A NOTE ON SOFT TOPOLOGICAL SPACES

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Abstract. This paper demonstrates the redundancies concerning the increasing popular "soft set" approaches to general topologies. It is shown that there is a complement preserving isomorphism (preserving arbitrary $\widetilde{\bigcup}$ and arbitrary $\widetilde{\bigcap}$) between the lattice $(\mathcal{ST}_E(X,E), \widetilde{\subset})$ of all soft sets on X with the whole parameter set E as domains and the powerset lattice $(\mathcal{P}(X \times E), \subseteq)$ of all subsets of $X \times E$. It therefore follows that soft topologies are redundant and unnecessarily complicated in theoretical sense.

1. Introduction

In 1999, Molodtsov [14] initiated the theory of soft sets, which provided a new mathematical tool for dealing with some uncertainties that traditional tools can not handle efficiently. Research works on soft set theory and its applications are progressing rapidly in various fields, including topology [3, 5, 8, 13, 15, 16], algebra [1, 2, 6, 9, 10], decision making [4, 11, 12] and so on.

Shabir and Naz [16] first introduced the concept of soft topological spaces. They defined basic notions of soft topological spaces such as soft open, soft closure, soft subspace and soft separation axioms. Consequently, Hussian and Ahmad [8], Çağan and Karataş [3], continued introducing new concepts and investigated their properties. All these related works are not only fundamental for further research on soft topological spaces, but also strengthen the foundations of the theory of soft topology. Although appealing ideas lying in approach to soft set theory and soft topology were embraced by many researchers working in the field of fuzzy set theory and general topology, there remains one principal question concerning the theoretical background of these programs of research:

Question: Are the ideas of soft topologies mathematically redundant, and if so, in which sense does the redundancy occur?

In [7], Gutierrez Garcia and Rodabaugh used this kind of idea to show the redundancies of interval-valued sets, grey sets, vague sets, interval-valued intuitionistic sets, intuitionistic fuzzy sets and topologies. In [17], Shi and Pang showed the redundancy of fuzzy soft topologies from a level of powerset. The aim of this paper is to resolve the similar question with respect to soft topologies. We shall show that soft topologies are redundant and unnecessarily complicated in theoretical sense. Moreover, we also show that the lattice $(\mathcal{ST}(X,E), \widetilde{\subset})$ of all soft sets on X with

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respect to the parameter set E and the powerset lattice $(\mathcal{P}(X\times E),\subseteq)$ of all subsets of $X\times E$ are isomorphic (preserving arbitrary $\widetilde{\bigcup}$ and arbitrary $\widetilde{\bigcap}$).

2. Preliminaries

Let X be a set and E be a set of parameters. Let $\mathcal{P}(X)$ denote the powerset of X.

Definition 2.1. [14] A pair (F,A) is called a soft set over X if $A\subseteq E$ and F is a mapping given by $F:A\to\mathcal{P}(X)$. The family of all soft sets over X with respect to the parameter set E is denoted by $\mathcal{ST}(X,E)$.

For convenience, let $\mathcal{ST}_E(X,E)$ denote the subset of $\mathcal{ST}(X,E)$, consisted of all soft sets whose domains are the parameter set E.

Definition 2.2. [13] Let (F,A) and (G,B) be two soft sets. We call (F,A) a subset of (G,B), denoted by $(F,A)\widetilde{\subset}(G,B)$, if

(1) $A\subseteq B$;

(2) $F(e)\subseteq G(e)$ for all $e\in A$.

(F,A) equals to (G,B), denoted by $(F,A)=(G,B)$, if $(F,A)\widetilde{\subset}(G,B)$ and $(G,B)\widetilde{\subset}(F,A)$.

Definition 2.3. [12] A soft set (F,A) over X is called a null soft set, denoted by Φ, if $F(e)=\emptyset$ for all $e\in A$.

Definition 2.4. [16] Let Y be a nonempty subset of X. Then $\widetilde{Y}$ denotes the soft set (Y,E) over X in which $Y(e)=Y$ for all $e\in E$. In particular, (X,E) will be denoted by $\widetilde{X}$.

Definition 2.5. [12] The union of two soft sets (F,A) and (G,B) over X is the soft set (H,C), where $C=A\cup B$, and $\forall e\in C$,

$$H(e)=\begin{cases} F(e), & \text{if } e\in A-B,\\ G(e), & \text{if } e\in B-A,\\ F(e)\cup G(e), & \text{if } e\in A\cap B.\end{cases}$$

We write $(F,A)\widetilde{\cup}(G,B)=(H,C)$.

