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FIXED FUZZY POINTS OF GENERALIZED GERAGHTY TYPE FUZZY MAPPINGS ON COMPLETE METRIC SPACES

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Abstract. Generalized Geraghty type fuzzy mappings on complete metric spaces are introduced and a fixed point theorem that generalizes some recent comparable results for fuzzy mappings in contemporary literature is obtained. Example is provided to show the validity of obtained results over comparable classical results for fuzzy mappings in fixed point theory. As an application, existence of coincidence fuzzy points and common fixed fuzzy points for hybrid pair of single valued self mapping and a fuzzy mapping is also established.

1. Introduction

The fixed point theory for nonlinear mappings is an important subject of nonlinear functional analysis and is widely applied to nonlinear integral and differential equations. Existence of a fixed point of a self mapping on a complete metric space X depends upon the behavior of a sequence generated through Picard iterative process. If an iterative sequence thus obtained is a Cauchy sequence then an appropriate contractive condition gives the existence of fixed point of a mapping (see [8, 21]). So the choice of a contractive condition on a mapping is a subject of great interest in metric fixed point theory. For recent development in this area and its contribution from application point of view in various disciplines, we refer to [1, 4, 5, 20]) and references therein. Banach contraction principle [3] gives appropriate and simple conditions to establish the existence and uniqueness of a solution of an operator equation $Tx = x$. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There are many generalizations of Banach contraction principle in the literature. See for example, [1, 6, 9, 11, 17, 22, 23, 28, 30, 31]. These generalization were made either by using the contractive condition or by imposing some additional conditions on an ambient space. Among these results Geraghty's contractive condition [11] stirred a lot of activity (see for details [2, 7, 16, 23]). Extensions of Geraghty's result for multivalued mappings have also been obtained in different directions [12, 13, 26] which in turn generalize Nadler's theorem [19], a multivalued version of Banach contraction principle.

On the other hand, the evolution of fuzzy mathematics commenced with an introduction of the notion of fuzzy sets by Zadeh [29] in 1965, as a new way to

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represent vagueness in every day life. Heilpern [15] introduced a concept of fuzzy mappings on a metric space and proved a fixed point theorem for fuzzy contraction mappings as a generalization of Nadler's theorem [19]. For more results on fuzzy mappings we refer to [10, 25, 27].

The aim of this paper is to introduce the generalized Geraghty type fuzzy mapping and to prove some fixed point results in complete metric spaces. As an application, coincidence fuzzy point and common fixed fuzzy point of hybrid pair of single valued self mapping and a fuzzy mapping are obtained. We provide an example to support the result.

Let us recall some basic definitions and known results needed in the sequel.

Let X be a space of points with generic elements of X denoted by x and $I = [0, 1]$. A fuzzy set A in X is a membership function $A : X \to I$ such that each element in X is associated with a real number $A(x)$ in the interval I. Let (X, d) be a metric space and A be a fuzzy set in X. If $\alpha \in (0, 1]$, then $\alpha-$ level set of A, denoted by A_α is defined as

$$A_\alpha = \{x : A(x) \geq \alpha\}.$$

For $\alpha = 0$, we have

$$A_0 = \overline{\{x : A(x) > 0\}},$$

where $\overline{B}$ denotes the closure of the non-fuzzy set B. A fuzzy set A in a metric linear space X is said to be an approximate quantity if and only if for each $\alpha \in [0, 1]$, A_α is compact, convex subset of X and

$$\sup_{x \in X} A(x) = 1. \tag{1}$$

Let $W(X)$ be the family of all approximate quantities and the fuzzy set $A \in W_\alpha(X)$ whenever A_α is nonempty, convex and compact in metric linear space X. A fuzzy set A is said to be more accurate than fuzzy set B, denoted by $A \subset B$ (say B includes A) if and only if $A(x) \leq B(x)$ for each x in X, where $A(x)$ and $B(x)$ denote the membership function of A and B respectively. It is easy to see that if $0 < \alpha \leq \beta \leq 1$, then $A_\beta \subseteq A_\alpha$.

