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ORDER INTERVALS IN THE METRIC SPACE OF FUZZY NUMBERS

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Abstract. In this paper, we introduce a function in order to measure the distance between two order intervals of fuzzy numbers, and show that this function is a metric. We investigate some properties of this metric, and finally present an application. We think that this study could provide a more general framework for researchers studying on interval analysis, fuzzy analysis and fuzzy decision making.

1. Introduction and Preliminary Concepts

In a decision making problem in which we deal with vague data or requirements, the premises frequently do not lead to precise conclusions. In such a situation, in order to approximate to some uncertain quantity, rather than modelling the data via real (crisp) numbers or even fuzzy numbers, to construct an order interval of fuzzy numbers as data type is more informative and it may represent more alternatives for the final decision.

The upper and lower bounds of such an order interval of fuzzy numbers corresponds to the maximum and minimum imaginable decisions, respectively. Thus the order interval signifies the extent of tolerance that a quantity could possibly take. However, the ordering of fuzzy numbers is not always easy to specify, and hence it is a basic problem in fuzzy optimization and fuzzy decision making systems. Since the set of fuzzy numbers is not totally ordered, there are many different methods proposed in the literature to compare fuzzy numbers. Each method has its own advantages and disadvantages. Therefore, we need to choose the most appropriate ordering model to solve the problem in hand. Some of these ordering methods can be found in [1, 3, 6, 7] and [11].

In this work, to obtain an order interval of fuzzy numbers, we consider a partial order defined on the set of fuzzy numbers, which is widely applied and encountered in the literature.

On the other hand, with respect to the degree of fuzziness of the data handled, we may need to consider more than one order interval of fuzzy numbers to manipulate the data more efficiently. In such a case, in order to decide which is (are) the best possible optimal order interval(s), it may be quite natural to consider the distances between these order intervals. Such a decision regarding the distance could minimize the loss of information and allow the decision makers compare the possible alternative decisions and find their way easily. In this study, we will try

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to construct a mathematical model to solve this problem. For this purpose, we first define a function to measure the distance between two order intervals of fuzzy numbers, and prove that this function is a metric. We present some theoretical results and finally consider a possible application of the subject matter. While determining our machinery, we have paid attention to consider a wide range of types of fuzzy numbers (triangular, trapezoidal, etc).

We first recall some of the basic concepts and notations in the theory of fuzzy numbers. We refer to [4] and [8] for more details.

A *fuzzy number* is a function $X : \mathbb{R} \longrightarrow [0,1]$ satisfying the following properties:

(i) X is *normal*, i.e., there exists a $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$;
(ii) X is *fuzzy convex*, i.e., for any $t, u \in \mathbb{R}$ and $\lambda \in [0,1]$ we have

$$X(\lambda t + (1-\lambda)u) \geq \min\{X(t),\ X(u)\};$$

(iii) X is *upper semi-continuous*;
(iv) The *closure* of the set $\{t \in \mathbb{R} : X(t) > 0\}$, denoted by X^0, is *compact.*

These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$level set

$$X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\} = \left[\underline{X}^\alpha,\ \overline{X}^\alpha\right]$$

is a non-empty compact convex subset of $\mathbb{R}$, as the support X^0.

A real number r can be considered as a fuzzy number $\widetilde{r}$ defined by

$$\widetilde{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r \end{cases} .$$

The set of all fuzzy numbers is usually denoted by $L(\mathbb{R})$.

A partial order $\preceq$ on $L(\mathbb{R})$ can be defined via

$$X \preceq Y \text{ iff } \underline{X}^\alpha \leq \underline{Y}^\alpha \text{ and } \overline{X}^\alpha \leq \overline{Y}^\alpha$$

for each $\alpha \in [0,1]$, where $X, Y \in L(\mathbb{R})$. We write $X \prec Y$ if $X \preceq Y$ and there exists an $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$.

The equality of two fuzzy numbers is defined by

$$X = Y \Leftrightarrow \underline{X}^\alpha = \underline{Y}^\alpha \text{ and } \overline{X}^\alpha = \overline{Y}^\alpha$$

for each $\alpha \in [0,1]$.

One basic problem encountered in fuzzy analysis is the distance between fuzzy numbers. There are many alternatives for this (for a brief review, see [4]). Here we will consider the one which is mostly used and easy to manipulate, namely, the function $D : L(\mathbb{R}) \times L(\mathbb{R}) \longrightarrow \mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$ defined by

$$D(X,Y) = \sup_{\alpha \in [0,1]} \max \left\{ \left|\underline{X}^\alpha - \underline{Y}^\alpha\right|,\ \left|\overline{X}^\alpha - \overline{Y}^\alpha\right| \right\}.$$

The function D is called the *supremum metric*. Puri and Ralescu [10] proved that the pair $(L(\mathbb{R}), D)$ is a complete metric space.

