ON FUZZY HYPERIDEALS OF Γ-HYPERRINGS

R. AMERI, H. HEDAYATI AND A. MOLAEE

Abstract. The aim of this paper is the study of fuzzy Γ-hyperrings. In this regard the notion of \(\nu\)-fuzzy hyperideals of Γ-hyperrings are introduced and basic properties of them are investigated. In particular, the representation theorem for \(\nu\)-fuzzy hyperideals are given and it is shown that the image of a \(\nu\)-fuzzy hyperideal of a Γ-hyperring under a certain conditions is two-valued. Finally, the product of \(\nu\)-fuzzy hyperideals are studied.

1. Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analysis their properties and applied them to groups, rational algebraic functions [16]. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties and applied mathematics (for example see [5], [6], [22]).

Also, following the introduction of fuzzy sets by L. A. Zadeh in 1965 [23], the fuzzy set theory were developed by Zadeh himself and many researchers in mathematics and it was applied in many pure and applied areas. For example the concept of a fuzzy group was introduced by A. Rosenfeld and the notion of fuzzy ideal in a ring introduced and studied by W. J. Liu [15]. Recently fuzzy set theory have been had good develop in hyperstructures theory (for example see [7], [8], [9], [10], [11],[24]).

The notion of Γ-rings introduced by N. Nobosawa in [19] and immediately after him in 1966, Barnes extended this notion and obtained more results [4]. Kyuno investigated the new aspects of Γ-rings such as, prime Γ-rings and left and right unities of Γ-rings. Also in recent years Ozturk, Y. B. Jun and C. Y. Lee in [12] and [20] applied the concept of fuzzy sets to the theory of Γ-rings.

In this paper, first we introduce the notion of (\(\nu\)-)fuzzy hyperideals of Γ-hyperrings and, then we obtain some related basic results. We characterize (\(\nu\)-)fuzzy hyperideals based on their level subsets and associate a new (\(\nu\)-fuzzy) hyperideal from a given fuzzy hyperideal of a Γ-hyperring. In particular, we show that under certain conditions \(\nu\)-fuzzy hyperideals of Γ-hyperrings are two-valued. Finally we describe \(\nu\)-fuzzy hyperideals of product of Γ-hyperrings.

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2. Preliminaries

In this section we gather all definitions and simple properties of $\Gamma$-hyperrings that we require in the next notions.

Let $H$ be a nonempty set. A map $+: H \times H \rightarrow P_*(H)$ is called hyperoperation or join operation, where $P_*(H)$ denotes the set of all nonempty subsets of $H$.

**Definition 2.1.** [6] A nonempty set $M$ together a hyperoperation $+$ is called a polygroup if the following conditions are satisfied:

1. For all $x, y, z \in M$, $(x + y) + z = x + (y + z)$;
2. For all $x \in M$ there exist an unique element $e \in M$ such that $e + x = x = x + e$ (we denote $e$ by $0$);
3. For all $x \in M$ there exists an unique element $x' \in M$ such that $e \in x + x' \cap x' + x$ (we denote $x'$ by $-x$);
4. For all $x, y, z \in M$, $z \in x + y \Rightarrow x \in z - y \Rightarrow y \in z - x$.

By $U <_{p} M$, we mean $U$ is a subpolygroup of $M$. We denote the set of all subpolygroup of $M$, by $SP(M)$. A canonical hypergroup is a commutative polygroup.

**Definition 2.2.** [15, 19] An algebraic structure $(R, +, \cdot)$ is called a hyperring if the following statements are satisfied:

(i) $(R, +)$ is a canonical hypergroup;
(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0 = 0.x$;
(iii) The multiplication is distributive with respect to the hyperoperation $+$, i.e., $x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x \ \forall x, y, z \in R$.

**Remark 2.3.** (i) It can be easily proved that zero is unique.
(ii) For simplicity of notation, sometimes we write $xy$ instead of $x.y$ in Definition 2.2.

(iii) If $A, B \subseteq R$ and $x \in R$, then $A + B = \bigcup\{a + b | a \in A, b \in B\}$. Also, $A + x$ is used for $A + \{x\}$.
(iv) By axioms of Definition 2.2, it is easy to see that, $-(x) = x$ and $-(x + y) = -x - y$, where $-A = \{-a | a \in A\}$. Also, $(a + b).(c + d) \subseteq a.c + b.c + a.d + b.d$.

**Definition 2.4.** Let $R$ be a hyperring. Then

(i) $R$ is commutative if $x.y = y.x \ \forall x, y \in R$;
(ii) $R$ is called with identity, if there exists an element, say $1 \in R$, such that $1.x = x.1, \forall x \in R$;
(iii) A nonempty subset $A$ of $R$ is said to be a subhyperring of $R$ if $(A, +, \cdot)$ is itself a hyperring. If $R \setminus \{0\}$ is a multiplicative group, then $(R, +, \cdot)$ is a hyperfield.