Corresponding to each $\alpha \in [0, 1]$ and $x \in X$, the fuzzy point x_α of X is the fuzzy set $x_\alpha : X \to [0, 1]$ given by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

For $\alpha = 1$, we have

$$x_1(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} = \{x\}$$

Let I^X be the collection of all fuzzy subsets of X, then $W(X)$ a subset of I^X. For $A, B \in W(X)$ and $\alpha \in [0, 1]$, define

$$\begin{aligned} p_\alpha(A, B) &= \inf\{d(x, y), x \in A_\alpha, y \in B_\alpha\}, \\ D_\alpha(A, B) &= \max\{\sup_{x \in A_\alpha} d(x, B_\alpha), \sup_{y \in B_\alpha} d(y, A_\alpha)\}, \\ D(A, B) &= \sup_\alpha D_\alpha(A, B). \end{aligned}$$

Note that p_α is nondecreasing function of α and D a metric on $W(X)$. For $\alpha \in [0,1]$. Let (X,d) be a metric space and Y an arbitrary set. A mapping $F : Y \to W_\alpha(X)$ is called a fuzzy mapping, that is, $Fy \in W_\alpha(X)$ for each y in Y. Thus if we characterize a fuzzy set Fy in a metric space X by a membership function Fy , then $Fy(x)$ is the grade of membership of x in Fy. Therefore a fuzzy mapping F is a fuzzy subset of $Y \times X$ with membership function $Fy(x)$. In a more general sense than that given in [15], a mapping $F : X \to I^X$ is a fuzzy mapping over X ([24]) and $(F(x)x)$ is the fixed degree of x in $F(x)$.

Definition 1.1. [10] A fuzzy point x_α in X is called a fixed fuzzy point of fuzzy mapping F if $x_\alpha \subset Fx$, that is, $(Fx)x \geq \alpha$ or $x \in (Fx)_\alpha$. That is, the fixed degree of x in Fx is at least α. If $\{x\} \subset Fx$, then x is a fixed point of a fuzzy mapping F.

Definition 1.2. Let $F : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self mapping on X. A fuzzy point x_α in X is called:

- **(a):** coincidence fuzzy point of hybrid pair $\{F, g\}$ if $(gx)_\alpha \subset Fx$, that is $(Fx)gx \geq \alpha$ or $gx \in (Fx)_\alpha$. That is, the fixed degree of gx in Fx is at least α.
- **(b):** common fixed fuzzy point of the hybrid pair $\{F, g\}$ if $x_\alpha = (gx)_\alpha \subset Fx$, that is $x = gx \in (Fx)_\alpha$ (the fixed degree of x and gx in Fx is the same and is at least α).

We denote $C_\alpha(F, g)$ and $F_\alpha(F, g)$ by the set of all coincidence fuzzy point and set of all common fixed fuzzy point of the hybrid pair $\{F, g\}$, respectively.

Definition 1.3. Let $F : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self mapping on X, then

- **(c):** the hybrid pair $\{F, g\}$ is called $w-$*fuzzy compatible* if
$$g(Fx)_\alpha \subseteq (Fgx)_\alpha$$
whenever $x \in C_\alpha(F, g)$.
- **(d):** mapping g is called $F-$*fuzzy weakly commuting* at some point $x \in X$ if $g^2(x) \in (Fgx)_\alpha$.

Lemma 1.4. [14] *Let X be a nonempty set and $g : X \longrightarrow X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \longrightarrow X$ is one to one.*

Lemma 1.5. (Heilpern [15]) *Let (X,d) be a metric space, $x, y \in X$ and $A, B \in W(X)$:*

1. *if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$;*
2. *$p_\alpha(x, A) \leq d(x, y)+ p_\alpha(y, A)$;*
3. *if $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$.*

Theorem 1.6. [10] *Let (X,d) be a complete metric space, F a fuzzy mapping from X to $W_\alpha(X)$, where $\alpha \in (0,1)$. If there exists $q \in (0,1)$ such that*
$$D_\alpha(Fx, Fy) \leq qd(x, y),$$
holds for each $x, y \in X$. Then there exists $x \in X$ such that x_α is a fixed fuzzy point

Lemma 1.7. (Lee and Cho [18]) *Let (X, d) be a complete metric space and F a fuzzy mapping from X into $W(X)$ and $x_0 \in X$. Then there exists a $x_1 \in X$ such that $\{x_1\} \subset Fx_0$.*

Let $S = \{\psi : [0, \infty) \to [0, 1)$: $t_n \to 0$ whenever $\psi(t_n) \to 1\}$. If

$$\psi_1(t) = \begin{cases} e^{-3t} \text{ if } t > 0 \\ 0, \text{ otherwise} \end{cases},$$

$$\psi_2(t) = \frac{1}{t+1}$$

then $\psi_1, \psi_2 \in S$.

Geraghty proved the following result:

Theorem 1.8. *Let (X, d) be a complete metric space and $f : X \to X$. If there exists $\psi \in S$ such that*

$$d(fx, fy) \leq \psi(d(x, y))d(x, y) \tag{3}$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$, the Picard sequence $\{f^n x\}$ converges to z whenever $n \to \infty$.