2. **Main Results**

In this section, we consider order intervals in the metric space $(L(\mathbb{R}), D)$ of fuzzy numbers, and obtain some basic results.

First of all, as in the case of the classical theory of partially ordered sets, an *order interval* $[X, Y]$ of fuzzy numbers can be defined as

$$[X, Y] = \{Z \in L(\mathbb{R}) : X \preceq Z \preceq Y\},$$

where $X, Y \in L(\mathbb{R})$. This interval is an ordinary (non-fuzzy) set whose elements are fuzzy numbers (see also [5]).

We say that the order intervals of fuzzy numbers $[X_1, Y_1]$ and $[X_2, Y_2]$ are *equal* if, and only if, $X_1 = X_2$ and $Y_1 = Y_2$.

In fact, an order interval of fuzzy numbers is a generalization of the concept of an *interval number*, that is, a compact interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. If X is an interval number, its endpoints are usually denoted by $\underline{X}$ and $\overline{X}$, namely, $X = [\underline{X}, \overline{X}]$. We do not distinguish between the degenerate interval number $[x, x]$ and the real number x. Interval numbers are the main tools of interval analysis (see [9]), and they are also useful in approximate reasoning [2].

Note that an order interval $[X, Y]$ of fuzzy numbers is in fact *order bounded* in the language of partially ordered sets. It can easily be shown that this order interval is also a *bounded set* in the metric space $(L(\mathbb{R}), D)$, namely, it is *metrically bounded.*

Theorem 2.1. *We will denote the set of all order intervals of fuzzy numbers by* $\mathbf{I}[L(\mathbb{R})]$. *To take the advantage of the synergism between fuzzy numbers and interval numbers, we now introduce a function*

$$D_I : \mathbf{I}[L(\mathbb{R})] \times \mathbf{I}[L(\mathbb{R})] \longrightarrow \mathbb{R}_0^+$$

which is defined by

$$D_I([X_1, Y_1], [X_2, Y_2]) = \max\{D(X_1, X_2), D(Y_1, Y_2)\}. \tag{1}$$

The function D_I defined by (1) is a metric on $\mathbf{I}[L(\mathbb{R})]$.

Proof. Assume that $[X_i, Y_i] \in \mathbf{I}[L(\mathbb{R})]$ for $i = 1, 2, 3$.

(i) *Positive definiteness:* Since the function D is a metric on $L(\mathbb{R})$, we get

$$D_I([X_1, Y_1], [X_2, Y_2]) \geq 0.$$

Similarly, we have $D_I([X_1, Y_1], [X_2, Y_2]) = 0$ if, and only if, $[X_1, Y_1] = [X_2, Y_2]$.

(ii) *Symmetry:* Clear from the definition.

(iii) *Triangle inequality:* We can write

$$D_I([X_1, Y_1], [X_2, Y_2]) = \max\{D(X_1, X_2), D(Y_1, Y_2)\}$$

$$\begin{aligned}
&\leq \max\{D(X_1, X_3) + D(X_3, X_2), D(Y_1, Y_3) + D(Y_3, Y_2)\} \\
&\leq \max\{D(X_1, X_3), D(Y_1, Y_3)\} + \max\{D(X_3, X_2), D(Y_3, Y_2)\} \\
&= D_I([X_1, Y_1], [X_3, Y_3]) + D_I([X_3, Y_3], [X_2, Y_2]).
\end{aligned}$$

Hence the pair $(\mathbf{I}[L(\mathbb{R})], D_I)$ is a metric space. □

Note that $D_I([X,X],\ [Y,Y]) = D(X,Y)$, and therefore the metric D_I reduces to the supremum metric D on the set of fuzzy numbers. Thus it is consistent with our identification of a degenerate order interval $[X,X]$ with the fuzzy number X. The set of fuzzy numbers $L(\mathbb{R})$ is isometrically embedded in the metric space $(\mathbf{I}[L(\mathbb{R})],\ D_I)$.