**Example 2.5.** [18] (i) Let $(A, +, \cdot)$ be a ring and $N$ a normal semigroup of $(A, \cdot)$. Then the multiplicative classes $\overline{x} = xN, x \in A$ form a partition of $A$. Let $\overline{A} = A/N$ be the set of these classes. If we define the product $\overline{x} \cdot \overline{y}$ in $\overline{A}$ of $\overline{x}, \overline{y} \in \overline{A}$ as equal to their product as subsets of $A$, and their sum $\overline{x} + \overline{y}$ in $\overline{A}$ as the set of all $\overline{z} \in \overline{A}$
contained in their sum as subsets of $A$, i.e.,

$$\overline{x} \oplus \overline{y} = \{\overline{z} \in \overline{x} + \overline{y}\} \quad \text{and} \quad \overline{x} \odot \overline{y} = \overline{x.y}.$$  

Then $(\overline{A}, \oplus, \odot)$ is a hyperring.

(ii) Let $R$ be a commutative ring with identity. Letting $\overline{R} = \{\overline{x} = \{x, -x\} | x \in R\}$. Then $\overline{R}$ is a hyperring with respect to the hyperoperation $\overline{x} \odot \overline{y} = \{x + y, x - y\}$ and multiplication $\overline{x} \odot \overline{y} = \overline{x.y}$.

**Definition 2.6.** (i) A nonempty subset $I$ of a hyperring $R$ is called a (resp. left) **right hyperideal** of $R$ if (resp. $x.x \in I \, \forall r \in R \, \forall x \in I$).

(ii) $I$ is called a **hyperideal** if $I$ is both left and right hyperideal;

(iii) A proper hyperideal $I$ of $R$ ($I \neq R$) is called a **prime hyperideal** if $a.b \in I$ implies that $a \in I$ or $b \in I$ (for a study of prime hyperideals and prime subhypermodules see [36]). The set of all prime hyperideal of $R$ is called the **prime spectrum** of $R$ and it is denoted by $\text{Spec}(R)$.

**Definition 2.7.** Let $(M, +)$ and $(\Gamma, +)$ be canonical hypergroups. Then $M$ is said to be a $\Gamma$-**hyperring** if there exists a mapping $\cdot : M \times \Gamma \times M \rightarrow P_+(M)$ such that the following conditions are satisfied:

1. $(x + y)αz \subseteq xαz + yαz \text{, } xα(y + z) \subseteq xαy + xαz, \forall x, y, z \in M, \forall α \in \Gamma$;
2. $x(α + β)y \subseteq xαy + xβy, \forall x, y \in M, \forall α, β \in \Gamma$;
3. $(xαy)βz = xα(yβz), \forall x, y, z \in M, \forall α, β \in \Gamma$.

If in Definition 2.2, we replace all inclusions by equality, then $M$ is called a **strong $\Gamma$-hyperring**.

**Definition 2.8.** A right (resp. left) **hyperideal** of a $\Gamma$-hyperring $M$ is a subpolygroup $U$ of $M$ such that $UTM \subseteq U$ (resp. $MU \subseteq U$). Also if $\Delta$ is a subpolygroup of $\Gamma$, then the subpolygroup $I$ of $M$ is said to be a right (left) $\Delta$-hyperideal if $I \Delta M \subseteq I$ (resp. $M \Delta I \subseteq I$). By $U \ll M$, we mean $U$ is a hyperideal of $\Gamma$-hyperring $M$. Also we denote the set of all hyperideals of $M$ by $\text{HI}(M)$.

Clearly every hyperideal of a $\Gamma$-hyperring is a $\Delta$-hyperideal for some $\Delta \subseteq \Gamma$.

We use $I = [0, 1]$, the real unit interval as a chain with the usual ordering, in which $\wedge$ stands for minimum or infimum (inf) (or intersection) and $\vee$ stands for maximum or supremum (sup) (or union), for the degree of membership. A fuzzy subset of a given set $X$ is a mapping $\mu : X \rightarrow [0, 1]$. We denote the set of all fuzzy subset of $X$ by $FS(X)$, that is $FS(X) = \{\mu | \mu : X \rightarrow [0, 1] \text{ is a function}\}$. For $\mu \in FS(X)$, the level subset of $\mu$ is defined by $\mu_t = \{x \in X | \mu(x) \geq t\}$. For a fuzzy set $\mu$ of $X$ we denote by $Im(\mu)$ the image of $\mu$.

**Definition 2.9.** [20] Let $(M, +)$ be a canonical hypergroup and $\mu \in FS(M)$. Then $\mu$ is a fuzzy subpolygroup of $M$ if for all $a, b \in M$ the following conditions hold:

1. $\bigwedge_{z \in a + b} \mu(z) \geq \mu(a) \wedge \mu(b)$;
2. $\mu(-a) \geq \mu(a)$.

By $\mu \ll FP M$, we mean $\mu$ is a fuzzy subpolygroup of $M$. Also we denote the set of all fuzzy subpolygroups of $M$, by $FP(M)$.
3. \(\nu\)-Fuzzy Hyperideals of \(\Gamma\)-Hyperrings

In the sequel by \(M\) we mean a \(\Gamma\)-hyperring.

**Definition 3.1.** (i) A fuzzy subset \(\mu\) of \(M\) is said to be a left (resp. right) fuzzy hyperideal of \(M\) if and only if for all \(x, y \in M\) and \(\gamma \in \Gamma\) we have

(1) \(\mu \in FP(M)\);

(2) \(\bigwedge_{z \in x \gamma y} \mu(z) \geq \mu(y)\) (resp. \(\bigwedge_{z \in x \gamma y} \mu(z) \geq \mu(x)\)).

By \(\mu <_{FHI} M\), we mean \(\mu\) is a fuzzy hyperideal of \(M\). Also we denote the set of all fuzzy hyperideals of \(M\) by \(FHI(M)\).