Now we introduce generalized Geraghty type fuzzy mapping.

Definition 1.9. Let (X, d) be a complete metric space. A fuzzy mapping $F : X \to W_\alpha(X)$ is said to be generalized Geraghty type fuzzy mapping if there exists $\psi \in S$ such that

$$D_\alpha(Fx, Fy) \leq \psi(M_\alpha(x, y)) \max\{d(x, y), p_\alpha(x, Fx), p_\alpha(y, Fy)\} \tag{4}$$

for all $x, y \in X$, where

$$M_\alpha(x, y) = \max\{d(x, y), p_\alpha(x, Fx), p_\alpha(y, Fy), \frac{p_\alpha(x, Fy) + p_\alpha(y, Fx)}{2}\}. \tag{5}$$

2. **Main Results**

In this section we prove fixed fuzzy point theorem for generalized Geraghty type fuzzy mappings on a complete metric space.

Theorem 2.1. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a generalized Geraghty type fuzzy mapping. Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$.*

Proof. Let u_0 be an arbitrary element of X. As $(Fu_0)_\alpha$ is nonempty and compact so there exists $u_1 \in (Fu_0)_\alpha$ such that $d(u_0, u_1) = p_\alpha(u_0, Fu_1)$. If $u_0 = u_1$, then $u_0 = u_1 \in (Fu_0)_\alpha$ and the proof is finished. Suppose that $u_0 \neq u_1$. Since $(Fu_1)_\alpha$ is nonempty and compact, there exists $u_2 \in (Fu_1)_\alpha$ such that

$$d(u_1, u_2) = p_\alpha(u_1, Fu_1) \leq D_\alpha(Fu_0, Fu_1).$$

If $u_1 = u_2$, then $u_1 = u_2 \in (Fu_1)_\alpha$ and the proof is finished. Suppose $u_1 \neq u_2$, then by given assumption we have

$$\begin{aligned}
& d(u_1, u_2) \\
\leq\ & D_\alpha(Fu_0, Fu_1) \\
\leq\ & \psi(\max\{d(u_0, u_1), p_\alpha(u_0, Fu_0), p_\alpha(u_1, Fu_1), \\
& \frac{p_\alpha(u_0, Fu_1) + p_\alpha(u_1, Fu_0)}{2}\}) \max\{d(u_0, u_1), p_\alpha(u_0, Fu_0), p_\alpha(u_1, Fu_1)\} \\
\leq\ & \psi(\max\{d(u_0, u_1), d(u_0, u_1), d(u_1, u_2), \\
& \frac{d(u_0, u_2) + d(u_1, u_1)}{2}\}) \max\{d(u_0, u_1), d(u_0, u_1), d(u_1, u_2)\} \\
\leq\ & \psi(\max\{d(u_0, u_1), d(u_1, u_2), \\
& \frac{d(u_0, u_1) + d(u_1, u_2)}{2}\}) \max\{d(u_0, u_1), d(u_1, u_2)\}.
\end{aligned}$$

Note that

$$d(u_1, u_2) < d(u_0, u_1).$$

If not, then above inequality gives

$$d(u_1, u_2) \leq \psi(d(u_1, u_2)d(u_1, u_2)$$

and hence $d(u_1, u_2) < d(u_1, u_2)$, because $\psi \in S$, a contradiction. Hence

$$d(u_1, u_2) < d(u_0, u_1). \tag{6}$$

Continuing this process, we construct a sequence $\{u_n\}$ in X such that $u_n \in (Fu_{n-1})_\alpha$, and $u_{n+1} \in (Fu_n)_\alpha$ with

$$d(u_n, u_{n+1}) = p_\alpha(u_n, Fu_n) \leq D_\alpha(Fu_{n-1}, Fu_n).$$

If $u_n = u_{n+1}$ then $u_n = u_{n+1} \in (Fu_n)_\alpha$ and the proof is finished. Suppose $u_n \neq u_{n+1}$, then by given assumption we have

$$\begin{aligned}
& d(u_n, u_{n+1}) \\
\leq\ & D_\alpha(Fu_{n-1}, Fu_n) \\
\leq\ & \psi(\max\{d(u_{n-1}, u_n), p_\alpha(u_{n-1}, Fu_{n-1}), p_\alpha(u_n, Fu_n), \\
& \frac{p_\alpha(u_{n-1}, Fu_n) + p_\alpha(u_n, Fu_{n-1})}{2}\}) \max\{d(u_{n-1}, u_n), \\
& p_\alpha(u_{n-1}, Fu_{n-1}), p_\alpha(u_n, Fu_n)\} \\
\leq\ & \psi(\max\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_2), \\
& \frac{d(u_{n-1}, u_{n+1}) + d(u_n, u_n)}{2}\}) \max\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1})\} \\
\leq\ & \psi(\max\{d(u_{n-1}, u_n), d(u_n, u_{n+1}), \\
& \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2}\}) \max\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\}.
\end{aligned}$$

