DOUBLE FUZZY IMPLICATIONS-BASED RESTRICTION INFERENCE ALGORITHM

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Abstract. The main condition of the differently implicational inference algorithm is reconsidered from a contrary direction, which motivates a new fuzzy inference strategy, called the double fuzzy implications-based restriction inference algorithm. New restriction inference principle is proposed, which improves the principle of the full implication restriction inference algorithm. Furthermore, focusing on the new algorithm, we analyze the basic property of its solution, and then obtain its optimal solutions aiming at the problems of fuzzy modus ponens (FMP) as well as fuzzy modus tollens (FMT). Lastly, comparing with the full implication restriction inference algorithm, the new algorithm can make the inference closer, and generate more, better specific inference algorithms.

1. Introduction

Two basic forms of fuzzy inference (see [2,16,22]) are fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) as following:

FMP: for a given rule $A \rightarrow B$ and input $A^*$, to compute $B^*$ (output), \hspace{1cm} (1)

FMT: for a given rule $A \rightarrow B$ and input $B^*$, to compute $A^*$ (output), \hspace{1cm} (2)

where $A, A^* \in F(U)$, $B, B^* \in F(V)$, in which $F(U), F(V)$ denote the set of all fuzzy subsets of universe $U, V$, respectively. To deal with these forms, the classical method is the compositional rule of inference (CRI) algorithm, proposed by Zadeh in 1973 (see [7,9,10,32]). In order to improve the CRI algorithm, Wang [29] put forward the full implication inference algorithm, which is broadly investigated by lots of scholars, see e.g. [13,15,33]. It is found that the full implication inference algorithm possesses many advantages, which are embodied as strict logic basis, reversibility and so on (see [14,18,20,24]).

The basic idea of the full implication inference algorithm is to seek out the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) such that

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v))$$

takes its maximum for any $u \in U$, $v \in V$ (where $\rightarrow$ is a fuzzy implication on $[0, 1]$, see Definition 2.2), in which (3) is called the basic formula of the full implication inference algorithm. Moreover it is generalized to the $\alpha$-full implication inference algorithm.

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algorithm, whose solution is the smallest \( B^* \in F(V) \) (or the largest \( A^* \in F(U) \)) making

\[
(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \geq \alpha
\]

hold for any \( u \in U, v \in V \) (where \( \alpha \in (0, 1] \)). Here (4) implies that the basic formula (3) should be greater than or equal to \( \alpha \), and (4) is obviously equivalent to

\[
(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \in [\alpha, 1].
\]

(5)

Then, what happens if the basic formula (3) belongs to the complementary set of \([\alpha, 1]\)? That is,

\[
(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \in [0, \alpha).
\]

(6)

This motivates a new, interesting strategy called the full implication restriction inference algorithm [23] (which provides necessary theoretical foundation for obtaining some performance measure of a new type of fuzzy controllers [21]), whose solution is the largest \( B^* \in F(V) \) (or the smallest \( A^* \in F(U) \)) making (6) hold for any \( u \in U, v \in V \).

On the other hand, although the full implication inference algorithm has many acknowledged advantages mentioned above, the effect of the full implication inference algorithm is imperfect (due to its inferior response ability and practicability) from the viewpoint of some kind of fuzzy controller (see [8,11,12]). To solve this problem, Tang and Liu generalized in [25] the full implication inference algorithm to the differently implicational universal triple I method of (1, 2, 2) type (the differently implicational inference algorithm for short), and then proposed the \( \alpha \)-differently implicational inference algorithm. The \( \alpha \)-differently implicational inference algorithm seeks the smallest \( B^* \in F(V) \) (or the largest \( A^* \in F(U) \)) making

\[
(A(u) \rightarrow_{\rightarrow_1} B(v)) \rightarrow_{\rightarrow_2} (A^*(u) \rightarrow_{\rightarrow_2} B^*(v)) \in [\alpha, 1]
\]

(7)

hold for any \( u \in U, v \in V \), where double fuzzy implications \( \rightarrow_{\rightarrow_1} \) and \( \rightarrow_{\rightarrow_2} \) can be different (in which \( \alpha \in (0, 1] \)). Moreover, it was found that the differently implicational inference algorithm can obtain more usable fuzzy controllers comparing with the CRI algorithm and full implication inference algorithm (see [25]). Later, the reversibility of differently implicational inference algorithm was discussed for FMT (where \( \rightarrow_{\rightarrow_2} \) employed \( I_L \), and \( \rightarrow_{\rightarrow_1} \) respectively took seven familiar fuzzy implications), and obtained excellent conclusion (see [26]). In [27], the differently implicational inference algorithm was researched for FMP (where \( \rightarrow_{\rightarrow_2} \) employed \( I_{FD} \)) and applied to pattern recognition of textual emotion polarity.

For the differently implicational inference algorithm, its main condition (7) can be similarly reconsidered from a contrary direction, that is,

\[
(A(u) \rightarrow_{\rightarrow_1} B(v)) \rightarrow_{\rightarrow_2} (A^*(u) \rightarrow_{\rightarrow_2} B^*(v)) \in [0, \alpha).
\]

(8)

The fuzzy inference strategy derived from (8) is called the double fuzzy implications-based restriction inference algorithm (DFI-restriction inference algorithm for short), which is the research aim of this paper.
2. Preliminaries

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is said to be a t-norm if $T$ is associative, increasing and commutative and satisfies the condition $T(1,a) = a$ ($a \in [0,1]$).

Definition 2.2. [1, 5] A fuzzy implication on $[0,1]$ is a function $I : [0,1]^2 \to [0,1]$ satisfying the conditions as follows:

(C1) $I$ is decreasing in the first variable,
(C2) $I$ is increasing in the second variable,
(C3) $I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$.

$I(a,b)$ is also written as $a \rightarrow b$ for any $a, b \in [0,1]$. It is easy to find that for every fuzzy implication, $I(0,a) = I(a,1) = 1$ ($a \in [0,1]$) and obviously $I(0,1) = 1$.

Definition 2.3. A function $I : [0,1]^2 \to [0,1]$ is said to be an R-implication if there exist a t-norm $T$ such that

$$I(a,b) = \sup\{x \in [0,1] \mid T(a,x) \leq b\}, \quad a,b \in [0,1]. \quad (9)$$

Definition 2.4. Let $T$ and $I$ be two $[0,1]^2 \to [0,1]$ functions, $(T,I)$ is called a residual pair or, $T$ and $I$ are residual to each other, if the following residual condition holds (iff denotes “if and only if”):

$$T(a,b) \leq c \text{ iff } b \leq I(a,c) \quad (a,b,c \in [0,1]). \quad (10)$$

Proposition 2.5. [6] Let $T$ be a t-norm, then the following statements are equivalent:

(i) $T$ is left-continuous;
(ii) $T$ and $I$ form a residual pair, where $I$ is obtained from $(9)$.