(ii) A fuzzy subset \(\mu\) of \(M\) is said to be a left (resp. right) \(\nu\)-fuzzy hyperideal of \(M\) if and only if for all \(x, y \in M\) and \(\gamma \in \Gamma\) we have

(1) \(\mu \in FP(M)\) and \(\nu \in FP(\Gamma)\);

(2) \(\bigwedge_{z \in x \gamma y} \mu(z) \geq \mu(y) \land \nu(\gamma)\) (resp. \(\bigwedge_{z \in x \gamma y} \mu(z) \geq \mu(x) \land \nu(\gamma)\)).

By \(\mu <_{FHI_{\nu}} M\), we mean \(\mu\) is a \(\nu\)-fuzzy hyperideal of \(M\). Also we denote the set of all \(\nu\)-fuzzy hyperideals of \(M\) by \(FHI_{\nu}(M)\).

Clearly, every fuzzy hyperideal is a \(\nu\)-fuzzy hyperideal, for some \(\nu \in FP(\Gamma)\), by letting \(\nu = \chi_{\Gamma}\), where \(\chi_{\Gamma}\) denotes the characteristic function of \(\Gamma\).

**Example 3.2.** Let \((M, +, \cdot)\) be an hyperring and \(\Gamma\) be an hyperideal of \(M\). Define \(\circ : M \times \Gamma \times M \rightarrow P^+(M)\) by \((a, \gamma, b) \mapsto a \circ \gamma \circ b = \{ z \in M \mid z \in a, \gamma, b \}\). Then it is easy to verify that \(M\) is a strong \(\Gamma\)-hyperring. Also if \(I\) and \(\Delta\) are hyperideals of hyperring \((M, +, \cdot)\) and \(\Delta \subseteq \Gamma\), then \(I\) is a \(\Delta\)-hyperideal of \(\Gamma\)-hyperring \(M\), since \(I \Delta M \subseteq I\) and \(M \Delta I \subseteq I\). Now define \(\mu\) and \(\nu\) on \(I\) and \(\Delta\) respectively as follow:

\[
\mu(x) = \begin{cases} 
0.8 & \text{if } x \in I, \\
0 & \text{Otherwise}
\end{cases} \quad \nu(\delta) = \begin{cases} 
0.5 & \text{if } \delta \in \Delta, \\
0 & \text{Otherwise}
\end{cases}
\]

It is easy to verify that \(\mu\) and \(\nu\) are fuzzy subpolygroups of \(M\) and \(\Gamma\) respectively. Suppose that \(x, y \in M\) and \(\delta \in \Delta\) and \(z \in x \circ \delta \circ y\). We can consider two cases:

(1) \(x \in I\) or \(y \in I\) then we can say that \(x \circ \delta \circ y \subseteq I\) and so for all \(z \in x \circ \delta \circ y\), we have \(\mu(z) = 0.8 \geq 0.5 = (\mu(x) \lor \mu(y)) \land \nu(\delta)\).

(2) \(x, y \notin I\) then \(\mu(z) \geq 0.5 = (\mu(x) \lor \mu(y)) \land \nu(\delta)\).

Therefore \(\mu\) is a \(\nu\)-fuzzy hyperideal of \(M\) as a \(\Gamma\)-hyperring.

**Example 3.3.** Let \(R\) be a hyperring and let \(M_{m,n}(R)\) be the set of all matrices by the size \(m \times n\) with entries of \(R\). Define \(\circ : M_{m,n}(R) \times M_{m,n}(R) \times M_{m,n}(R) \rightarrow P^+(M_{m,n}(R))\) by:

\[
A \circ B \circ C = \{Z \in M_{m,n}(R) \mid Z \in ABC, \ A, C \in M_{m,n}(R), \ B \in M_{m,n}(R)\}.
\]

Then it is easy to verify that \(M_{m,n}(R)\) is a \(M_{m,n}(R)\)-hyperring. Also if \(I\) and \(J\) are hyperideal of hyperring \((R, +, \cdot)\), then it is easy to verify that \(M_{m,n}(I)\) is a \(M_{m,n}(J)\)-hyperideal of \(M_{m,n}(R)\) since \(M_{m,n}(I) \circ M_{m,n}(J) \circ M_{m,n}(R) \subseteq M_{m,n}(I)\) (by Definition 2.3) and \(M_{m,n}(R) \circ M_{m,n}(J) \circ M_{m,n}(I) \subseteq M_{m,n}(I)\) (by Definition
2.3). Now define $\mu$ and $\nu$ on $M_{m,n}(I)$ and $M_{n,m}(J)$ respectively as follow:

$$
\mu(X) = \begin{cases} 
4/5 & \text{if } X \in M_{m,n}(I), \\
7/10 & \text{if } X \not\in M_{m,n}(I)
\end{cases} \quad \nu(Y) = \begin{cases} 
1/2 & \text{if } Y \in M_{n,m}(J), \\
1/4 & \text{if } Y \not\in M_{n,m}(J)
\end{cases}
$$

It is routine to check that $\mu$ is a $\nu$–fuzzy hyperideal of $M_{m,n}(R)$ as an $M_{n,m}(R)$–hyperring.