We claim that

$$d(u_n, u_{n+1}) < d(u_{n-1}, u_n).$$

If not, then by above inequality we obtain that

$$d(u_n, u_{n+1}) \leq \psi(d(u_n, u_{n+1})d(u_n, u_{n+1}).$$

Since $\psi \in S$, therefore $d(u_n, u_{n+1}) < d(u_n, u_{n+1})$ gives a contradiction. Hence

$$d(u_n, u_{n+1}) < \psi(d(u_{n-1}, u_n))d(u_{n-1}, u_n) \tag{7}$$

implies that

$$d(u_n, u_{n+1}) < d(u_{n-1}, u_n). \tag{8}$$

This shows that $\{d(u_n, u_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Hence

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = \lambda \geq 0$$

for some $\lambda \in [0, +\infty)$. Next we will prove that $\lambda = 0$. On contrary let $\lambda > 0$. From (2), we obtain

$$\frac{d(u_n, u_{n+1})}{d(u_{n-1}, u_n)} \leq \psi(d(u_{n-1}, u_n)) < 1. \tag{9}$$

On taking limit n tends to ∞, we get

$$1 = \frac{\lambda}{\lambda} = \lim_{n\to\infty} \frac{d(u_n, u_{n+1})}{d(u_{n-1}, u_n)} \leq \lim_{n\to\infty} \psi(d(u_{n-1}, u_n)) < 1,$$

which implies that $\lim_{n\to\infty} \psi(d(u_{n-1}, u_n)) = 1$. As $\psi \in S$, so $\lambda = \lim_{n\to\infty} d(u_{n-1}, u_n) = 0$. This contradicts our assumption $\lambda > 0$. Therefore, $\lim_{n\to\infty} d(u_{n-1}, u_n) = 0$.

Now we prove that $\lim_{n,m\to\infty} d(u_n, u_m) = 0$. Using triangular inequality and assumption (2), we obtain

$$\begin{aligned}
d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_m) \\
&\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{m+1}) + d(u_{m+1}, u_m) \\
&= d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + d(u_{n+1}, u_{m+1}) \\
&\leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + p_\alpha(u_{n+1}, Fu_m) \\
&\leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + D_\alpha(Fu_n, Fu_m) \\
&\leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + \psi(M_\alpha(u_n, u_m)) \\
&\quad \max\{d(u_n, u_m), p_\alpha(u_n, Fu_n), p_\alpha(u_m, Fu_m)\}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + \psi(M_\alpha(u_n, u_m)) \\
&\quad \max\{d(u_n, u_m), d(u_n, u_{n+1}), d(u_m, u_{m+1})\}.
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
M_\alpha(u_n, u_m) &= \max\{d(u_n, u_m), p_\alpha(u_n, Fu_{n-1}), p_\alpha(u_m, Fu_m), \\
&\quad \frac{p_\alpha(u_n, Fu_m) + p_\alpha(u_m, Fu_n)}{2}\}.
\end{aligned}$$

From (1), we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m) + \psi(M_\alpha(u_n, u_m))d(u_n, u_m) \\ &\quad + \max\{d(u_n, u_{n+1}), d(u_m, u_{m+1})\} \\ &\leq 3\max\{d(u_n, u_{n+1}), d(u_m, u_{m+1})\} + \psi(M_\alpha(u_n, u_m))d(u_n, u_m), \end{aligned}$$

which further gives

$$[1 - \psi(M_\alpha(u_n, u_m))]d(u_n, u_m) \leq 3\max\{d(u_n, u_{n+1}), d(u_m, u_{m+1})\}. \tag{11}$$

On taking limit as n, m tend to ∞, we get

$$\lim_{n,m\to\infty}[1 - \psi(M_\alpha(u_n, u_m))] \lim_{n,m\to\infty} d(u_n, u_m) = 0. \tag{12}$$