Lemma 2.6. [30, 31] Suppose that $T$ is a left-continuous t-norm on $[0,1]$, and that $I$ is an R-implication derived from $T$, then $(T,I)$ is a residual pair, and $I$ satisfies the following conditions:

(C4) $a \leq I(b,c) \iff b \leq I(a,c)$,
(C5) $I(a, I(b,c)) = I(b, I(a,c))$,
(C6) $I(T(a,b),c) = I(a, I(b,c))$,
(C7) $I(1,a) = a$ ($a \in [0,1]$),
(C8) $a \leq b \iff I(a,b) = 1$,
(C9) $I(\sup_{x \in X} x, a) = \inf_{x \in X} I(x,a)$,
(C10) $I(a, \inf_{x \in X} x) = \inf_{x \in X} I(a,x)$,

where $a, b, c, x \in [0,1]$ and $X \subset [0,1]$, $X \neq \emptyset$.

It is easy to get Proposition 2.7 from Theorem 3.1 in [25].

Proposition 2.7. Let $I$ be a fuzzy implication satisfying

(C11) $I(a,b)$ is right-continuous w.r.t. $b$ ($a \in [0,1]$), $b \in [0,1]$),
(C12) $x \in [0,1] \mid I(a,x) = 1 \neq \emptyset$ ($a \in [0,1]$),

then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(a,b) = \inf\{x \in [0,1] \mid b \leq I(a,x)\}, \quad a,b \in [0,1]$$
is residual to \( I \), and (9) holds.

**Remark 2.8.** It is easy to verify that the \( R \)-implication derived from left-continuous \( t \)-norm also satisfies (C11) and (C12).

**Definition 2.9.** Let \( Z \) be any nonempty set and \( F(Z) \) the set of all fuzzy subsets on \( Z \), define partial order relation \( \leq_F \) on \( F(Z) \) (according to pointwise order) as: \( A \leq_F B \) iff \( A(z_0) \leq B(z_0) \) for any \( z_0 \in Z \), where \( A, B \in F(Z) \).

**Lemma 2.10.** \( < F(Z), \leq_F > \) is a complete lattice.

### 3. The DFI-restriction Inference Algorithm for FMP

For the FMP problem (1), from the viewpoint of the double fuzzy implications-based restriction inference algorithm, we can obtain the following restriction inference principle:

**DFI-restriction Inference Principle for FMP:** The conclusion \( B^* \) of FMP problem (1) is the largest fuzzy set satisfying (8) in \( < F(V), \leq_F > \).

It is obvious that such DFI-restriction inference principle for FMP improves the full implication restriction inference principle w.r.t. (6) for FMP in [23].

**Definition 3.1.** Let \( A, A^* \in F(U), B \in F(V) \), if \( B^* \) (in \( < F(V), \leq_F > \)) makes (8) hold for any \( u \in U, v \in V \), then \( B^* \) is called a DFI-restriction inference solution of FMP (DFI-solution of FMP for short).

**Theorem 3.2.** Assume that \( A, A^* \in F(U), B \in F(V), \alpha \in (0, 1] \). Then there exists a \( B^* \in F(V) \) as a DFI-solution of FMP iff the following inequality holds for any \( u \in U, v \in V \):

\[
(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) < \alpha. \tag{11}
\]

**Proof.** On the one hand, if (11) holds, then we take \( B^*(v) \equiv 0 \), thus \( B^* \) obviously satisfies (8), and hence \( B^* \) is a DFI-solution of FMP. On the other hand, if there exists a \( B^* \in F(V) \) which is a DFI-solution of FMP, then \( B^* \) satisfies (8), and because \( \to_2 \) satisfies (C2) we have that \( A^*(u) \to_2 B^*(v) \geq A^*(u) \to_2 0 \), and \( \alpha > (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \geq (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) \), i.e. (11) holds.

Similar to Theorem 3.2, we can prove Proposition 3.3.

**Proposition 3.3.** Suppose that \( D_1 \) is a DFI-solution of FMP, and that \( D_2 \leq_F D_1 \) (in which \( D_1, D_2 \in F(V), < F(V), \leq_F > \)). Then \( D_2 \) is a DFI-solution of FMP.

**Remark 3.4.** Suppose that (11) holds. For a DFI-solution of FMP \( D_1^* \), every fuzzy set \( D_2^* \) which is less than \( D_1^* \), will be a DFI-solution of FMP (see Proposition 3.3, where \( D_1^*, D_2^* \in F(V), < F(V), \leq_F > \)). This means that there are many DFI-solutions of FMP, including \( D_2^*(v) \equiv 0 \) \((v \in V)\). The last \( D_2^* \) is a special solution, for which (8) always holds no matter what major premise \( A \to_1 B \) and minor premise \( A^* \) are adopted. Therefore, when the optimal DFI-solution of FMP exists, it should be the largest one; in other words, it should be the supremum of all solutions.
Theorem 3.5. If \( \rightarrow_2 \) is a fuzzy implication satisfying (C11) and (C12), and \( T \) is the function residual to \( \rightarrow_2 \), and \( A, A^* \in F(U), B \in F(V), \alpha \in (0, 1] \), (11) holds, then the supremum of DFI-solutions of FMP can be computed as follows:

\[
B^*(v) = \inf_{u \in U} T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha)), \ v \in V.
\]  

(12)

Proof. Since \( \rightarrow_2 \) is a fuzzy implication satisfying (C11) and (C12), it follows from Proposition 2.7 that the residual condition (10) holds.

Let \( G_1 = \{ v \in V \mid B^*(v) = 0 \} \), and \( G_2 = \{ v \in V \mid B^*(v) > 0 \} \). Suppose that \( C \in F(V) \), and that \( C(v) = 0 \) for \( v \in G_1 \), and that \( C(v) < B^*(v) \) for \( v \in G_2 \). We shall show that \( C \) is a DFI-solution of FMP, that is, the following inequality holds for any \( u \in U, v \in V \):

\[
(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 C(v)) < \alpha.
\]  

(13)

If \( v \in G_1 \), then it follows from (11) that \( C(v) = 0 \) satisfies (13) for any \( u \in U \).

If \( v \in G_2 \), then it follows from (12) and \( C(v) < B^*(v) \) that

\[
C(v) < T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha))
\]  

(14)

holds for any \( u \in U \). Suppose, on the contrary, that (13) does not hold. Then there exist \( u_0 \in U \) and \( v_0 \in V \) such that

\[
(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (A^*(u_0) \rightarrow_2 C(v_0)) \geq \alpha
\]

holds (obviously \( v_0 \in G_2 \)). Thus it follows from the residual condition (10) that

\[
T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq A^*(u_0) \rightarrow_2 C(v_0), \text{ and } T(A^*(u_0), T(A(u_0) \rightarrow_1 B(v_0), \alpha)) \leq C(v_0),
\]

which contradicts (14). Thus (13) holds for any \( u \in U, v \in V \). As a result, \( C \) is a DFI-solution of FMP.

Next, we shall check that \( B^* \) determined by (12) is the supremum of DFI-solutions of FMP. Assume that \( D \in F(V) \), and that there exists \( v_0 \in V \) such that \( D(v_0) > B^*(v_0) \). We shall prove that \( D \) is not a DFI-solution of FMP. In fact, it follows from (12) that there exists \( u_0 \in U \) such that

\[
D(v_0) > T(A^*(u_0), T(A(u_0) \rightarrow_1 B(v_0), \alpha))
\]

holds. We have from the residual condition (10) that

\[
T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq A^*(u_0) \rightarrow_2 D(v_0), \text{ and } \alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (A^*(u_0) \rightarrow_2 D(v_0)) \text{. Thus, } D \text{ is not a DFI-solution of FMP.}
\]

To sum up, \( B^* \) determined by (12) is the supremum of DFI-solutions of FMP. \( \square \)

It follows from Theorem 3.5 and Remark 2.8 that we have Corollary 3.6.