**Lemma 3.4.** Let $\mu$ be a $\nu$–fuzzy hyperideal of $M$. Then $\mu(x) \leq \mu(0_M)$, for all $x \in M$.

**Proof.** For any $x \in M$ we have $0_M \in x - x$. Thus $\mu(0_M) \geq \mu(x) \wedge \mu(-x) = \mu(x)$. □

**Theorem 3.5.** (Representation Theorem) Let $\mu$ be a fuzzy set in a $\Gamma$-hyperring $M$.

Then $\mu$ is a left (resp. right) $\nu$-fuzzy hyperideal of $M$ if and only if each level subset $\mu_t$ of $\mu$ is a left (resp. right) $\nu_t$-hyperideal of $M$, for each $t \in [0, \mu(0_M) \wedge \nu(0_M)]$.

**Proof.** Suppose that $\mu$ is a left (resp. right) $\nu$-fuzzy hyperideal of $M$ and let $\mu_t \neq \emptyset$.

We have $\mu_t \subseteq M$, then for any $x, y, z \in \mu_t$, $(x + y) + z = x + (y + z)$. We show that

$$
\forall a \in \mu_t, \exists 0_M \in \mu_t : a + 0_M = a.
$$

Since $a \in \mu_t$ and $\mu_t \subseteq M$, so $a \in M$ then there exists an unique $0_M \in M$ such that $a + 0_M = a$. Also we have $0_M \in a - a$, thus $\mu(0_M) \geq \mu(a) \wedge \mu(-a) \geq t$, therefore $0_M \in \mu_t$. Similarly for all $x \in \mu_t$, there exists $-x \in \mu_t$, such that $0_M \in x - x$. We now show that

$$
M \varepsilon \mu_t \subseteq \mu_t \quad \text{(resp. } \mu_t \varepsilon M \subseteq \mu_t).
$$

Let $m \in M, \gamma \in \nu_t, u \in \mu_t$, and $z \in m \gamma u$, then we have

$$
\mu(z) \geq \bigwedge_{z \in m \gamma u} \mu(z) \geq \mu(u) \wedge \nu(\gamma) \geq t;
$$

thus $z \in \mu_t$. Therefore $M \varepsilon \mu_t \subseteq \mu_t$. Similarly we can prove that $\mu_t \varepsilon M \subseteq \mu_t$.

Conversely, suppose that $\mu_t$ is a left (resp. right) $\nu_t$-hyperideal of $M$. We show that for all $a, b \in M, \bigwedge_{z \in a + b} \mu(z) \geq \mu(a) \wedge \mu(b)$.

If $a, b \in M$, then there exist $t_1, t_2 \in [0, 1], \mu(a) = t_1, \mu(b) = t_2$. Put $t = t_1 \wedge t_2$, thus $a, b \in \mu_t$, and $a + b \in \mu_t$. Also if $z \in a + b$, we have $\mu(z) \geq t = \mu(a) \wedge \mu(b)$, therefore $\bigwedge_{z \in a + b} \mu(z) \geq \mu(a) \wedge \mu(b)$. Obviously for all $x \in M$, we have $\mu(x) \geq \mu(-x)$.

Let $x, y \in M$ and $\gamma \in \Gamma$ and $\mu(y) = t_1$ and $\nu(\gamma) = t_3$. Put $t = t_1 \wedge t_3$, thus $y \in \mu_t$ and $\gamma \in \nu_t$. So $x \gamma y \subseteq \mu_t$, since $\mu_t$ is a $\nu_t$-hyperideal. Then for all $z \in x \gamma y$ we have

$$
\mu(z) \geq t = \mu(y) \wedge \nu(\gamma).
$$

Similarly, we obtain that $\bigwedge_{z \in x \gamma y} \mu(z) \geq \mu(x) \wedge \nu(\gamma)$. This completes the proof. □
Example 3.6.

(1) Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be a strictly increasing sequence of left (resp. right) hyperideals of an arbitrary $\Gamma$-hyperring $M$ and $\{t_j\}_{j=1}^\infty$ be a strictly increasing sequence in $[0, 1]$. Define $\mu$ on $M$ as follows:

$$\mu(x) = t_j \text{ if } x \in I_j \setminus I_{j-1}, \text{ where } t_{j-1} < t_j, j = 1, 2, \ldots \text{ and } \mu(x) = 0, \text{ if } x \in M \setminus \bigcup_{j=1}^\infty I_j.$$ 

It is easy to verify that $\mu_{t_{j+1}} \subseteq \mu_{t_j}$ and the only level subsets of $M$ are $M$, and $\mu_{t_j} = I_j$, $j = 1, 2, \ldots$. Then by Theorem 3.5 $\mu$ is a left (resp. right) fuzzy hyperideal of $M$.

(2) Let $I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ be a strictly decreasing sequence of left (resp. right) hyperideals of an arbitrary $\Gamma$-hyperring $M$ and $\{t_j\}_{j=1}^\infty$ be a strictly decreasing sequence in $[0, 1]$.

Define fuzzy subset $\mu$ on $V$ by $\mu(x) = t_{j-1}$, if $x \in I_{j-1} \setminus I_j$ where

$$t_{j-1} > t_j, j = 1, 2, \ldots \text{ and } \mu(x) = 1 \text{ if } x \in \bigcap_{j=1}^\infty I_j.$$ 

Again by Theorem 3.5 it is easy to verify that $\mu$ is a left (resp. right) fuzzy hyperideal of $M$, since the only level subsets of $M$ are $M$ and $\mu_{t_j} = I_j$, $j = 1, 2, \ldots$.