So either $\lim\limits_{n,m\to\infty} d(u_n, u_m) = 0$, or $\lim\limits_{n,m\to\infty}[1 - \psi(M_\alpha(u_n, u_m))] = 0$. If

$$\lim_{n,m\to\infty} d(u_n, u_m) = 0$$

then we are done. If

$$\lim_{n,m\to\infty}[1 - \psi(M_\alpha(u_n, u_m))] = 0,$$

then $\lim\limits_{n,m\to\infty} \psi(M_\alpha(u_n, u_m)) = 1$ and $\lim\limits_{n,m\to\infty}(M_\alpha(u_n, u_m) = 0$. That is

$$\begin{aligned} 0 &= \lim_{n,m\to\infty} M_\alpha(u_n, u_m) = \lim_{n,m\to\infty} \max\{d(u_n, u_m), p_\alpha(u_n, Fu_{n-1}), \\ &\quad p_\alpha(u_m, Fu), \frac{p_\alpha(u_n, Fu_m) + p_\alpha(u_m, Fu_n)}{2}\} = \lim_{n,m\to\infty} d(u_n, u_m). \end{aligned}$$

So we obtain that $\lim\limits_{n,m\to\infty} d(u_n, u_m) = 0$. This means that $\{u_n\}$ is a Cauchy sequence in the complete metric space (X, d), so there exists an element z in X such that $d(u_n, z) = 0$. Now it is left to show that $z_\alpha \subset Fz$ for each $\alpha \in [0, 1]$. Assume on contrary that $z_\alpha \nsubseteq Fz$, that is $z \notin (Fz)_\alpha$. In this case by Lemma 1.5, we have $p_\alpha(z, Fz) \neq 0$, that is, $p_\alpha(z, Fz) > 0$. Now consider

$$\begin{aligned} p_\alpha(z, Fz) &\leq \lim_{n\to\infty} p_\alpha(u_{n+1}, Fz) \leq \lim_{n\to\infty} D_\alpha(Fu_n, Fz) \\ &\leq \lim_{n\to\infty} \psi(\max\{d(u_n, z),\ d(u_n, u_{n+1}), p_\alpha(z, Fz), \\ &\quad \frac{p_\alpha(u_n, Fz) + p_\alpha(z, Fu_n)}{2}\}) \max\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, Fz)\} \\ &= \psi(p_\alpha(z, Fz))p_\alpha(z, Fz). \end{aligned}$$

So

$$1 \leq \psi(p_\alpha(z, Fz)) < 1, . \tag{13}$$

Hence $\psi(p_\alpha(z, Fz)) = 1$, which implies as $p_\alpha(z, Fz) = 0$, a contradiction. So we have $p_\alpha(z, Fz) = 0$ consequently by Lemma 1.5, $z_\alpha \subset Fz$. $\square$

Corollary 2.2. *Let (X,d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exists $\psi \in S$ such that*

$$D_\alpha(Fx, Fy) \leq \psi(m_\alpha(x,y))m_\alpha(x,y) \tag{14}$$

for all $x, y \in X$, where

$$m_\alpha(x,y) = \max\{d(x,y), p_\alpha(x,Fx), p_\alpha(y,Fy)\}. \tag{15}$$

Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$.

Corollary 2.3. *Let (X,d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exists $\psi \in S$ such that*

$$D_\alpha(Fx, Fy) \leq \psi(M_\alpha(x,y))d(x,y) \tag{16}$$

for all $x, y \in X$, where

$$M_\alpha(x,y) = \max\{d(x,y), p_\alpha(x,Fx), p_\alpha(y,Fy), \frac{p_\alpha(x,Fy)+p_\alpha(y,Fx)}{2}\}. \tag{17}$$

Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$.

Corollary 2.4. *Let (X,d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exists $\psi \in S$ such that*

$$D_\alpha(Fx, Fy) \leq \psi(d(x,y))d(x,y) \tag{18}$$

for all $x, y \in X$,then there exists a point $x \in X$ such that $x_\alpha \subset Fx$.

Corollary 2.4 is a generalization of Theorem 1.8 as fuzzy mappings are more general than single valued self mappings.