Remark 2.2. It is natural to regard two order intervals

$$I_1 = [X_1, Y_1],\ I_2 = [X_2, Y_2] \in \mathbf{I}[L(\mathbb{R})]$$

as subsets of $L(\mathbb{R})$. In this case, it is also possible to consider the *metric distance* between these order intervals, that is,

$$dist\ (I_1,\ I_2) = \inf_{X \in I_1, Y \in I_2} D(X,Y).$$

It is clear that if $I_1 \cap I_2 \neq \emptyset$, then $dist\ (I_1,\ I_2) = 0$. However, this does not hold in the case of the metric D_I on $\mathbf{I}[L(\mathbb{R})]$. In general, we have

$$dist\,(I_1,\ I_2) \leq D_I\,(I_1,\ I_2)\,.$$

To see this, let us consider the following example.

Example 2.3. Let $I_1 = [X_1, Y_1],\ I_2 = [X_2, Y_2] \in \mathbf{I}[L(\mathbb{R})]$ be defined via the following boundary fuzzy numbers:

$$\begin{aligned} X_1(t) &= \begin{cases} t-1, & 1 \leq t \leq 2 \\ -t+3, & 2 \leq t \leq 3 \end{cases}, \\ Y_1(t) &= \begin{cases} t-9, & 9 \leq t \leq 10 \\ -t+11, & 10 \leq t \leq 11 \end{cases}, \\ X_2(t) &= \begin{cases} t-5, & 5 \leq t \leq 6 \\ -t+7, & 6 \leq t \leq 7 \end{cases}, \\ Y_2(t) &= \begin{cases} t-12, & 12 \leq t \leq 13 \\ -t+14, & 13 \leq t \leq 14 \end{cases}. \end{aligned}$$

Then $I_1 \cap I_2 \neq \emptyset$ and hence $dist\ (I_1,\ I_2) = 0$, but $D_I\,(I_1,\ I_2) = 4$. Note that the mapping $dist : \mathbf{I}[L(\mathbb{R})] \times \mathbf{I}[L(\mathbb{R})] \longrightarrow \mathbb{R}_0^+$ causes loss of information about the distances between the numerical data carried by the order intervals. This is why we need to introduce a new type of distance function on $\mathbf{I}[L(\mathbb{R})]$.

Theorem 2.4. *Let* $[X_1, Y_1], [X_2, Y_2] \in \mathbf{I}[L(\mathbb{R})]$ *and* $\varepsilon > 0$. *Then*

$$D_I([X_1, Y_1],\ [X_2, Y_2]) < \varepsilon \tag{2}$$

if, and only if, the following two statements hold:

(i) For every $X \in [X_1, Y_1]$, *there is a* $Y \in [X_2, Y_2]$ *such that* $D(X,Y) < \varepsilon$,

(ii) For every $Y \in [X_2, Y_2]$, *there is an* $X \in [X_1, Y_1]$ *such that* $D(X,Y) < \varepsilon$.

Proof. Necessity: Assume that (2) holds for an $\varepsilon > 0$. Then we have

$$\underline{X_2}^{\alpha} < \underline{X_1}^{\alpha} + \varepsilon \tag{3}$$

and

$$\underline{Y_1}^{\alpha} < \underline{Y_2}^{\alpha} + \varepsilon \tag{4}$$

for each $\alpha \in [0,1]$. Similarly, we have

$$\overline{X_2}^{\alpha} < \overline{X_1}^{\alpha} + \varepsilon \tag{5}$$

and

$$\overline{Y_1}^{\alpha} < \overline{Y_2}^{\alpha} + \varepsilon \tag{6}$$

for each $\alpha \in [0,1]$. Now suppose that there exists an $X \in [X_1, Y_1]$ such that $D(X,Y) \geq \varepsilon$ for all $Y \in [X_2, Y_2]$. In this case there exists an $\alpha \in [0,1]$ such that at least one of the following holds:

$$\underline{X}^{\alpha} \geq \underline{Y}^{\alpha} + \varepsilon, \tag{7}$$

$$\underline{X}^{\alpha} \leq \underline{Y}^{\alpha} - \varepsilon, \tag{8}$$

$$\overline{X}^{\alpha} \geq \overline{Y}^{\alpha} + \varepsilon, \tag{9}$$

$$\overline{X}^{\alpha} \leq \overline{Y}^{\alpha} - \varepsilon. \tag{10}$$

On the other hand, we have

$$\underline{X_1}^{\alpha} \leq \underline{X}^{\alpha} \leq \underline{Y_1}^{\alpha} \tag{11}$$

and

$$\overline{X_1}^{\alpha} \leq \overline{X}^{\alpha} \leq \overline{Y_1}^{\alpha} \tag{12}$$