(3) In Example 3.3, $\mu$ is $\nu$-fuzzy hyperideal of $M_{m,n}(R)$, since $\mu_{4/5} = M_{m,n}(R)$ and $\nu_{\infty} = M_{m,n}(I)$ and $\nu_{j/10} = \nu_{j/4} = M_{n,m}(J)$, which are hyperideals.

Lemma 3.7. If $\mu \in FHIL(M)$ and $\bigwedge_{t \in x-y} \mu(t) = \mu(0_M)$, then $\mu(x) = \mu(y)$.

Proof. We have

$$\mu(x) \geq \bigwedge_{t \in x-y+y} \mu(t) \geq \bigwedge_{t \in x-y} \mu(t') \wedge \mu(y) = \mu(0_M) \wedge \mu(y) = \mu(y).$$

Then, $\mu(x) \geq \mu(y)$. Similarly, we have $\mu(y) \geq \mu(x)$. Therefore $\mu(x) = \mu(y)$. \hfill $\square$

In next propositions we construct new ($\nu$-fuzzy) hyperideals by given fuzzy hyperideals of $\Gamma$-hyperrings.

Proposition 3.8. Let $\mu$ be a left (resp. right) $\nu$-fuzzy hyperideal of $M$ and $\mu(0_M) = \nu(0_t)$. Then the set

$$M_{\mu} = \{x \in M \mid \mu(x) = \mu(0_M)\}$$

is a left (resp. right) $\nu_{\mu(0_M)}$-hyperideal of $M$.

Proof. A direct verification shows that $M_{\mu}$ is a canonical hypergroup and $M_{\mu} \subseteq M$. We show that $M_{\nu_{\mu(0_M)}} M_{\mu} \subseteq M_{\mu}$. Let $z \in x \gamma y$ such that $x \in M, \gamma \in \nu(0_{\mu(0_M)})$ and $y \in M_{\mu}$. We have $\mu(z) \geq \mu(y) \wedge \nu(\gamma) \geq \mu(0_M)$. Then by Lemma 3.4, $\mu(z) = \mu(0_M)$, thus $z \in M_{\mu}$. Similarly, we obtain $M_{\nu_{\mu(0_M)}} M \subseteq M_{\mu}$.

Proof. \hfill $\square$

Proposition 3.9. Let $\mu$ be a left (resp. right) $\nu$-fuzzy hyperideal of $M$, then

$$supp(\mu) = \{x \in M \mid \mu(x) > 0\}$$
is a left (resp. right) \( \text{supp}(\nu) \)-hyperideal of \( M \).

**Proof.** The proof is similar to the proof of Proposition 3.5 by some modification. \( \square \)

**Proposition 3.10.** If \( \mu \) is a \( \nu \)-fuzzy hyperideal of \( \Gamma \)-hyperring \( M \), then

\[
R(\mu)(x) = \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(a) \mid a \in nz, \exists n \in \mathbb{N} \} \}
\]

is a \( \nu \)-fuzzy hyperideal of \( M \).

**Proof.** Let \( z \in x + y \). We prove that

\[
R(\mu)(z) \geq R(\mu)(x) \land R(\mu)(y).
\]

For this we have

\[
R(\mu)(z) = \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(a) \mid a \in nz, \exists n \in \mathbb{N} \} \}
\]

\[
\geq \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(a) \mid a \in nx + ny, \exists n \in \mathbb{N} \} \} \quad \text{(since} \; z \in x + y)\]

\[
\geq \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(t_1) \mid t_1 \in nx, \exists n \in \mathbb{N} \} \} \land \{ \bigwedge \{ \mu(t_2) \mid t_2 \in ny, \exists n \in \mathbb{N} \} \}
\]

\[
= \left[ \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(t_1) \mid t_1 \in nx, \exists n \in \mathbb{N} \} \} \right] \land \left[ \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(t_2) \mid t_2 \in ny, \exists n \in \mathbb{N} \} \} \right]
\]

\[
= R(\mu)(x) \land R(\mu)(y).
\]

Also we have

\[
R(\mu)(x) = \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(z) \mid z \in nx, \exists n \in \mathbb{N} \} \}
\]

\[
\geq \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(\pm z) \mid z \in n(-x), \exists n \in \mathbb{N} \} \}
\]

\[
= R(\mu)(\pm x).
\]

Now suppose that \( z \in x \gamma y \). We prove that

\[
R(\mu)(z) \geq (R(\mu)(x) \lor R(\mu)(y)) \land \nu(\gamma).
\]

For this we have

\[
R(\mu)(z) = \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(a) \mid a \in nz, \exists n \in \mathbb{N} \} \}
\]

\[
\geq \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(a) \mid a \in (nx) \gamma y, \exists n \in \mathbb{N} \} \} \quad \text{(since} \; z \in x \gamma y)\]

\[
\geq \bigvee_{n \in \mathbb{N}} \{ \bigwedge \{ \mu(b) \mid b \in nx, \exists n \in \mathbb{N} \} \} \land \nu(\gamma) \quad \text{(since} \; \mu \in \text{FHI}(M))
\]

\[
= R(\mu)(x) \land \nu(\gamma).
\]

Similarly, we can prove that \( R(\mu)(z) \geq R(\mu)(y) \land \nu(\gamma) \). Therefore \( R(\mu) \in \text{FHI}_\nu(M) \). \( \square \)
Proposition 3.11. Let $\mu \in FHI_{\nu}(M)$ and $\mu^+(x) = \mu(x) + 1 - \mu(0_M)$.