Corollary 3.6. If \( \rightarrow_2 \) is an \( R \)-implication derived from left-continuous \( t \)-norm \( T \), and \( A, A^* \in F(U), B \in F(V), \alpha \in (0, 1] \), (11) holds, then the supremum of DFI-solutions of FMP is as follows:

\[
B^*(v) = \inf_{u \in U} T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha)), \ v \in V.
\]
\[ I_L(a, b) = \begin{cases} 1, & a \leq b \\ a' + b, & a > b \end{cases} \] (Łukasiewicz implication);
\[ I_G(a, b) = \begin{cases} 1, & a \leq b \\ b, & a > b \end{cases} \] (Gödel implication);
\[ I_{Go}(a, b) = \begin{cases} 1, & a = 0 \\ a' \wedge b, & a \neq 0 \end{cases} \] (Goguen implication);
\[ I_{FD}(a, b) = \begin{cases} 1, & a \leq b \\ a' \vee b, & a > b \end{cases} \] (Fodor implication);
\[ I_{GR}(a, b) = \begin{cases} 1, & a \leq b \\ 0, & a > b \end{cases} \] (Gaines-Rescher implication);
\[ I_{Y}\omega(a, b) = \frac{a}{1 - \min\{1, ((1 - a)\omega + (1 - b)\omega)^{1/\omega}\}} \] (Yager implication);
\[ I_{R}(a, b) = \frac{a' + a \times b}{2 - (a + b - ab)^{1/\omega}} \] (Reichenbach implication);
\[ I_{E}\omega(a, b) = \begin{cases} 1, & a \leq b \\ 1 - \frac{1}{\omega}(a - \sqrt{1 - a} + \sqrt{1 - b}), & a > b \end{cases} \] where \( g(a, b) = \sqrt{1 - a} + \sqrt{1 - b} \);
\[ I_{R}(a, b) = \begin{cases} 1, & a \leq b \\ 1 - a + ab, & a > b \end{cases} \] (revised Reichenbach implication).

Moreover, the functions respectively residual to \( I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Y}\omega, I_{E}\omega \) are as follows.

\[ T_L(a, b) = \begin{cases} a + b - 1, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases} \]
\[ T_G(a, b) = a \wedge b, \]
\[ T_{Go}(a, b) = a \times b, \]
\[ T_{FD}(a, b) = \begin{cases} a \wedge b, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases} \]
\[ T_{GR}(a, b) = \begin{cases} a, & b > 0 \\ 0, & b = 0 \end{cases} \]
\[ T_{Y}\omega(a, b) = \begin{cases} \sqrt{a}, & a > 0 \\ 0, & a = 0 \end{cases} \]
\[ T_{R}(a, b) = \begin{cases} (a + b - 1)/a, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases} \]
\[ T_{Y}\omega(a, b) = \begin{cases} 1 - (g(a, b))^2, & g(a, b) \leq 1 \\ 0, & g(a, b) > 1 \end{cases} \] where \( g(a, b) = \sqrt{1 - a} + \sqrt{1 - b} \),
\[ T_{E}\omega(a, b) = ab/[2 - (a + b - ab)^{1/\omega}] \]
\[ T_{R}(a, b) = \begin{cases} \lfloor (a + b - 1)/a \rfloor \wedge a, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases} \]

Here \( T_{E}\omega \) is the Einstein product; and \( T_{Y}\omega \) is the t-norm of Yager (where \( \omega \) takes 0.5), which is defined as

\[ T_{Y}\omega(a, b) = I_1 - \min\{1, ((1 - a)\omega + (1 - b)\omega)^{1/\omega}\} \] in which \( \omega \in (0, \infty) \).

It is easy to know that \( T_L, T_G, T_{Go}, T_{FD}, T_{E}\omega, T_{Y}\omega \) are left-continuous t-norm, thus \( I_L, I_G, I_{Go}, I_{FD}, I_{Y}\omega, I_{E}\omega \) are R-implications derived from left-continuous t-norm.
Proposition 3.8. Suppose that $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$ in (8), and that (11) holds, then the supremum of DFI-solutions of FMP is as follows, respectively ($v \in V$):

(i) If $\rightarrow_2$ takes $I_L$, then $B^*(v) = \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}$;

(ii) If $\rightarrow_2$ takes $I_G$, then $B^*(v) = \inf_{u \in U} \{A^*(u) \land (A(u) \rightarrow_1 B(v))\} \land \alpha$;

(iii) If $\rightarrow_2$ takes $I_{Go}$, then $B^*(v) = \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v))\} \land \alpha$;

(iv) If $\rightarrow_2$ takes $I_{FD}$, then $B^*(v) = \inf_{u \in U} \{A^*(u) \land (A(u) \rightarrow_1 B(v))\} \land \alpha$;

(v) If $\rightarrow_2$ takes $I_{GR}$, then $B^*(v) = \inf_{u \in U} \{A^*(u)\}$;

(vi) If $\rightarrow_2$ takes $I_Y$, then $B^*(v) = \inf_{u \in U} \{\alpha \land (A(u) \rightarrow_1 B(v))\}$;

(vii) If $\rightarrow_2$ takes $I_R$, then $B^*(v) = \inf_{u \in U} \{[A^*(u) + \varphi_R(u,v) - 1]/A^*(u)]\}$ where $\varphi_R(u,v) = \{(A(u) \rightarrow_1 B(v)) + (\alpha - 1)/(A(u) \rightarrow_1 B(v))\}$;

(viii) If $\rightarrow_2$ takes $I_{Ya}$, then $B^*(v) = \inf_{u \in U} \{1 - (\sqrt{1 - \alpha} + \sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2\}$;

(ix) If $\rightarrow_2$ takes $I_{EP}$, then $B^*(v) = \inf_{u \in U} \{A^*(u) \land \varphi_{EP}(u,v)/[2 - A^*(u) - \varphi_{EP}(u,v) + A^*(u) \land \varphi_{EP}(u,v)]\}$ where $\varphi_{EP}(u,v) = \alpha \times (A(u) \rightarrow_1 B(v))/[2 - (A(u) \rightarrow_1 B(v)) - \alpha + \alpha \times (A(u) \rightarrow_1 B(v))]$;

(x) If $\rightarrow_2$ takes $I_{RR}$, then $B^*(v) = \inf_{u \in U} \{[A^*(u) + \varphi_{RR}(u,v) - 1]/A^*(u)] \land A^*(u)\}$ where $\varphi_{RR}(u,v) = \{(A(u) \rightarrow_1 B(v)) + (\alpha - 1)/(A(u) \rightarrow_1 B(v))\} \land (A(u) \rightarrow_1 B(v))$.

Proof. If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, then $\rightarrow_2$ is a fuzzy implication satisfying (C11) and (C12), thus it follows from Theorem 3.5 that the supremum of DFI-solutions of FMP is $B^*(v) = \inf_{u \in U} (T^*(u,T(A(u) \rightarrow_1 B(v),\alpha)), v \in V$ (where $T$ is the function residual to $\rightarrow_2$). Then, we need to get the specific expression of $B^*$. We only prove the cases of $B_L, I_{RR}$ as examples, the remainders can be proved similarly.