for each $\alpha \in [0,1]$. Thus (3), (4) and (11) yield

$$\underline{X_2}^{\alpha} - \varepsilon < \underline{X_1}^{\alpha} \leq \underline{X}^{\alpha} \leq \underline{Y_1}^{\alpha} < \underline{Y_2}^{\alpha} + \varepsilon \tag{13}$$

for each $\alpha \in [0,1]$. If (7) holds, then (13) yields

$$\underline{Y}^{\alpha} + \varepsilon \leq \underline{X}^{\alpha} < \underline{Y_2}^{\alpha} + \varepsilon \tag{14}$$

for each $Y \in [X_2, Y_2]$. However, (14) does not hold for $Y = Y_2$, which is a contradiction. Similarly, if (8) holds, then (13) yields

$$\underline{Y}^{\alpha} - \varepsilon \geq \underline{X}^{\alpha} > \underline{X_2}^{\alpha} - \varepsilon \tag{15}$$

for each $Y \in [X_2, Y_2]$. However, (15) does not hold for $Y = X_2$. Similar contradictions can be obtained by using (5), (6), (9), (10) and (12). This completes the proof of (i). Since D_I has the symmetry property, statement (ii) also holds.

Sufficiency: Now assume that (i) and (ii) hold. Suppose on the contrary that $D_I([X_1, Y_1], [X_2, Y_2]) \geq \varepsilon$ for a given $\varepsilon > 0$. Then we have either $D(X_1, X_2) \geq \varepsilon$ or $D(Y_1, Y_2) \geq \varepsilon$. If $D(X_1, X_2) \geq \varepsilon$, then there exists an $\alpha \in [0,1]$ such that either $\left|\underline{X_1}^{\alpha} - \underline{X_2}^{\alpha}\right| \geq \varepsilon$ or $\left|\overline{X_1}^{\alpha} - \overline{X_2}^{\alpha}\right| \geq \varepsilon$ holds. Now without loss of generality, assume that $\left|\underline{X_1}^{\alpha} - \underline{X_2}^{\alpha}\right| \geq \varepsilon$. If $\underline{X_1}^{\alpha} \leq \underline{X_2}^{\alpha} - \varepsilon$, then there is not a $Y \in [X_2, Y_2]$ such that $D(Y, X_1) < \varepsilon$. This contradicts (i). If $\underline{X_2}^{\alpha} + \varepsilon \leq \underline{X_1}^{\alpha}$, then there is not an $X \in [X_1, Y_1]$ such that $D(X, X_2) < \varepsilon$, which contradicts (ii). Similar contradictions can be obtained for the case $D(Y_1, Y_2) \geq \varepsilon$. Therefore, (i) and (ii) together imply (2). □

The following result is straightforward.

Theorem 2.5. *A sequence* $([X_n, Y_n])_{n\in\mathbb{N}}$ *in* $(\mathbf{I}[L(\mathbb{R})],\ D_I)$ *is convergent to* $[X,Y] \in \mathbf{I}[L(\mathbb{R})]$ *if, and only if,* $D(X_n, X) \longrightarrow 0$ *and* $D(Y_n, Y) \longrightarrow 0$.

Theorem 2.6. *The pair* $(\mathbf{I}[L(\mathbb{R})],\ D_I)$ *is a complete metric space.*

Proof. Let $([X_n, Y_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbf{I}[L(\mathbb{R})],\ D_I)$. Then $D(X_m, X_n) \longrightarrow 0$ and $D(Y_m, Y_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$. Since $(L(\mathbb{R}),\ D)$ is complete (see [10]), there exist $X, Y \in L(\mathbb{R})$ such that $D(X_n, X) \longrightarrow 0$ and $D(Y_n, Y) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence we have $[X_n, Y_n] \longrightarrow [X, Y] \in \mathbf{I}[L(\mathbb{R})]$ in the metric D_I, which completes the proof. □

Proposition 2.7. *The metric space* $(\mathbf{I}[L(\mathbb{R})],\ D_I)$ *is not separable.*

Proof. We will construct an uncountable subset $A \subset \mathbf{I}[L(\mathbb{R})]$ with the property that $D_I(X,Y) = 1$ for all $X, Y \in A$. To this end, let us define a fuzzy number φ_y via

$$\varphi_y(t) = \begin{cases} 1, & t = 1 \\ y, & t \in [0,1) \\ 0, & t \notin [0,1] \end{cases}$$

for each $y \in (0,1)$. Then it is easy to see that

$$\varphi_y^\alpha = \begin{cases} [0,1], & \text{if } 0 \le \alpha \le y \\ [1,1], & \text{if } y < \alpha \le 1 \end{cases}.$$