(i) Then $\mu^+$ is a $\nu$-fuzzy hyperideal of $M$.

(ii) If $\mu(0_M) = \nu(0_M)$, then $\mu^+$ is a $\nu^+$-fuzzy hyperideal of $M$, where $\nu^+(x) = \nu(x) + 1 - \nu(0_M)$.

Proof. (i) Let $z \in x + y$, then we have

$$\mu^+(z) = \mu(z) + 1 - \mu(0_M) \geq (\mu(x) \land \mu(y)) + 1 - \mu(0_M) \quad \text{(since $\mu \in FHI_{\nu}(M)$)}$$

$$= (\mu(x) + 1 - \mu(0_M)) \land (\mu(y) + 1 - \mu(0_M))$$

$$= \mu^+(x) \land \mu^+(y).$$

Also we have

$$\mu^+(z) = \mu(z) + 1 - \mu(0_M) \geq \mu(-z) + 1 - \mu(0_M) \quad \text{(since $\mu \in FHI_{\nu}(M)$)}$$

$$= \mu^+(-z).$$

Now suppose $z \in x\gamma y$, then we have

$$\mu^+(z) = \mu(z) + 1 - \mu(0_M) \geq (\mu(x) \land \nu(\gamma)) + 1 - \mu(0_M).$$

(1)

We consider the following cases.

Case 1. If $\mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) \land \nu(\gamma)) + 1 - \mu(0_M) = \nu(\gamma) + 1 - \mu(0_M)$$

(2)

we have $\mu(x) + 1 - \mu(0_M) \geq \mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) + 1 - \mu(0_M)) \land \nu(\gamma) = \nu(\gamma).$$

(3)

Then from (1), (2) and (3) it is concluded that $\mu^+(z) \geq \nu(\gamma) + 1 - \mu(0_M) \geq \nu(\gamma)$. Thus $\mu^+(z) \geq \mu^+(x) \land \nu(\gamma)$.

Case 2. If $\mu(x) \leq \nu(\gamma)$, then

$$\mu^+(z) \geq (\mu(x) \land \nu(\gamma)) + 1 - \mu(0_M)$$

$$= \mu(x) + 1 - \mu(0_M)$$

$$= \mu^+(x)$$

$$\geq \mu^+(x) \land \nu(\gamma).$$

Similarly to the both cases 1 and 2 we can obtain $\mu^+(z) \geq \mu^+(y) \land \nu(\gamma)$. Thus

$$\mu^+(z) \geq (\mu^+(x) \lor \mu^+(y)) \land \nu(\gamma).$$

Therefore $\mu^+ \in FHI_{\nu}(M)$. 


(ii) Let $\mu$ be a $\nu$-fuzzy hyperideal of $\Gamma$-hyperring $M$ and $\mu(0_M) = \nu(0_{\Gamma})$ and $z \in x - y$, for all $x, y \in M$. Obviously $\mu^+(z) \geq \mu^+(x) \land \mu^+(y)$. Suppose that $z \in x\gamma y$, for $x, y \in M$ and $\gamma \in \Gamma$. Then

$$
\mu^+(z) \geq (\mu(x) \lor \mu(y)) + 1 - \mu(0_M)
$$

$$
= [\mu(x) + 1 - \mu(0_M)] \lor (\mu(y) + 1 - \mu(0_M)) \land (\nu(\gamma) + 1 - \mu(0_M))
$$

$$
= [\mu^+(x) \lor \mu^+(y)] \land \nu^+(\gamma) \quad \text{(since } \mu(0_M) = \nu(0_{\Gamma})\text{)}.
$$

Therefore $\mu^+$ is a $\nu^+$-fuzzy hyperideal of $M$.

\[\Box\]

**Proposition 3.13.** Let $M$ be a $\Gamma$-hyperring and $\mu \in FHI_\nu(M)$.

(i) If $f : [0, \mu(0_M) \lor \nu(0_{\Gamma})] \rightarrow [0, 1]$ is an increasing map, then $\mu_f : M \rightarrow [0, 1]$ defined by $\mu_f(x) = f(\mu(x))$ for all $x \in M$ is a $\nu_f$-fuzzy hyperideal of $M$, where $\nu_f : \Gamma \rightarrow [0, 1]$ is defined by $\nu_f(\gamma) = f(\nu(\gamma))$ for all $\gamma \in \Gamma$.

(ii) If $\mu(0_M) = \nu(0_{\Gamma})$ and $\tilde{\mu} : M \rightarrow [0, 1]$ defined by $\tilde{\mu}(x) = \mu(x)\mu(0_M)$ for all $x \in M$ is a $\tilde{\nu}$-fuzzy hyperideal of $M$, where $\tilde{\nu} : \Gamma \rightarrow [0, 1]$ is defined by $\tilde{\nu}(\gamma) = \nu(\gamma)\nu(0_{\Gamma})$ for all $\gamma \in \Gamma$.