(i) Suppose that $\rightarrow_2$ takes $I_L$. It follows from Example 3.7 that $T_L$ is the function residual to $I_L$. By (11), we have that $A^*(u) > 0, A(u) \rightarrow_1 B(v) > 1 - A^*(u)$, and $1 - (A(u) \rightarrow_1 B(v)) > 1 - A^*(u) < \alpha$ hold for any $u \in U, v \in V$. We further get that $\{A(u) \rightarrow_1 B(v)\} + \alpha - 1$ and $A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 1 > 1$ hold for any $u \in U, v \in V$. Thus,

$$B^*(v) = \inf_{u \in U} T_L(A^*(u), T_L(A(u) \rightarrow_1 B(v),\alpha))$$

$$= \inf_{u \in U} T_L(A^*(u), ((A(u) \rightarrow_1 B(v)) + \alpha - 1))$$

$$= \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}, v \in V.$$
Then \( A^*(u) + \{(A(u) \rightarrow B(v)) + \alpha - 1\}/(A(u) \rightarrow B(v)) \} > 1 \) holds for any \( u \in U, v \in V \). Summarizing above, we obtain that

\[
B^*(v) = \inf_{u \in U} T_{RR}(A^*(u), T_{RR}(A(u) \rightarrow B(v), \alpha)) = \inf_{u \in U} T_{RR}(A^*(u), \varphi_{RR}(u, v)) = \inf_{u \in U} [(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u), v \in V
\]

where \( \varphi_{RR}(u, v) = [(A(u) \rightarrow B(v)) + \alpha - 1]/(A(u) \rightarrow B(v)) \wedge (A(u) \rightarrow B(v)). \)

**Theorem 3.9.** If \( \rightarrow \in \{ I_L, I_G, I_{GR}, I_{GB}, I_R, I_{Yu}, I_{EP}, I_{RR} \}, A, A^* \in F(U), B \in F(V), \alpha \in (0, 1], (11) holds, then the condition, which the supremum of DFI-solutions of FMP is the maximum, is as follows, respectively (for any \( u \in U, v \in V \)):

(i) Let \( \rightarrow \in I_L, A^*(u) + (A(u) \rightarrow B(v)) > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow B(v))\} \) holds;

(ii) Let \( \rightarrow \in I_G, (A^*(u) \wedge (A(u) \rightarrow B(v)))\wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow B(v))\} \) holds;

(iii) Let \( \rightarrow \in I_{GR}, A^*(u) \times (A(u) \rightarrow B(v)) > \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow B(v))\} \) holds;

(iv) Let \( \rightarrow \in I_{GB}, A^*(u) \times (A(u) \rightarrow B(v)) \wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow B(v))\} \) holds;

(v) Let \( \rightarrow \in I_{Yu}, A^*(u) > \inf_{u \in U} \{A^*(u)\} \) holds;

(vi) Let \( \rightarrow \in I_{EP}, A^*(u) \times \varphi_{EP}(u, v) > \inf_{u \in U} \{A^*(u) \times \varphi_{EP}(u, v)\} \) holds;

(vii) Let \( \rightarrow \in I_R, A^*(u) \wedge (A(u) \rightarrow B(v)) > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow B(v))\} \) holds;

(viii) Let \( \rightarrow \in I_{GR}, 1 - (\sqrt{1} - A^*(u) + \sqrt{1} - (A(u) \rightarrow B(v)) + \sqrt{1} - \alpha)^2 > \inf_{u \in U} [1 - (\sqrt{1} - A^*(u) + \sqrt{1} - (A(u) \rightarrow B(v)) + \sqrt{1} - \alpha)^2] \) holds;

(ix) Let \( \rightarrow \in I_{GB}, A^*(u) \times \varphi_{EP}(u, v) > \inf_{u \in U} [A^*(u) \times \varphi_{EP}(u, v)] \) holds;

(x) Let \( \rightarrow \in I_{Yu}, [(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u) > \inf_{u \in U} [(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u) \) holds.

**Proof.** Note that if the supremum \( B^* \) of DFI-solutions of FMP is a DFI-solution of FMP, then \( B^* \) is the maximum of DFI-solutions of FMP. Thus it is enough to prove that \( B^* \) is a DFI-solution of FMP (i.e., \( B^* \) should make (8) hold for any \( u \in U, v \in V \)). We still prove the cases of \( I_L, I_{RR} \) as examples.

(i) Let \( \rightarrow \in I_L \). It follows from Proposition 3.8 that the supremum of DFI-solutions of FMP is \( B^*(v) = \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow B(v)) + \alpha - 2\} \). By the condition given in (i), we obtain

\[
B^*(v) = \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow B(v)) + \alpha - 2\} = \alpha - 2 + \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow B(v))\} < \alpha - 2 + A^*(u) + (A(u) \rightarrow B(v)).
\]
Note that $0 \geq \alpha - 2 + (A(u) \rightarrow_1 B(v))$, then $A^*(u) \geq A^*(u) + \alpha - 2 + (A(u) \rightarrow_1 B(v)) > B^*(v)$ holds. Thus

$$A(u) \rightarrow_1 B(v) \geq \alpha - 1 + (A(u) \rightarrow_1 B(v))$$

$$= (1 - A^*(u)) + [\alpha - 2 + A^*(u) + (A(u) \rightarrow_1 B(v))]$$

$$> 1 - A^*(u) + B^*(v).$$

To sum up, we get $(u \in U, v \in V)$

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v))$$

$$= (A(u) \rightarrow_1 B(v)) \rightarrow_2 [1 - A^*(u) + B^*(v)]$$

$$= 1 - (A(u) \rightarrow_1 B(v)) + 1 - A^*(u) + B^*(v)$$

$$< 1 - (A(u) \rightarrow_1 B(v)) + 1 - A^*(u) + \alpha - 2 + A^*(u) + (A(u) \rightarrow_1 B(v))$$

$$= \alpha.$$

Thus $B^*$ makes (8) hold for any $u \in U, v \in V$, and hence it is a DFI-solution of FMP.

(x) Let $\rightarrow_2$ take $I_{RR}$. It follows from Proposition 3.8 that the supremum of DFI-solutions of FMP is as follows:

$$B^*(v) = \inf_{u \in U} \{ (A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u) \} \wedge A^*(u).$$

For convenience, denote $R_1(u, v) = A(u) \rightarrow_1 B(v)$. It follows from (11) that $A^*(u) > 0, R_1(u, v) > 1 - A^*(u)$, and $1 - R_1(u, v) \times A^*(u) < \alpha$ hold for any $u \in U, v \in V$. By the condition given in (x), we obtain

$$B^*(v) < \frac{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)}{A^*(u)}$$

$$= A^*(u) + \frac{[(R_1(u, v) + \alpha - 1)/R_1(u, v)] \wedge R_1(u, v)] - 1}{A^*(u)} \wedge A^*(u).$$

Further, we get $A^*(u) > B^*(v)$, and

$$B^*(v) < A^*(u) + \frac{[R_1(u, v) + \alpha - 1]/R_1(u, v)] - 1}{R_1(u, v) \times A^*(u)}$$

$$= A^*(u) + \frac{R_1(u, v) \times A^*(u) + \alpha - 1}{R_1(u, v) \times A^*(u)}$$

$$< A^*(u) + 1 - A^*(u) + A^*(u) \times B^*(v).$$

i.e., $R_1(u, v) > 1 - A^*(u) + A^*(u) \times B^*(v)$.