Definition 2.6. [6] The intersection of two soft sets (F,A) and (G,B) over X is the soft set (H,C), where $C=A\cap B$, and $\forall e\in C$, $H(e)=F(e)\cap G(e)$. We write $(F,A)\widetilde{\cap}(G,B)=(H,C)$.

Definition 2.7. [16] For a soft set (F,A) over X, the relative complement of (F,A) is denoted by $(F,A)^{'}$ and is defined by $(F,A)^{'}=(F^{'},A)$, where $F^{'}:A\to\mathcal{P}(X)$ is a mapping given by $F^{'}(e)=F(e)^c$ for all $e\in A$ and $F(e)^c=X-F(e)$.

3. Main Results

We first generalize finite union and finite intersection of soft sets to arbitrary union and arbitrary intersection, respectively.

Definition 3.1. The union of a family of soft sets $\{(F_i, A_i)\}_{i\in I}$ (I is arbitrary) is the soft set (F, A), where $A = \bigcup_{i\in I} A_i$ and $F : A \to \mathcal{P}(X)$ is defined by

$$\forall e \in A, \quad F(e) = \bigcup_{e\in A_i} F_i(e).$$

We write $(F, A) = \widetilde{\bigcup}_{i\in I}(F_i, A_i)$.

Definition 3.2. The intersection of a family of soft sets $\{(F_i, A_i)\}_{i\in I}$ (I is arbitrary) is the soft set (F, A), where $A = \bigcap_{i\in I} A_i$ and $F : A \to \mathcal{P}(X)$ is defined by

$$\forall e \in A, \quad F(e) = \bigcap_{e\in A_i} F_i(e).$$

We write $(F, A) = \widetilde{\bigcap}_{i\in I}(F_i, A_i)$. In fact, $F(e) = \bigcap_{i\in I} F_i(e)$.

Remark 3.3. It is routine to check that Definition 2.5 (resp. Definition 2.6) is a special case of Definition 3.1 (resp. Definition 3.2) whenever I consists of two elements.

By Definition 2.2, we know $\widetilde{\subset}$ is in fact a partial order on $\mathcal{ST}(X, E)$, i.e., $(\mathcal{ST}(X, E), \widetilde{\subset})$ is a poset. Moreover, Definitions 2.5 and 2.6 define the union $\widetilde{\cup}$ and the intersection $\widetilde{\cap}$. This brings up a natural question: Is $\widetilde{\subset}$ coordinated with $\widetilde{\cup}$ and $\widetilde{\cap}$? The following result demonstrates it.

Theorem 3.4. *Binary relation $\widetilde{\subset}$ is in accordance with $\widetilde{\cup}$ and $\widetilde{\cap}$, i.e., $\widetilde{\subset}$, $\widetilde{\cup}$ and $\widetilde{\cap}$ satisfy the connecting lemma, i.e., for each (F, A), $(G, B)\in \mathcal{ST}(X, E)$, the following are equivalent:*

(1) $(F, A)\widetilde{\subset}(G, B)$.
(2) $(F, A)\widetilde{\cup}(G, B) = (G, B)$.
(3) $(F, A)\widetilde{\cap}(G, B) = (F, A)$.

Proof. We prove only (1)⇔(2). (1)⇔(3) can be proved similarly.

(1)⇒(2) Since $(F, A)\widetilde{\subset}(G, B)$, it follows that $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Put $(H, C) = (F, A)\widetilde{\cup}(G, B)$. Then $C = A \cup B$ and for each $e \in C = B$,

$$H(e) = \begin{cases} G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B, \end{cases} = \begin{cases} G(e), & \text{if } e \in B - A, \\ G(e), & \text{if } e \in A \cap B, \end{cases} = G(e).$$

This proves $(H, C) = (G, B)$, i.e., $(F, A)\widetilde{\cup}(G, B) = (G, B)$.