**Proof.** (i) Let $z \in x + y$ then $\mu(z) \geq \mu(x) \land \mu(y)$. Since $f$ is increasing then, $f(\mu(z)) \geq f(\mu(x)) \land f(\mu(y))$, therefore $\mu_f(z) \geq f(\mu(-z)) = f(\mu(z)) = \mu_f(z)$. Also we have

$$
\mu_f(z) = f(\mu(z)) \geq f(\mu(-z)) = \mu_f(z).
$$

Suppose that $z \in x\gamma y$, then we have

$$
\mu(z) \geq (\mu(x) \lor \mu(y)) \land \nu(\gamma) \quad \text{(since } \mu \in FHI_\nu(M)\text{)}
$$

$$
\Rightarrow f(\mu(z)) \geq [f(\mu(x)) \lor f(\mu(y))] \land f(\nu(\gamma)) \quad \text{(since } f \text{ is increasing)}
$$

$$
\Rightarrow \mu_f(z) \geq (\mu_f(x) \lor \mu_f(y)) \land \nu_f(\gamma).
$$

Therefore $\mu_f \in FHI_{\nu_f}(M)$.

(ii) Let $z \in x + y$ then we have

$$
\tilde{\mu}(z) = \mu(z)/\mu(0_M)
$$

$$
= (1/\mu(0_M))\mu(z)
$$

$$
\geq (1/\mu(0_M))(\mu(x) \land \mu(y)) \quad \text{(since } \mu \in FHI_\nu(M)\text{)}
$$

$$
= (\mu(x)/\mu(0_M)) \land (\mu(y)/\mu(0_M))
$$

$$
= \tilde{\mu}(x) \land \tilde{\mu}(y).
$$

Also we have

$$
\tilde{\mu}(z) = (1/\mu(0_M))\mu(z)
$$

$$
\geq (1/\mu(0_M))\mu(-z) \quad \text{(since } \mu \in FHI_\nu(M)\text{)}
$$

$$
= \tilde{\mu}(-z).
$$
Suppose that $z \in x \gamma y$, then we have
\[ \tilde{\mu}(z) = \frac{1}{\mu(0_M)} \mu(z) \]
\[ \geq (1/\mu(0_M)) \mu(x) \lor \mu(y) \land \nu(\gamma) \] (since $\mu \in FHI_{\nu}(M)$)
\[ = [\tilde{\mu}(x) \lor \tilde{\mu}(y)] \land \tilde{\nu}(\gamma). \]
Therefore, $\tilde{\mu} \in FHI_{\nu}(M)$. \hfill \Box

In the next theorem, we prove that under certain conditions, fuzzy hyperideal of $\Gamma$-hyperring is two-valued.

**Theorem 3.14.** Let $\mu \in FHI_{\nu}(M)$, $\eta = 1/2\eta'$ and $\mu$ be maximal in the set $X = \{ \nu \in FHI_{\nu}(M) \mid \nu(x) = 1, \exists x \in M \}$ under conclusion. Then $\mu$ is two-valued fuzzy hyperideal of $M$ and it takes just 0 and 1.

**Proof.** Clearly $\mu \in FHI_{\nu}(M)$. We know that there exists $x \in M$ such that $\mu(x) = 1$, thus $\mu(0_M) \geq \mu(x) = 1$, hence $\mu(0_M) = 1$.

Let $x \in M$ be such that $\mu(x) \neq 1$. We show that $\mu(x) = 0$. Suppose that there exists $a \in M$ such that $0 < \mu(a) < 1$. Define $\nu : M \to [0,1]$ by $\nu(x + y) = 1/2(\mu(x) + \mu(a))$, for all $x \in M$. We show that $\nu \in FHI_{\nu}(M)$. Suppose that $z \in x + y$, then we have
\[ \nu(z) = 1/2(\mu(z) + \mu(a)) \]
\[ \geq 1/2[(\mu(x) \land \mu(y)) + \mu(a)] \] (since $\mu \in FHI_{\nu}(M)$)
\[ = 1/2(\mu(x) + \mu(a)) \land 1/2(\mu(y) + \mu(a)) \]
\[ = 1/2(\mu(x) + \mu(a)) \land 1/2(\mu(y) + \mu(a)) \]
\[ = \nu(x) \land \nu(y). \]
Also it is easy to verify that if $z \in M$, then $\nu(z) \geq \nu(-z)$. Now suppose $z \in x\gamma y$, we prove $\nu(z) \geq \nu(x) \land \nu(\gamma)$. We have
\[ \nu(z) = 1/2(\mu(z) + \mu(a)) \]
\[ \geq 1/2[(\mu(x) \land \eta(\gamma)) + \mu(a)] \] (since $\mu \in FHI_{\nu}(M)$)
\[ = 1/2[\mu(x) + \mu(a)] \land 1/2[\eta(\gamma)] + \mu(a) \]
\[ = \nu(x) \land (\eta(\gamma) + 1/2[\eta(\gamma)]) \]
\[ \geq \nu(x) \land \eta(\gamma). \]
Similarly we can prove that $\nu(z) \geq \nu(\gamma) \land \eta(\gamma)$. Therefore $\nu \in FHI_{\nu}(M)$.