To sum up, we achieve $(u \in U, v \in V)$

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v))$$

$$= R_1(u, v) \rightarrow_2 [1 - A^*(u) + A^*(u)B^*(v)]$$

$$= 1 - R_1(u, v) + R_1(u, v) \times [1 - A^*(u) + A^*(u)B^*(v)]$$

$$= 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) \times B^*(v)$$

$$< 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) \times$$

$$[(R_1(u, v) \times A^*(u) + \alpha - 1)/(R_1(u, v) \times A^*(u))]$$

$$= 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) + \alpha - 1$$

$$= \alpha.$$

Therefore $B^*$ makes (8) hold for any $u \in U, v \in V$, and hence it is a DFI-solution of FMP. \qed
\[
\supremum \text{ of } \text{DFI-solutions of FMP}
\]

The supremum of DFI-solutions of FMP is as follows

\[
\begin{align*}
&I_2 \quad \text{The supremum } B^* \quad \text{The condition which } B^* \text{ is the maximum} \\
&I_L \quad \inf_{u \in U} \{ A^*(u) + (A(u) \rightarrow B(v)) \} + \alpha - 2 \quad A^*(u) + (A(u) \rightarrow B(v)) > \\
&I_G \quad \inf_{u \in U} \{ A^*(u) \wedge (A(u) \rightarrow B(v)) \} \wedge \alpha \quad (A^*(u) \wedge (A(u) \rightarrow B(v))) \wedge \alpha > \\
&I_Go \quad \inf_{u \in U} \{ A^*(u) \times (A(u) \rightarrow B(v)) \} \wedge \alpha \quad (A^*(u) \times (A(u) \rightarrow B(v))) \wedge \alpha > \\
&I_FD \quad \inf_{u \in U} \{ A^*(u) \wedge (A(u) \rightarrow B(v)) \} \wedge \alpha \quad (A^*(u) \wedge (A(u) \rightarrow B(v))) \wedge \alpha > \\
&I_GR \quad \inf_{u \in U} \{ A^*(u) \} \quad A^*(u) > B^* \\
&I_V \quad \inf_{u \in U} \{ A^*(u) \times (A(u) \rightarrow B(v)) \} \wedge \alpha \quad (A^*(u) \times (A(u) \rightarrow B(v))) \wedge \alpha > \\
&I_R \quad \inf_{u \in U} \{ A^*(u) + \varphi_R(u,v) - 1 \} / A^*(u) \quad [A^*(u) + \varphi_R(u,v) - 1] / A^*(u) > B^* \\
&I_{Vu} \quad \inf_{u \in U} \{ 1 - (\sqrt{1 - A^*(u)}) + \sqrt{1 - (A(u) \rightarrow B(v))} \} \quad 1 - (\sqrt{1 - A^*(u)}) + \sqrt{1 - (A(u) \rightarrow B(v))} > B^* \\
&I_{EP} \quad \inf_{u \in U} \{ A^*(u) \times \varphi_{EP}(u,v) / [2 - A^*(u)] \} \quad A^*(u) \times \varphi_{EP}(u,v) / [2 - A^*(u)] > B^* \\
&I_{RR} \quad \inf_{u \in U} \{ (A^*(u) + \varphi_{RR}(u,v) - 1) / A^*(u) \} \quad [(A^*(u) + \varphi_{RR}(u,v) - 1) / A^*(u)] > B^* \\
\end{align*}
\]

Table 1. Some Conclusions of the DFI-restriction Inference Algorithm for FMP

In the following Table 1 we show some conclusions of the DFI-restriction inference algorithm for FMP from Proposition 3.8 and Theorem 3.9.

**Example 3.10.** Let \( U = V = [0, 1] \), \( A(u) = (2 + u)/4 \), \( B(v) = (1 + 2v)/4 \), \( A^*(u) = (2 - u)/2 \), \( \alpha = 1/2 \), in which \( u, v \in V \). Assume that \( \rightarrow_2 = I_{G0}, \rightarrow_1 = I_L \) in the DFI-restriction inference algorithm for FMP. We now compute the supremum of DFI-solutions of FMP.

\[
A(u) \rightarrow B(v) = I_L(A(u), B(v)) = \begin{cases} 
1 - \frac{2 + u + 1 + 2v}{4}, & \text{if } \frac{2 + u + 1 + 2v}{4} > 1 + u > 2v; \\
1, & \text{if } \frac{2 + u + 1 + 2v}{4} \leq 1 + u \leq 2v.
\end{cases}
\]

Here (11) evidently holds. Then it follows from Proposition 3.8 (iii) that the supremum of DFI-solutions of FMP is as follows

\[
B^*(v) = \inf_{u \in [0,1]} \{ A^*(u) \times (A(u) \rightarrow B(v)) \} \wedge \alpha
\]

\[
= \inf_{u \in [0,1]} \{ A^*(u) \times (A(u) \rightarrow B(v)) \} \wedge \alpha \big\{ 1 + u > 2v \big\} \wedge \inf_{u \in [0,1]} \big\{ \frac{2 - u}{4} \big\} \big\{ 1 + u \leq 2v \big\}.
\]
(i) Suppose \( v = 1 \), then \( \{ u \in [0, 1] \mid 1 + u > 2v \} = \emptyset \), and \( 1 \in \{ u \in [0, 1] \mid 1 + u \leq 2v \} \). Note that \( \frac{2 - u}{u} \) is decreasing w.r.t. \( u \), thus we obtain \( B^*(v) = (\inf \emptyset) \land \frac{1}{4} = 1 \land \frac{1}{4} = \frac{1 + v}{8} \).

(ii) Suppose \( 1 > v \geq 1/2 \), then \( 1 \in \{ u \in [0, 1] \mid 1 + u > 2v \} \). Since \( \frac{2 - u}{u} \), \( \frac{2 - u + 2v}{4} \) are decreasing w.r.t. \( u \), we get \( B^*(v) = \frac{2 + 2v}{16} \land \frac{2 - (2v - 1)}{4} = \frac{1 + v}{8} \land \frac{3 - 2v}{4} = \frac{1 + v}{8} \), where \( \frac{1 + v}{8} < \frac{3 - 2v}{4} \) since \( 1 > v \geq 1/2 \).

(iii) Suppose \( 0 \leq v < 1/2 \), then \( 1 \in \{ u \in [0, 1] \mid 1 + u > 2v \} \), and \( \{ u \in [0, 1] \mid 1 + u \leq 2v \} = \emptyset \), thus we have \( B^*(v) = \frac{1 + v}{8} \land (\inf \emptyset) = \frac{1 + v}{8} \). Together we achieve \( B^*(v) = \frac{1 + v}{8} \), \( v \in V \).