Now consider the order intervals $\left[\widetilde{0},\ \varphi_y\right]$ and $\left[\varphi_z,\ \widetilde{1}\right]$ such that

$$0 < z < y < 1.$$

Thus we have

$$D_I\left(\left[\widetilde{0},\ \varphi_y\right],\ \left[\varphi_z,\ \widetilde{1}\right]\right) = 1.$$

Now we can say that the intersection of the open balls $B\left(\left[\widetilde{0},\ \varphi_y\right]; \frac{1}{3}\right)$ and $B\left(\left[\varphi_z,\ \widetilde{1}\right]; \frac{1}{3}\right)$ in $\mathbf{I}[L(\mathbb{R})]$ is empty since $\varphi_y \prec \varphi_z$ and $D\left(\varphi_y,\ \varphi_z\right) = 1$. We can find uncountably many of such open balls. Now if M is a dense subset of $\mathbf{I}[L(\mathbb{R})]$, then each of such open balls must contain an element of M. Hence M cannot be countable. Since M is arbitrary, we can say that the pair $(\mathbf{I}[L(\mathbb{R})],\ D_I)$ cannot have a countable dense subset, and hence the result. □

Corollary 2.8. *The metric space* $(\mathbf{I}[L(\mathbb{R})],\ D_I)$ *is not compact and hence not totally bounded, since it is complete by Theorem 2.6.*

Remark 2.9. Let us reconsider the order intervals

$$\left[\widetilde{0},\ \varphi_y\right] \text{ and } \left[\varphi_z,\ \widetilde{1}\right],$$

constructed in the proof of Proposition 2.7. We see that although the order intervals $\left[\widetilde{0},\ \varphi_y\right]$ and $\left[\varphi_z,\ \widetilde{1}\right]$ are not degenerate, for every $X \in \left[\widetilde{0},\ \varphi_y\right]$ and $Y \in \left[\varphi_z,\ \widetilde{1}\right]$ we have $D(X,Y) = 1$. In other words, we can find order intervals $[X_1, Y_1],\ [X_2, Y_2] \in \mathbf{I}[L(\mathbb{R})]$ with $D(Y_1, X_2) = d$ such that there exists an $X \in [X_1, Y_1]$ with $X \neq Y_1$ and

$D(X, X_2) = d$, which is impossible in the case of interval numbers. This situation shows that the theory of order intervals of fuzzy numbers may significantly differ from the theory of interval numbers.

In what follows, we consider some geometric properties of an order interval of fuzzy numbers.

Proposition 2.10. *The metric diameter of an order interval* $[X_1, Y_1] \in \mathbf{I}\,[L(\mathbb{R})]$ *is equal to* $D\,(X_1, Y_1)$. *We denote this by*

$$diam\ [X_1, Y_1] = D\,(X_1, Y_1)\,.$$

Proof. By definition of the diameter of a set in a metric space, we can write

$$\text{diam}\ [X_1, Y_1] = \sup_{X,Y \in [X_1, Y_1]} D(X, Y).$$

On the other hand, we have

$$D\,(X_1, Y_1) = \sup_{\alpha\in[0,1]} \max\ \left\{ \left|\underline{X_1}^{\alpha} - \underline{Y_1}^{\alpha}\right|,\ \left|\overline{X_1}^{\alpha} - \overline{Y_1}^{\alpha}\right| \right\}.$$

Since $X, Y \in [X_1, Y_1]$, we have

$$\sup_{\alpha\in[0,1]} \max \left\{ \left|\underline{X}^{\alpha} - \underline{Y}^{\alpha}\right|,\ \left|\overline{X}^{\alpha} - \overline{Y}^{\alpha}\right| \right\} \leq \sup_{\alpha\in[0,1]} \max \left\{ \left|\underline{X_1}^{\alpha} - \underline{Y_1}^{\alpha}\right|,\ \left|\overline{X_1}^{\alpha} - \overline{Y_1}^{\alpha}\right| \right\},$$

that is, $D(X, Y) \leq D\,(X_1, Y_1)$ for every $X, Y \in [X_1, Y_1]$. Hence

$$\text{diam}\ [X_1, Y_1] \leq D\,(X_1, Y_1)\,.$$

Moreover, we have $D\,(X_1, Y_1) \leq \text{diam}\ [X_1, Y_1]$, and hence the equality follows. □

Analogous to the case of interval numbers, we will call the *diameter* of an order interval $[X, Y] \in \mathbf{I}\,[L(\mathbb{R})]$ as the *width* of $[X, Y]$, and we will write $w\,[X, Y]$ instead of diam $[X, Y]$.

