^s(X^*)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ can be shown. $\Phi \leq \Psi$ implies $\Phi^\sigma \leq \Psi^\sigma$, $(\Phi \wedge \Psi)^\sigma = \Phi^\sigma \wedge \Psi^\sigma$, if $\Phi \vee \Psi$ exists, then $\Phi^\sigma \vee \Psi^\sigma$ exists, $(j(\mathcal{F}))^\sigma = \mathcal{F}$ and $([\langle \mathcal{F} \rangle])^\sigma = \sigma(\langle \mathcal{F} \rangle)$ and $[x]^\sigma = [x]$. We define now, for $\Phi \in \mathcal{F}_L^s(X^*)$, $C^\sigma(\Phi) = C(\Phi^\sigma)$. It is then not difficult to see that $(X^*, C^\sigma) \in |SL\text{-}CHY|$ and that $C^\sigma(j(\mathcal{F})) = C(\mathcal{F})$ for all $\mathcal{F} \in \mathcal{F}_L^s(X)$. We show that (X^*, C^σ) is complete. Let $\Phi \in \mathcal{F}_L^s(X^*)$. If $\lim_C \Phi^\sigma(x) \not\geq C(\Phi^\sigma)$ for all $x \in X$, then by the completion axiom, there is $\mathcal{V} \in \mathcal{N}_C$ such that $C(\Phi^\sigma \wedge \mathcal{V}) \geq C(\Phi^\sigma)$. From $\sigma(\langle \mathcal{V} \rangle) \in \langle \mathcal{V} \rangle$ we conclude $C(\sigma(\langle \mathcal{V} \rangle) \wedge \mathcal{V}) = \top$ and hence $C(\Phi^\sigma \wedge \sigma(\langle \mathcal{V} \rangle)) \geq C(\sigma(\langle \mathcal{V} \rangle) \wedge \mathcal{V}) \wedge C(\Phi^\sigma \wedge \mathcal{V}) \geq C(\Phi^\sigma)$. Therefore $C((\Phi \wedge [\langle \mathcal{V} \rangle])^\sigma) \geq C(\Phi^\sigma)$ which means $C^\sigma(\Phi \wedge [\langle \mathcal{V} \rangle]) \geq C^\sigma(\Phi)$. If there is $x \in X$ such that $\lim_C \Phi^\sigma(x) \geq C(\Phi^\sigma)$, then as $([x])^\sigma = \sigma([x]) = [x]$ we conclude $C^*(\Phi \wedge [x]) = C(\Phi^\sigma \wedge [x]^\sigma) = C(\Phi^\sigma \wedge [x]) \geq C(\Phi^\sigma) = C^\sigma(\Phi)$. Furthermore, $j(X) = X$ is dense in (X^*, C^σ): If $x^* = x$, then $[x]_{j(X)} = [x] \in \mathcal{F}_L^s(X)$ and $\lim_{C^\sigma}[x](x) = C^\sigma([x]) = C([x]^\sigma) = C([x]) = \top$. If $x^* = \langle \mathcal{V} \rangle$ with $\mathcal{V} \in \mathcal{N}_C$ then the trace $j(\mathcal{V})_{j(X)}$ exists and $\lim_{C^\sigma} j(\mathcal{V})(\langle \mathcal{V} \rangle) = C^\sigma(j(\mathcal{V}) \wedge [\langle \mathcal{V} \rangle]) = C(\mathcal{V} \wedge \sigma(\langle \mathcal{V} \rangle)) = \top$. Hence $((X^*, C^\sigma), j)$ is a completion of (X, C). □

It is not difficult to translate the construction of (X^*, C^σ) to the category *SL-CTS*. We leave this to the interested reader.

We finally show that the completion constructed in the previous lemma preserves the T2-property.

Lemma 5.4. *Let $(X, C) \in |SL\text{-}CHY|$ satisfy the completion axiom. If (X, C) is a T2-space, then also (X^*, C^σ) is a T2-space.*

Proof. It suffices to show that $\lim_{C^\sigma}[x^*](y^*)$ implies $x^* = y^*$, see [7]. We distinguish four cases. Let first $x^* = x, y^* = y$ with $x, y \in X$. Then $C^\sigma([x^*] \wedge [y^*]) = C([x] \wedge [y])$ and hence, by the T2-property, $x^* = x = y = y^*$. If $x^* = \langle \mathcal{V} \rangle$ and $y^* = \langle \mathcal{U} \rangle$ with $\mathcal{V}, \mathcal{U} \in \mathcal{N}_C$, then $C([\langle \mathcal{V} \rangle]^\sigma \wedge [\langle \mathcal{U} \rangle]^\sigma) = C(\sigma(\mathcal{V}) \wedge \sigma(\mathcal{V})) = \top$ and hence, because $\mathcal{V} \sim \sigma(\mathcal{V}) \sim \sigma(\mathcal{U}) \sim \mathcal{U}$, we conclude $\mathcal{V} \sim \mathcal{U}$ and therefore $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$. For the case $x^* = \langle \mathcal{V} \rangle$ and $y^* = y$ with $\mathcal{V} \in \mathcal{N}_C$ and $y \in X$ we have $\top = C(\sigma(\mathcal{V}) \wedge [y]) = \lim_C \sigma(\mathcal{V})(y)$. As $\sigma(\mathcal{V}) \in \langle \mathcal{V} \rangle$ we have $C(\sigma(\mathcal{V}) \wedge \mathcal{V}) = \top$ and hence $C(\mathcal{V} \wedge \sigma(\mathcal{V}) \wedge [y]) = C(\mathcal{V} \wedge \sigma(\mathcal{V})) \wedge C(\sigma(\mathcal{V}) \wedge [y]) = \top$. This implies $\top = \lim_C \mathcal{V} \wedge \sigma(\mathcal{V})(y) \leq \lim_C \mathcal{V}(y)$, a contradiction to $\mathcal{V} \in \mathcal{N}_C$. Similarly, the case $x^* = x$ and $y^* = \langle \mathcal{V} \rangle$ with $x \in X$ and $\mathcal{V} \in \mathcal{N}_C$ leads to a contradiction and, therefore, cannot occur. □

6. **Conclusions**

In this paper, we had a more careful look into the concept of completeness for lattice-valued Cauchy spaces. We pointed out that there are two slightly different definitions for the category of lattice-valued Cauchy tower spaces [1]. These definitions coincide in the left-continuous case. We gave two examples, one with a complete Boolean algebra and the other with the unit interval as the valued lattice, which show that the left-continuity is necessary here. For the category of lattice-valued Cauchy spaces as defined in [7], we also studied two different definitions of completeness. It turns out that a construction in the literature yields, for a non-complete space, only a weakly complete space as "completion". In order to confirm a previous result about the existence of a completion, a different completion was constructed. This also allows to avoid the assumption of a prime bottom element of the lattice. Up to now it is an open question to give a necessary and sufficient condition for the existence of a weak completion for a non-complete lattice-valued Cauchy space.

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Gunther Jäger, School of Mechanical Engineering, University of Applied Sciences Stralsund, 18435 Stralsund, Germany

E-mail address: gunther.jaeger@fh-stralsund.de