Example 2.5. Let $X = \{0,1,2\}$ be endowed with metric d defined as:

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(0,2) &= d(2,0) = 6, \\ d(0,1) &= d(0,1) = 10, \\ d(1,2) &= d(2,1) = 16. \end{aligned}$$

Let $\alpha \in (0, \frac{1}{3})$ and $\psi(t) = \frac{7}{8}$. Define a fuzzy mapping F from X into $W_\alpha(X)$ as:

$$(F0)(x) = \begin{cases} \alpha \text{ if } x = 0, \\ \frac{\alpha}{2} \text{ if } x = 1, \\ 0 \text{ if } x = 2, \end{cases} \quad \text{and } (F1)(x) = \begin{cases} \alpha \text{ if } x = 0 \\ 0 \text{ if } x = 1, \\ \frac{\alpha}{3} \text{ if } x = 2, \end{cases}$$

and

$$(F2)(x) = \begin{cases} \frac{\alpha}{3} \text{ if } x = 0, \\ \alpha \text{ if } x = 1, \\ \frac{\alpha}{4} \text{ if } x = 2, \end{cases} .$$

Then $(F0)_\alpha = (F1)_\alpha = \{0\}$, $(F2)_\alpha = \{1\}$. Note that for all $x, y \in \{0, 1\}$, we have $D_\alpha(Fx, Fy) = H((Fx)_\alpha, (Fy)_\alpha) = 0$. For $x = 0$ and $y = 2$, we obtain

$$D_\alpha(F1, F2) = D_\alpha(F0, F2) = H((F0)_\alpha, (F2)_\alpha) = d(0, 1) = 10$$

and

$$\begin{aligned} M_\alpha(0,2) &= \max\{d(0,2), p_\alpha(0, F0), p_\alpha(2, F2), \frac{p_\alpha(0, F2) + p_\alpha(2, F0)}{2}\} \\ &= \max\{d(0,2), d(0,0), d(2,1), \frac{d(0,1) + d(2,0)}{2}\} \\ &= \max\{6, 0, 16, 8\} = 16. \end{aligned}$$

So

$$D_\alpha(F0, F2) = 10 \leq 14 = \psi(M_\alpha(0,2)) \max\{d(0,2), p_\alpha(0, F0), p_\alpha(2, F2)\}.$$

Now for $x = 1$ and $y = 2$, we obtain

$$\begin{aligned} M_\alpha(1,2) &= \max\{d(1,2), p_\alpha(1, F1), p_\alpha(2, F2), \frac{p_\alpha(1, F2) + p_\alpha(2, F1)}{2}\} \\ &= \max\{d(1,2), d(1,0), d(2,1), \frac{d(1,1) + d(2,0)}{2}\} \\ &= \max\{16, 10, 16, 3\} = 16. \end{aligned}$$

Hence

$$D_\alpha(F1, F2) = 10 \leq 14 = \psi(M_\alpha(1,2)) \max\{d(1,2), p_\alpha(1, F1), p_\alpha(2, F2)\}.$$

Consequently

$$D_\alpha(Fx, Fy) \leq \psi(M_\alpha(x,y)) \max\{d(x,y), p_\alpha(x, Fx), p_\alpha(y, Fy)\}$$

is satisfied for all $x, y \in X$, where

$$M_\alpha(x,y) = \max\{d(x,y), p_\alpha(x, Fx), p_\alpha(y, Fy), \frac{p_\alpha(x, Fy) + p_\alpha(y, Fx)}{2}\}$$

Hence all the conditions of Theorem 2.1 are satisfied. Moreover, for $x = 0$, we have $x_\alpha \subset F(x)$ as $(F0)0 \geq \alpha$. Hence $\{0\} \subset (F0)_\alpha$. This implies that $x = 0$ is the fixed fuzzy point of fuzzy mapping F.

Remark 2.6. Note that Theorem 1.6 is not applicable to the mapping F from X into $W_\alpha(X)$ defined above. Indeed

$$D_\alpha(F0, F2) = d(1, 0) = 10$$

$d(0, 2) = 6$, and for any choice of $q \in]0, 1[$ we have

$$D_\alpha(F0, F2) \nleq q d(0, 2).$$

Theorem 1.8 also does not hold true for fuzzy mapping which is the Corollary 2.4 of Theorem 2.1. Hence Theorem 3.1 is a proper generalization of results given in [10, 11, 15, 19].

3. **Application**

Let $F : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self mapping on X, then the pair $\{F, g\}$ is said to be a generalized Geraghty type fuzzy hybrid pair if there exists $\psi \in S$ such that

$$D_\alpha(Fx, Fy) \leq \psi(M^g_\alpha(x, y)) \max\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy)\} \tag{19}$$

for all $x, y \in X$, where

$$M^g_\alpha(x, y) = \max\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy), \frac{p_\alpha(gx, Fy) + p_\alpha(gy, Fx)}{2}\}. \tag{20}$$

Now as an application of Theorem 2.1, we obtain coincidence fuzzy points and common fixed fuzzy points of the hybrid pair $\{F, g\}$.