Example 3.11. Let \( U, V, A, B, A^*, \alpha \) be the same as in Example 3.10. Assume that \( \rightarrow_2 = I_{G_0} \rightarrow_1 = I_{G_0} \) in the DFI-restriction inference algorithm for FMP, which degenerates into the full implication restriction inference algorithm for FMP (taking \( I_{G_0} \)). We now compute the supremum of DFI-solutions of FMP.

\[
A(u) \rightarrow_1 B(v) = I_{G_0}(A(u), B(v)) = \left\{ \begin{array}{ll}
\frac{1 + 2u}{4}, & \text{if } \frac{2 + u}{4} > \frac{1 + 2v}{4} \\
1, & \text{if } \frac{2 + u}{4} \leq \frac{1 + 2v}{4}
\end{array} \right.
\]

Here (11) obviously holds. Then we get from Proposition 3.8 (iii) that the supremum of DFI-solutions of FMP is as follows (\( v \in V \)):

\[
B^*(v) = \inf_{u \in U} \left( A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \right)
\]

\[
= \inf_{u \in [0, 1]} \left( A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1 + u > 2v \right) \land \inf_{u \in [0, 1]} \left( A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1 + u \leq 2v \right)
\]

\[
= \inf_{u \in [0, 1]} \left\{ \frac{2 - u}{4} \times \frac{1 + 2u}{2 + u} \mid 1 + u > 2v \right\} \land \inf_{u \in [0, 1]} \left\{ \frac{2 - u}{4} \mid 1 + u \leq 2v \right\}
\]

\[
= \inf_{u \in [0, 1]} \left\{ \frac{(2 - u)(1 + 2v)}{4(2 + u)} \mid 1 + u > 2v \right\} \land \inf_{u \in [0, 1]} \left\{ \frac{2 - u}{4} \mid 1 + u \leq 2v \right\}
\]

(i) Suppose \( v = 1 \), then \( \{ u \in [0, 1] \mid 1 + u > 2v \} = \emptyset \), and \( 1 \in \{ u \in [0, 1] \mid 1 + u \leq 2v \} \). Since \( \frac{2 - u}{u} \) is decreasing w.r.t. \( u \), we have

\[
B^*(v) = (\inf \emptyset) \land \frac{1}{4} = 1 \land \frac{1}{4} = \frac{1 + 2v}{12}.
\]

(ii) Suppose \( 1 > v \geq 1/2 \), then \( 1 \in \{ u \in [0, 1] \mid 1 + u > 2v \} \). Noting that \( \frac{2 - u}{4(2 + u)} \) are decreasing w.r.t. \( u \), we get

\[
B^*(v) = \frac{1 + 2v}{12} \land \frac{2 - (2v - 1)}{4} = \frac{1 + 2v}{12} \land \frac{3 - 2v}{4} = \frac{1 + 2v}{12},
\]

where \( \frac{1 + 2v}{12} < \frac{3 - 2v}{4} \) since \( 1 > v \geq 1/2 \).

(iii) Suppose \( 0 \leq v < 1/2 \), then \( 1 \in \{ u \in [0, 1] \mid 1 + u > 2v \} \), and \( \{ u \in [0, 1] \mid 1 + u \leq 2v \} = \emptyset \), thus we have

\[
B^*(v) = \frac{1 + 2v}{12} \land (\inf \emptyset) = \frac{1 + 2v}{12} \land 1 = \frac{1 + 2v}{12}.
\]
Together we obtain $B^*(v) = \frac{1 + 2v}{12}$, $v \in V$.

**Remark 3.12.** For the same $U, V, A, B, A^*, \alpha$ in Example 3.10 and Example 3.11, because

$$0 \leq v < 1 \Rightarrow \frac{1 + v}{8} > \frac{1 + 2v}{12},$$
$$v = 1 \Rightarrow \frac{1 + v}{8} = \frac{1 + 2v}{12},$$

the supremum of DFI-solutions of FMP (from the DFI-restriction inference algorithm) in Example 3.10 is larger than the one (from the full implication restriction inference algorithm) in Example 3.11. From the viewpoint of DFI-restriction inference principle for FMP (which seeks the largest $B^*$ satisfying (8)), the DFI-restriction inference algorithm for FMP in Examples 3.10 lets the inference closer, then it is more reasonable than the full implication restriction inference algorithm for FMP in Examples 3.11.

**Remark 3.13.** For the set of fuzzy implications $\{I_L, I_G, I_{G_0}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, it follows from the DFI-restriction inference algorithm that we can get

$$10 \times 10 = 100$$

kinds of specific fuzzy inference algorithms for FMP, where $\rightarrow_2, \rightarrow_1$ respectively employ these 10 fuzzy implications. Furthermore, there are

$$8 \times 10 + 1 = 81$$

kinds of actual specific algorithms for FMP (derived from the DFI-restriction inference algorithm), because the expression is independent of $\rightarrow_1$ for the case $\rightarrow_2 = I_{GR}$, and the expressions are the same for the cases $\rightarrow_2 = I_G$ and $\rightarrow_2 = I_{FD}$ (according to Table 1). But, from the full implication restriction inference algorithm, it is easy to find that there are only 10 kinds of specific algorithms for FMP (including 9 kinds of actual specific algorithms for FMP), which is shown as Table 2. As a result, the DFI-restriction inference algorithm can provide more and better specific algorithms for FMP (than the full implication restriction inference algorithm), so it is better than the full implication restriction inference algorithm.

### 4. The DFI-restriction Inference Algorithm for FMT

For the FMT problem (2), aiming at the double fuzzy implications-based restriction inference algorithm, we can get the following restriction inference principle (which similarly improves the full implication restriction inference principle w.r.t. (6) for FMT in [23]):

<table>
<thead>
<tr>
<th>Specific ones</th>
<th>100 kinds</th>
<th>10 kinds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual specific ones</td>
<td>81 kinds</td>
<td>9 kinds</td>
</tr>
</tbody>
</table>

**Table 2. Specific Fuzzy Inference Algorithms for FMP**
DFI-restriction Inference Principle for FMT: The conclusion $A^*$ of FMT problem (2) is the smallest fuzzy set satisfying (8) in $<F(U), \leq_F>$.  

**Definition 4.1.** Let $A \in F(U), B, B^* \in F(V)$, if $A^*$ (in $<F(U), \leq_F>$) makes (8) hold for any $u \in U$ and $v \in V$, then $A^*$ is called a DFI-restriction inference solution of FMT (DFI-solution of FMT for short).

Similar to Theorem 3.2 and Proposition 3.3, we can get Theorem 4.2 and Proposition 4.3.

**Theorem 4.2.** Assume that $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1]$. Then there exists an $A^* \in F(U)$ as a DFI-solution of FMT iff the following inequality holds for any $u \in U, v \in V$:

$$ (A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_2 B^*(v)) < \alpha. \quad (15) $$

**Proposition 4.3.** Suppose that $C_1$ is a DFI-solution of FMT, and that $C_1 \leq_F C_2$ (in which $C_1, C_2 \in <F(U), \leq_F>$). Then $C_2$ is a DFI-solution of FMT.