Hence, by Proposition 3.10, $\nu^+ \in FHI_{\nu}(M)$. Also we have
\[ \nu^+(x) = \nu(x) + 1 - \nu(0_M) \]
\[ = 1/2(\mu(x) + \mu(a)) + 1 - 1/2(\mu(0_M) + \mu(a)) \]
\[ = 1/2(\mu(x) + 1). \] (since $\mu(0_M) = 1$)

So we have
\[ \nu^+(0_M) = 1/2(\mu(0_M) + 1) = 1/2(1 + 1) = 1. \]
Proposition 4.1. Let \( \nu^+ \) be \( \nu \)-fuzzy hyperideal of \( M_i \) as \( \Gamma_i \)-hyperring \( \forall i \in I \). Then \( \prod_{i \in I} M_i \) is a \( \prod_{i \in I} \nu_i \)-fuzzy hyperideal of \( \prod_{i \in I} M_i \) as \( \prod_{i \in I} \Gamma_i \)-hyperring.

Proof. Suppose \( (z_i)_{i \in I} \in (x_i)_{i \in I} + (y_i)_{i \in I} \). Then \( z_i \in x_i + y_i \), so \( \mu_i(z_i) \geq \mu_i(x_i) \land \mu_i(y_i) \), for all \( i \in I \). Also we have

\[
(\prod_{i \in I} \mu_i)((z_i)_{i \in I}) = \bigwedge_{i \in I} \mu_i(z_i) \\
\geq \bigwedge_{i \in I} (\mu_i(x_i) \land \mu_i(y_i)) \quad \text{(since } \mu_i \in FHI_{\nu_i}(M_i)) \\
= (\bigwedge_{i \in I} \mu_i(x_i)) \land (\bigwedge_{i \in I} \mu_i(y_i)) \\
= (\prod_{i \in I} \mu_i)((x_i)_{i \in I}) \land (\prod_{i \in I} \mu_i)((y_i)_{i \in I}).
\]

Thus \( \nu^+ \in X \). Also we have

\[
\nu^+(0_M) = 1 > \nu^+(a) = 1/2(\mu(a) + 1) > \mu(a) \neq 1.
\]

Hence \( \nu^+ \) is non-constant and \( \nu^+(a) > \mu(a) \). So \( \mu \) is not maximal, this is a contradiction. Therefore there is not any \( a \in M \) such that \( 0 < \mu(a) < 1 \). \( \square \)

4. Fuzzy Product of \( \nu \)-Fuzzy Hyperideals

Suppose that \( (M_i, +, \cdot, \text{Hyperring})_{i \in I} \) is a family of canonical hypergroups. Then \( \prod_{i \in I} M_i = \{ (x_i)_{i \in I} \mid x_i \in M_i \} \), the cartesian product of \( (M_i, +, \cdot, \text{Hyperring})_{i \in I} \), with following hyperoperation is a canonical hypergroup:

\[
(x_i)_{i \in I} \oplus (y_i)_{i \in I} = \{(z_i)_{i \in I} \mid z_i \in x_i + y_i \}.
\]

It is easy to verify that if \( M_i \) is a \( \Gamma_i \)-hyperring, then \( \prod_{i \in I} M_i \) is \( \prod_{i \in I} \Gamma_i \)-hyperring by the following rule:

\[
\circ : (\prod_{i \in I} M_i) \times (\prod_{i \in I} \Gamma_i) \times (\prod_{i \in I} M_i) \to P^*(\prod_{i \in I} M_i),
\]

which is defined by

\[
(x_i)_{i \in I} \circ (\gamma_i)_{i \in I} \circ (y_i)_{i \in I} = \{(z_i)_{i \in I} \mid z_i \in x_i \gamma_i y_i, \forall i \in I \}.
\]

Notation. In the next proposition, by \( \prod_{i \in I} \mu_i \), we mean the fuzzy product of \( \mu_i \)'s, which is defined as follows:

\[
(\prod_{i \in I} \mu_i)((x_i)_{i \in I}) = \bigwedge_{i \in I} \mu_i(x_i).
\]
Also it is easy to verify that \( \prod_{i \in I} \mu_i((x_i)_i) \leq \prod_{i \in I} \mu_i((-x_i)_i) \).

Suppose that \((z_i)_{i \in I} \in (x_i)_{i \in I} \cap (y_i)_{i \in I} \), then we have \( z_i \in x_i y_i, \forall i \in I \)

\[ \Rightarrow \mu_i(z_i) \geq (\mu_i(x_i) \lor \mu_i(y_i)) \land \nu_i(\gamma_i), \forall i \in I \quad \text{(since} \mu_i \in \text{FHI}_{\nu_i}(M_i)\text{)} \]

\[ \Rightarrow \prod_{i \in I} \mu_i(z_i) \geq \prod_{i \in I} \[ (\mu_i(x_i) \lor \mu_i(y_i)) \land \nu_i(\gamma_i) \] \]

\[ \Rightarrow \prod_{i \in I} \mu_i((z_i)_{i \in I}) \geq \prod_{i \in I} \mu_i((x_i)_{i \in I}) \lor \prod_{i \in I} \mu_i((y_i)_{i \in I}) \land \prod_{i \in I} \nu_i((\gamma_i)_{i \in I}). \]

Therefore \( \prod_{i \in I} \mu_i \) is a \( \prod_{i \in I} \nu_i \)-fuzzy hyperideal of \( \prod_{i \in I} M_i \).

\[\square\]

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**References**


Reza Ameri*, Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: ameri@umz.ac.ir

Hossein Hedayati, Department of Basic Sciences, Babol University of Technology, Babol, Iran

E-mail address: h.hedayati@umz.ac.ir

A. Molaei, Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran

*Corresponding author