Theorem 3.1. *Let (X, d) be a complete metric space and $\{F, g\}$ a generalized Geraghty type fuzzy hybrid pair. Then $C_\alpha(F, g) \neq \phi$ provided that $(F(X))_\alpha \subseteq g(X)$ for each α. Moreover F and g have common fixed fuzzy point if any of the following conditions holds:*

- **(f):** F and g are $w-$ *fuzzy compatible*, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $x \in C_\alpha(F, g)$, $u \in X$ and g is continuous at u.
- **(g):** g is $F-$ *fuzzy weakly commuting* for some $x \in C_\alpha(g, F)$, and $g^2 x = gx$.
- **(h):** g is continuous at x for some $x \in C_\alpha(g, F)$ and for some $u \in X$, such that $\lim_{n\to\infty} g^n u = x$.

Proof. By Lemma 1.4, there exists $E \subseteq X$ such that $g : E \longrightarrow X$ is one to one and $g(E) = g(X)$. Define a mapping $\mathcal{A} : g(E) \longrightarrow W_\alpha(X)$ by

$$\mathcal{A}gx = Fx \text{ for all } gx \in g(E). \tag{21}$$

As g is one to one on E, so $\mathcal{A}$ is well defined. Therefore (3) becomes

$$D_\alpha(\mathcal{A}gx, \mathcal{A}gy) \leq \psi(M^g_\alpha(x, y)) \max\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy)\} \tag{22}$$

for all $gx, gy \in g(E)$. Hence $\mathcal{A}$ satisfies (2) and all the conditions of Theorem 2.1. Using Theorem 2.1 with mapping $\mathcal{A}$, it follows that $\mathcal{A}$ has fixed fuzzy point $u \in g(E)$. Now it is left to prove that F and g have coincidence fuzzy point. Since $\mathcal{A}$ has fixed fuzzy point $u_\alpha \subset \mathcal{A}u$, therefore $u \in (\mathcal{A}u)_\alpha$. As $(F(X))_\alpha \subseteq g(X)$, there exists $u_1 \in X$ such that $gu_1 = u$, thus it follows that

$$gu_1 \in (\mathcal{A}gu_1)_\alpha = (Fu_1)_\alpha.$$

This implies that $u_1 \in X$ is coincidence fuzzy point of F and g. Hence $C_\alpha(F, g) \neq \phi$. Suppose now that (a) holds. Then for some $x_\alpha \in C_\alpha(F, g)$, we have $\lim_{n\to\infty} g^n x = u$, where $u \in X$. Thus $(g^{n-1}x, u) \in \nabla$. Since g is continuous at u, so we have that u is a fixed points of g. As F and g are $w-$ *fuzzy compatible* and $(g^n x)_\alpha \in C_\alpha(F, g)$ for all $n \geq 1$. That is, $g^n x \in F(g^{n-1}x)_\alpha$ for all $n \geq 1$. Now we show that $gu \in (Fu)_\alpha$. Assume on contrary that $gu \notin (Fu)_\alpha$, then by Lemma 1.5 $p_\alpha(gu, Fu) > 0$

$$\begin{aligned} p_\alpha(gu, Fu) &\leq p_\alpha(gu, g^n x) + p_\alpha(g^n x, Fu) \\ &\leq p_\alpha(gu, g^n x) + D_\alpha(F(g^{n-1}x), Fu) \\ &\leq p_\alpha(gu, g^n x) + \psi(\max\{d(gg^{n-1}x, gu), p_\alpha(g^n x, Fg^{n-1}x), p_\alpha(gu, Fu), \\ &\quad \frac{p_\alpha(gu, Fg^{n-1}x) + p_\alpha(g^n x, Fu)}{2}\}) \\ &\quad \max\{d(gg^{n-1}x, gu), p_\alpha(g^n x, Fg^{n-1}x), p_\alpha(gu, Fu)\}. \end{aligned}$$