**Remark 4.4.** Suppose that (15) holds. For a DFI-solution of FMT $C_1^*$, every fuzzy set $C_2^*$ which is larger than $C_1^*$, will be a DFI-solution of FMT (see Proposition 4.3, where $C_1^*, C_2^* \in <F(U), \leq_F>$). This implies that there are many DFI-solutions of FMT, including $C_2^*(u) \equiv 1 \{u \in U\}$. The last $C_2^*$ is a special solution, since (8) always holds no matter what $A \rightarrow_1 B$ and $B^*$ are adopted. As a result, when the optimal DFI-solution of FMT exists, it should be the smallest one; in other words, it should be the infimum of all solutions.

**Example 4.5.** From Theorem 4.2, we know that if $\rightarrow_2 \in \{I_L, I_G, I_Go, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, then there exists an $A^* \in F(U)$ which is a DFI-solution of FMT iff (15) holds for any $u \in U, v \in V$.

**Theorem 4.6.** If $\rightarrow_2$ is a fuzzy implication satisfying (C4), (C11) and (C12), and $T$ is the function residual to $\rightarrow_2$, and $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1]$, (15) holds, then the infimum of DFI-solutions of FMT can be computed as follows:

$$ A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\}, u \in U. \quad (16) $$

**Proof.** Since the fuzzy implication $\rightarrow_2$ satisfies (C11) and (C12), it follows from Proposition 2.7 that the residual condition (10) holds.

Let $H_1 = \{u \in U \mid A^*(u) = 1\}$ and $H_2 = \{u \in U \mid A^*(u) < 1\}$. Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, and that $C(u) > A^*(u)$ for $u \in H_2$. We shall show that $C$ is a DFI-solution of FMT, that is, the following inequality holds for any $u \in U, v \in V$:

$$ (A(u) \rightarrow_1 B(v)) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) < \alpha. \quad (17) $$

If $u \in H_1$, then it follows from (15) that $C(u) = 1$ satisfies (17) for any $v \in V$. If $u \in H_2$, then it follows from (16) and $C(u) > A^*(u)$ that

$$ C(u) > T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v) \quad (18) $$
holds for any $v \in V$. Suppose, on the contrary, that (17) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ such that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (C(u_0) \rightarrow_2 B^*(v_0)) \geq \alpha$ holds (obviously $u_0 \in H_2$). Thus it follows from residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq C(u_0) \rightarrow_2 B^*(v_0)$ holds, and considering that $\rightarrow_2$ satisfies (C4), we have $C(u_0) \leq T(A(u_0) \rightarrow_1 B(v_0), \alpha) \rightarrow_2 B^*(v_0)$, which contradicts (18). Therefore (17) holds for any $u \in U, v \in V$. As a result, $C$ is a DFI-solution of FMT.

Next, we shall check that $A^*$ determined by (16) is the infimum of DFI-solutions of FMT. Assume that $D \in F(U)$, and that there exists $u_0 \in U$ such that $D(u_0) < A^*(u_0)$. We shall prove that $D$ is not a DFI-solution of FMT. In fact, it follows from (16) that there exists $v_0 \in V$ such that

$$D(u_0) < T(A(u_0) \rightarrow_1 B(v_0), \alpha) \rightarrow_2 B^*(v_0)$$

holds. Since $\rightarrow_2$ satisfies (C4), it follows that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq D(u_0) \rightarrow_2 B^*(v_0)$, and we have from the residual condition (10) that $\alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (D(u_0) \rightarrow_2 B^*(v_0))$. Thus $D$ is not a DFI-solution of FMT.

To sum up, $A^*$ determined by (16) is the infimum of DFI-solutions of FMT. □

**Example 4.7.** It is easy to get that $I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP}$ are fuzzy implications satisfying (C4), (C11) and (C12). From Theorem 4.6, we obtain that if $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP}\}$ and (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\}, u \in U$.

**Proposition 4.8.** Suppose that $\rightarrow_2$ is a fuzzy implication satisfying (C4), (C6), (C11) and (C12), and that $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1], (15)$ holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$.

*Proof.* Suppose that $T$ is the function residual to $\rightarrow_2$. Since $\rightarrow_2$ satisfies (C6), we have from Theorem 4.6 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\} = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$. □

**Proposition 4.9.** If $\rightarrow_2$ is an R-implication derived from left-continuous t-norm $T$, and $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1], (15)$ holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{A(u) \rightarrow_1 B(v) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$.

*Proof.* Since $\rightarrow_2$ is an R-implication derived from left-continuous t-norm $T$, it follows that $\rightarrow_2$ is a fuzzy implication satisfying (C4), (C5), (C6), (C11) and (C12) (see Remark 2.8). Thus we get by Proposition 4.8 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\} = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$. □

**Example 4.10.** Note that $I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP}$ are R-implications derived from left-continuous t-norm. Thus it follows from Proposition 4.9 that if $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP}\}$ and (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)\}, u \in U$. 


Theorem 4.11. If $\rightarrow_2$ is a fuzzy implication satisfying (C11) and (C12), and the following condition:

(C13) $I(a,b) = I(b',a')$ ($a, b \in [0,1]$),

and $T$ is the function residual to $\rightarrow_2$, and $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0,1]$, (15) holds, then the infimum of DFI-solutions of FMT can be computed as follows:

$$A^*(u) = \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}, \; u \in U,$$

where $T'(a,b) = 1 - T(a,b)$ ($a, b \in [0,1]$).

Proof. Since the fuzzy implication $\rightarrow_2$ satisfies (C11) and (C12), we get that the residual condition (10) holds.

Let $H_1 = \{u \in U | A^*(u) = 1\}$ and $H_2 = \{u \in U | A^*(u) < 1\}$. Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, and that $C(u) > A^*(u)$ for $u \in H_2$. We shall show that $C$ is a DFI-solution of FMT, that is, the following inequality holds for any $u \in U, v \in V$:

$$A(u) \rightarrow_1 B(v) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) < \alpha. \quad (20)$$

If $u \in H_1$, then it follows from (15) that $C(u) = 1$ satisfies (20) for any $v \in V$.

If $u \in H_2$, then it follows from (19) and $C(u) > A^*(u)$ that

$$C(u) > T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \quad (21)$$

holds for any $v \in V$. Suppose, on the contrary, that (20) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ such that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (C(u_0) \rightarrow_2 B^*(v_0)) \geq \alpha$ holds (obviously $u_0 \in H_2$). Since $\rightarrow_2$ satisfies (C13), we get that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow ((B^*(v_0))' \rightarrow_2 (C(u_0))') \geq \alpha$, and it follows from residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq (B^*(v_0))' \rightarrow_2 (C(u_0))'$, and $T((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha)) \leq (C(u_0))'$, thus

$$C(u_0) \leq T'((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha)),$$

which contradicts (21). Therefore (20) holds for any $u \in U, v \in V$. As a result, $C$ is a DFI-solution of FMT.

Next, we shall check that $A^*$ determined by (19) is the infimum of DFI-solutions of FMT. Assume that $D \in F(U)$, and that there exists $u_0 \in U$ such that $D(u_0) < A^*(u_0)$. We shall prove that $D$ is not a DFI-solution of FMT. In fact, it follows from (19) that there exists $v_0 \in V$ such that

$$D(u_0) < T'((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha))$$

holds, thus we have $(D(u_0))' > T((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha))$. From the residual condition (10) and the fact that $\rightarrow_2$ satisfies (C13), we have that

$$T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq (B^*(v_0))' \rightarrow_2 (D(u_0))' = D(u_0) \rightarrow_2 B^*(v_0),$$

and $\alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (D(u_0) \rightarrow_2 B^*(v_0))$. Thus, $D$ is not a DFI-solution of FMT.