Proposition 2.11. *The function* $f : (\mathbf{I}\,[L(\mathbb{R})]\,,\ D_I) \longrightarrow \mathbb{R}$ *defined by*

$$f\,([X, Y]) = w\,[X, Y]$$

is Lipschitz continuous.

Proof. Let $[X_1, Y_1]\,,\ [X_2, Y_2] \in \mathbf{I}\,[L(\mathbb{R})]$. Then we can write

$$\begin{aligned} |D\,(X_1, Y_1) - D\,(X_2, Y_2)| &\leq D\,(X_1, X_2) + D\,(Y_1, Y_2) \\ &\leq 2\ \max \{D\,(X_1, X_2)\,,\ D\,(Y_1, Y_2)\}\,, \end{aligned}$$

that is,

$$|f\,([X_1, Y_1]) - f\,([X_2, Y_2])| \leq 2\ D_I\,([X_1, Y_1]\,,\ [X_2, Y_2])\,,$$

as desired. Hence we can say that if $[X_1, Y_1]\,,\ [X_2, Y_2] \in \mathbf{I}\,[L(\mathbb{R})]$ such that

$$D_I\,([X_1, Y_1]\,,\ [X_2, Y_2]) < \varepsilon,$$

then $|w\,[X_1, Y_1] - w\,[X_2, Y_2]| < 2\varepsilon.$ □

3. An Application

In this section, we give a possible application of the metric D_I.

Suppose that we observe a sensitive item (think of an elementary particle in the study of quantum mechanics), and we want to measure a certain property P of this particle via 4 different measurement devices. In such a case, it is highly possible that we encounter some uncertainties due to the measurement process. For instance, different devices may produce different results or the measurement process itself may produce uncertainties, etc.

Assume that we measure the property P of the particle via the device D_1 sufficiently many times in regular periods because the particle is so sensitive to the observations and outer conditions and that the device prints out fuzzy number shaped functions (waves) to model the property P of the particle. After the measurement process, let us record these fuzzy numbers obtained from the device D_1 and place them in an order interval $[X_1, Y_1]$ of fuzzy numbers. Hence we get fuzzy bounds for the property P of the particle. Similarly, let us repeat the measurement with the devices D_2, D_3, D_4 and place the fuzzy numbers obtained from each device into the order intervals $[X_2, Y_2]$, $[X_3, Y_3]$ and $[X_4, Y_4]$, respectively.

Now suppose that we use the metric D_I in order to compute the distances between these order intervals, and find that

$$\begin{aligned} D_I([X_1, Y_1], [X_2, Y_2]) &= 10.14 \\ D_I([X_1, Y_1], [X_3, Y_3]) &= 0.2 \\ D_I([X_1, Y_1], [X_4, Y_4]) &= 0.9 \\ D_I([X_2, Y_2], [X_3, Y_3]) &= 11.9 \\ D_I([X_2, Y_2], [X_4, Y_4]) &= 9.0 \\ D_I([X_3, Y_3], [X_4, Y_4]) &= 0.7. \end{aligned}$$

By definition of D_I, the distance between two order intervals is close to zero means that these order intarvals are close to each other. Hence this situation may give an idea about the consistency of the measurements and the devices. For instance, the fact that the order interval $[X_2, Y_2]$ is far from the other order intervals may give an idea about the error range and consistency of D_2. Moreover, if we consider the fact that the devices D_1 and D_3 produce order intervals close to each other, we may predict that these two devices would be more efficient on the final decision about the results of the experiment.

4. Conclusion

A vast majority of information we encounter on most processes tends to be fuzzy and linguistic in form. Thus we need to reduce such complicated processes to rigorous mathematical forms. There are many alternative solutions to such problems in the theory of fuzzy sets. In this work, we have considered a mathematical modelling of fuzzy data by using order intervals of fuzzy numbers, which might be very useful in processes that carry high uncertainty and fuzziness. As a first step, we have introduced a metric on the set of all order intervals of fuzzy numbers in order to discriminate between possible decisions to be made at the end of the process. We have also investigated some geometric properties of such order intervals, and finally presented an application. We think that the modelling of fuzzy data by using order intervals of fuzzy numbers could provide many useful applications in most areas of fuzzy decision making.

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