On taking limit as $n \to \infty$, we get

$$p_\alpha(gu, Fu) \leq \psi(p_\alpha(gu, Fu)) p_\alpha(gu, Fu). \tag{23}$$

Since $p_\alpha(gu, Fu) > 0$, so $\psi(p_\alpha(gu, Fu)) = 1$. Consequently we get $p_\alpha(gu, Fu) = 0$, a contradiction. Hence $u = gu \in (Fu)_\alpha$. That is, u_α is common fixed fuzzy point of F and g. Suppose now that (b) holds. If for some $x_\alpha \in C_\alpha(F, g)$, g is $F-$*fuzzy weakly commuting* and $g^2 x = gx$ then $gx = g^2 x \in (Fgx)_\alpha$. Hence $(gx)_\alpha$ is a common fixed fuzzy point of F and g. Suppose now that (c) holds and assume that for some $x_\alpha \in C_\alpha(F, g)$ and for some $u \in X$, $\lim\limits_{n\to\infty} g^n u = x$ and $\lim\limits_{n\to\infty} g^n v = y$. By continuity of g at x and y, we get $x = gx \in (Fx)_\alpha$.The result follows. □

Example 3.2. Let $X = [0, 1]$ be endowed with usual metric. Let $\alpha \in (0, \frac{1}{3})$ and $\psi(t) = e^{-2t}$. Define a fuzzy mapping F from X into $W_\alpha(X)$ as:

$$(F0)(x) = \begin{cases} 1 \text{ if } x = 0 \\ \alpha \text{ if } x \in (0, \frac{1}{3}] \\ \frac{\alpha}{3} \text{ if } x \in (\frac{1}{3}, 1] \end{cases} \quad \text{and } (F1)(x) = \begin{cases} 1 \text{ if } x = 0 \\ 3\alpha \text{ if } x \in (0, \frac{1}{3}] \\ \frac{\alpha}{3} \text{ if } x \in (\frac{1}{3}, 1] \end{cases}$$

and for $z \in (0, 1)$,

$$(Fz)(x) = \begin{cases} 1 \text{ if } x = 0 \\ \alpha \text{ if } x \in (0, \frac{1}{3}] \\ 0 \text{ if } x \in (\frac{1}{3}, 1] \end{cases}.$$

Define a self maping $g : X \to X$ by $g(x) = x^2$. Then

$$\begin{aligned} (F0)_1 &= (F1)_1 = (Fz)_1 = \{0\}, \\ (F0)_\alpha &= (F1)_\alpha = (Fz)_\alpha = [0, \frac{1}{3}], \\ (F0)_{\frac{\alpha}{3}} &= (F1)_{\frac{\alpha}{3}} = [0, 1] \text{ and } (Fz)_{\frac{\alpha}{3}} = [0, \frac{1}{3}]. \end{aligned}$$

Note that for all $x, y \in X$, we have

$$\begin{aligned} D_1(Fx, Fy) &= H((Fx)_1, (Fy)_1) = 0, \\ D_\alpha(Fx, Fy) &= H((Fx)_\alpha, (Fy)_\alpha) = 0. \end{aligned}$$

So, for all $x, y \in X$, we have $D_\alpha(Fx, Fy) = 0$. Hence for all $x, y \in X$,

$$D_\alpha(Fx, Fy) \leq \psi(M_\alpha^g(x, y)) \max\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy)\}$$

hold true, where

$$M_\alpha^g(x, y) = \max\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy), \frac{p_\alpha(gx, Fy) + p_\alpha(gy, Fx)}{2}\}.$$

Hence all the conditions of Theorem 3.1 are satisfied. Moreover for each $x \in [0, \frac{1}{3}]$, we have $x_\alpha \subset F(x)$ and $(gx)_\alpha \subset F(x)$. For $\alpha = 1$, we have

$$\{0\} = \{g0\} \subset (F0)_1.$$

4. **Conclusion**

In attempt to model the real world problems, we have to deal with uncertainties and vagueness of the data, tools or conditions in the form of constraints . Fuzzy set theory has provided many important tools in mathematics and related disciplines to resolve the issues of uncertainty and ambiguity. Fuzzy sets and mappings play important roles in the process of fuzzification of systems and fuzzy optimization.

We extended the Geraghty type fixed point theorems for fuzzy mappings (mappings on an arbitrary nonempty set to the subfamily of fuzzy sets on complete metric spaces) to the generalized Geraghty type fuzzy mappings. Fixed fuzzy point theorems for newly introduced mappings have been proved in this paper. An example is provided to illustrate the main result and to show that main theorem is a proper generalization of Geraghty type mappings (compare [11]) self mappings on metric spaces for the set of fuzzy mappings. As a result we also generalized, unify and extended the well known generalizations [10, 11, 15] of Nadler's theorem [19]. Further as an application of Theorem 2.1, we obtained the existence of coincidence fuzzy points and common fixed fuzzy points of the hybrid pair of single valued self mapping and a fuzzy mapping. Fixed point theorems for fuzzy mappings obtained in this article can be used in solving the real world problems involving fuzzy situations.

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