(2)⇒(1) Since $(F, A)\widetilde{\cup}(G, B) = (G, B)$, it follows that $A \cup B = B$, which means $A \subseteq B$. Then for each $e \in A$,

$$F(e) \subseteq F(e) \cup G(e) = G(e).$$

This shows $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Hence $(F, A)\widetilde{\subset}(G, B)$. □

By Theorem 3.4, we conclude that $(\mathcal{ST}(X, E), \widetilde{\subset})$ is a lattice, and the supremum $\vee$ and the infimum $\wedge$ are just $\widetilde{\cup}$ and $\widetilde{\cap}$, respectively. Moreover, we have the following result.

Theorem 3.5. *$(\mathcal{ST}(X, E), \widetilde{\subset})$ and $(\mathcal{ST}_E(X, E), \widetilde{\subset})$ are complete lattices.*

Proof. It is obvious and the proof is omitted. We only point out that the arbitrary $\bigvee$ and arbitrary $\bigwedge$ in $\mathcal{ST}(X,E)$ and $\mathcal{ST}_E(X,E)$ both agree with $\widetilde{\bigcup}$ and $\widetilde{\bigcap}$, respectively. □

Definition 3.6. [16] Let $\tau \subseteq \mathcal{ST}_E(X,E)$. Then τ is called a soft topology on X if it satisfies the following axioms:

(T1) $\Phi, \widetilde{X}$ belong to τ;
(T2) The union of any number of soft sets in τ belongs to τ;
(T3) The intersection of any two soft sets in τ belongs to τ.

The triple (X,τ,E) is called a soft topological space over X. The members of τ are called open soft sets. A soft set (F,E) is called closed soft if its relative complement $(F,E)'$ belongs to τ.

Remark 3.7. (1) Since the soft topology τ is a subset of $\mathcal{ST}_E(X,E)$, i.e., all open soft sets are soft sets with E as domains. Especially, the null soft set Φ in the axiom (T1) is the soft set (F,E) defined by $F(e)=\emptyset$ for all $e\in E$.

(2) The union of any number of soft sets in τ is a special case of $\widetilde{\bigcup}$. Concretely, for $(F_i,E)\in\tau$ $(i\in I)$, the union $(F,E)=\widetilde{\bigcup}_{i\in I}(F_i,E)$ is defined by

$$\forall e\in E,\quad F(e)=\bigcup_{i\in I}F_i(e).$$

(3) Similarly, the intersection (F,E) of any number of soft sets $(F_i,E)\in\tau$ $(i\in I)$ has the following form:

$$\forall e\in E,\quad F(e)=\bigcap_{i\in I}F_i(e).$$

Next we discuss the relationship between $(\mathcal{ST}_E(X,E),\widetilde{\subset})$ and $(\mathcal{P}(X\times E),\subseteq)$. Firstly, the following lemma is necessary.

Lemma 3.8. *Let (F,A) be a soft set over X and define $\varphi((F,A))$ as follows:*

$$\varphi((F,A))=\bigcup_{e\in A}F(e)\times\{e\}.$$

Then $(x,e)\in\varphi((F,A))$ if and only if $e\in A$ and $x\in F(e)$.

Proof. Assume that $(x,e)\in X\times E$, then

$$\begin{aligned}(x,e)\in\varphi((F,A)) &\Leftrightarrow \exists e_1\in A \text{ s.t. } (x,e)\in F(e_1)\times\{e_1\}\\ &\Leftrightarrow e=e_1\in A \text{ and } x\in F(e_1)\\ &\Leftrightarrow e\in A \text{ and } x\in F(e).\end{aligned}$$

□

Corollary 3.9. *Let $(F,E)\in\mathcal{ST}_E(X,E)$. Then $(x,e)\in\varphi((F,E))$ if and only if $x\in F(e)$.*

Next, we shall show that the restriction mapping $\varphi|_{\mathcal{ST}_E(X,E)}:(\mathcal{ST}_E(X,E),\widetilde{\subset})\to(\mathcal{P}(X\times E),\subseteq)$ is an isomorphism. Moreover, the following result comprises the heart of the claim of restrictive redundancy for soft topologies.

Theorem 3.10. *The restriction mapping* $\varphi|_{\mathcal{ST}_E(X,E)} : (\mathcal{ST}_E(X,E), \widetilde{\bigcup}, \widetilde{\bigcap}, (\cdot)^{'}) \to (\mathcal{P}(X \times E), \bigcup, \bigcap, (\cdot)^c)$ *is an isomorphism.*

Proof. For convenience, we still denote $\varphi|_{\mathcal{ST}_E(X,E)}$ by φ. We first construct a mapping $\phi : (\mathcal{P}(X \times E), \subseteq) \to (\mathcal{ST}_E(X,E), \widetilde{\subset})$ by $\phi(U) = (F_U, E)$ for each $U \in \mathcal{P}(X \times E)$ such that $\forall e \in E,\ F_U(e) = \{x \in X : (x,e) \in U\}$.

For each $(F,E) \in \mathcal{ST}_E(X,E)$, $\phi \circ \varphi((F,E)) = (F_{\varphi((F,E))}, E)$. By Corollary 3.9, for each $e \in E$,

$$x \in F_{\varphi((F,E))}(e) \Leftrightarrow (x,e) \in \varphi((F,E)) \Leftrightarrow x \in F(e),$$

i.e., $F_{\varphi((F,E))} = F$. This shows $\phi \circ \varphi((F,E)) = (F,E)$.

For each $U \in \mathcal{P}(X \times E)$, $\varphi \circ \phi(U) = \varphi((F_U,E))$. By Corollary 3.9, it follows that for each $(x,e) \in X \times E$,

$$(x,e) \in \varphi \circ \phi(U) \Leftrightarrow x \in F_U(e) \Leftrightarrow (x,e) \in U,$$

i.e., $\varphi \circ \phi(U) = U$. This proves φ is a bijective mapping.

Considering the preservation of arbitrary $\widetilde{\bigcup}$ and arbitrary $\widetilde{\bigcap}$, let $(F_i,E) \in \mathcal{ST}_E(X,E)$ $(i \in I)$. Then denote $(F,E) = \widetilde{\bigcup}_{i\in I}(F_i,E)$ and $(G,E) = \widetilde{\bigcap}_{i\in I}(F_i,E)$. By Corollary 3.9, we have

$$\begin{aligned} & (x,e) \in \varphi((F,E)) \\ \Leftrightarrow\quad & x \in F(e) = \bigcup_{i\in I} F_i(e) \quad \text{(by Remark 3.7)} \\ \Leftrightarrow\quad & \exists i_0 \in I,\ \text{s.t. } x \in F_{i_0}(e) \\ \Leftrightarrow\quad & \exists i_0 \in I,\ (x,e) \in \varphi((F_{i_0},E)) \\ \Leftrightarrow\quad & (x,e) \in \bigcup_{i\in I} \varphi((F_i,E)), \end{aligned}$$

and

$$\begin{aligned} & (x,e) \in \bigcap_{i\in I} \varphi((F_i,E)) \\ \Leftrightarrow\quad & \forall i \in I,\ (x,e) \in \varphi((F_i,E)) \\ \Leftrightarrow\quad & \forall i \in I,\ x \in F_i(e) \\ \Leftrightarrow\quad & x \in \bigcap_{i\in I} F_i(e) = G(e) \\ \Leftrightarrow\quad & (x,e) \in \varphi((G,E)). \end{aligned}$$

This proves $\varphi|_{\mathcal{ST}_E(X,E)}$ preserves arbitrary $\widetilde{\bigcup}$ and arbitrary $\widetilde{\bigcap}$.

To check that φ is complement preserving, take any $(F,E) \in \mathcal{ST}_E(X,E)$ and $(x,e) \in X \times E$. Then

$$\begin{aligned} & (x,e) \in \varphi((F,E)^{'}) = \varphi((F^{'},E)) \\ \Leftrightarrow\quad & x \in F^{'}(e) = X - F(e) \quad \text{(by Corollary 3.9)} \\ \Leftrightarrow\quad & x \notin F(e) \\ \Leftrightarrow\quad & (x,e) \notin \varphi((F,E)) \quad \text{(by Corollary 3.9)} \\ \Leftrightarrow\quad & (x,e) \in \varphi((F,E))'. \end{aligned}$$

This shows $\varphi((F,E)^{'}) = \varphi((F,E))^{'}$, as desired. □

Claim. Theorem 3.10 implies that, soft topologies on X are equivalent to general topologies on the set $X \times E$. Therefore, all claims concerning general topology can be adopted to preclude any possible scheme based on soft topology.

4. Conclusion

As judged by the above criteria, we claim that soft topology is exactly a special subcase of general topology. Therefore, it makes no sense if researchers go on studying theoretical aspects of soft topology. Moreover, applied researchers should take care of soft set theory in applications.

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