To sum up, $A^*$ determined by (19) is the infimum of DFI-solutions of FMT.
Example 4.12. It is easy to obtain that \( I_L, I_{Fb}, I_{GR}, I_R, I_{RR} \) are fuzzy implications satisfying \((C11),\ (C12)\) and \((C13)\). Thus it follows from Theorem 4.11 that if \( \rightarrow_2 \in \{ I_L, I_{Fb}, I_{GR}, I_R, I_{RR} \} \) and \((15)\) holds, then the infimum of DFI-solutions of FMT is \( A^\ast(u) = \sup_{v \in V} \{ T'((B^\ast(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \} \), \( u \in U \).

Lemma 4.13. [19] Suppose that \( I \) is an R-implication satisfying \((C13)\) derived from left-continuous t-norm \( T \), then the following formula holds:
\[
(C14) \ T'(b', a) = T'(a', b') = I(a, b) \quad \forall a, b \in [0, 1].
\]

Proposition 4.14. Suppose that \( \rightarrow_2 \) is an R-implication satisfying \((C13)\) derived from left-continuous t-norm \( T \), and \( A \in F(U) \), \( B, B^\ast \in F(V) \), \( \alpha \in (0, 1] \), \((15)\) holds, then \( (u \in U) \)
\[
\sup_{v \in V} \{ \alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^\ast(v)] \} = \sup_{v \in V} \{ T'((B^\ast(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \}.
\]

Proof. It follows from Lemma 4.13 that \((C14)\) holds, and we get from Remark 2.8 that \( \rightarrow_2 \) satisfies \((C6)\). \( T \) obviously is commutative, then it follows from Proposition 4.9 that the infimum of DFI-solutions of FMT is
\[
A^\ast(u) = \sup_{v \in V} \{ \alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^\ast(v)] \} = \sup_{v \in V} \{ T(\alpha, A(u) \rightarrow_1 B(v)) \rightarrow_2 B^\ast(v) \} = \sup_{v \in V} \{ T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^\ast(v) \} = \sup_{v \in V} \{ T'((B^\ast(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \}, \quad u \in U.
\]

Remark 4.15. If \( \rightarrow_2 \) is an R-implication satisfying \((C13)\) derived from left-continuous t-norm, then Theorem 4.11 obviously holds. This moment, Proposition 4.14 demonstrates that the infimum of DFI-solutions of FMT achieved by Theorem 4.11 is equivalent to the one from Proposition 4.9.

Remark 4.16. For \( \rightarrow_2 \in \{ I_G, I_L, I_{FD}, I_{GR}, I_{Y_a}, I_{EP} \} \), it follows from Theorem 4.6, Proposition 4.8 and Proposition 4.9 that the expressions of infimum of DFI-solutions of FMT achieved from Example 4.7 is the same as the one from Example 4.10. For \( \rightarrow_2 \in \{ I_L, I_{FD} \} \) (which is an R-implication satisfying \((C13)\) derived from left-continuous t-norm), it follows from Proposition 4.14 that the expressions of infimum of DFI-solutions of FMT achieved by Example 4.10 and Example 4.12 (and also Example 4.7) are consistent.

Proposition 4.17. Suppose that \( \rightarrow_2 \in \{ I_L, I_G, I_{FD}, I_{GR}, I_R, I_{Y_a}, I_{EP}, I_{RR} \} \) in \((8)\), and that \((15)\) holds, then the infimum of DFI-solutions of FMT is as follows respectively \( (u \in U) \):
\[
\begin{align*}
(i) \text{ If } \rightarrow_2 \text{ takes } I_L, & \text{ then } A^\ast(u) = \sup_{v \in V} \{ B^\ast(v) - (A(u) \rightarrow_1 B(v)) + 2 - \alpha \}; \\
(ii) \text{ If } \rightarrow_2 \text{ takes } I_G, & \text{ then } A^\ast(u) = \sup_{v \in V} \{ B^\ast(v) \}; \\
(iii) \text{ If } \rightarrow_2 \text{ takes } I_{GR}, & \text{ then } A^\ast(u) = \sup_{v \in V} \{ B^\ast(v)/[\alpha \times (A(u) \rightarrow_1 B(v))] \}; \\
(iv) \text{ If } \rightarrow_2 \text{ takes } I_{FD}, & \text{ then } A^\ast(u) = \sup_{v \in V} \{ (1 - (A(u) \rightarrow_1 B(v))) \lor B^\ast(v) \} \lor (1 - \alpha); \\
\end{align*}
\]
(v) If $\rightarrow_2$ takes $I_{GR}$, then $A^*(u) = \sup_{v \in V} \{ B^*(v) \} $;
(vi) If $\rightarrow_2$ takes $I_R$, then $A^*(u) = \sup_{v \in V} \{ \alpha' / [ (A(u) \rightarrow_1 B(v)) \times (B^*(v))'] \} $;
(vii) If $\rightarrow_2$ takes $I_{Ya}$, then $A^*(u) = \sup_{v \in V} \{ 1 - \sqrt{1 - B^*(v) - \sqrt{1 - \alpha - \sqrt{1 - (A(u) \rightarrow_1 B(v))}}^2} \} $;
(viii) If $\rightarrow_2$ takes $I_{EP}$, then $A^*(u) = \sup_{v \in V} \{ [2 \phi_{EP}(u,v) - \alpha \times \phi_{EP}(u,v)] / [\alpha + \phi_{EP}(u,v) - \alpha \times \phi_{EP}(u,v)] \} $, where $\phi_{EP}(u,v) = B^*(v) \times [2 - (A(u) \rightarrow_1 B(v))] / [(A(u) \rightarrow_1 B(v)) + B^*(v) - B^*(v) \times (A(u) \rightarrow_1 B(v))]$;
(ix) If $\rightarrow_2$ takes $I_{RR}$, then $A^*(u) = \sup_{v \in V} \{ \alpha' / ( (A(u) \rightarrow_1 B(v)) \times (B^*(v))') \} \vee \{ (A(u) \rightarrow_1 B(v)) / (B^*(v))' \} \vee B^*(v) \} $.

**Proof.** If $\rightarrow_2 \in \{ I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP} \} $, then it follows from Example 4.10 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{ \alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \vee B^*(v)] \} $, $u \in U $. If $\rightarrow_2 \in \{ I_{GR}, I_R, I_{RR} \} $, suppose that $T$ is the function residual to $\rightarrow_2$, then it follows from Example 4.12 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{ T''((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \} $, $u \in U $. Then we need to get the specific expression of $A^*(u) $. We only prove the cases of $I_{FD}, I_{GR}$ as examples, the remainder can be proved similarly.

(iv) Suppose that $\rightarrow_2$ takes $I_{FD}$. It follows from (15) that we have $1 > B^*(v)$, $A(u) \rightarrow_1 B(v) > B^*(v)$, $[1 - (A(u) \rightarrow_1 B(v))] \vee B^*(v) < \alpha $ hold for any $u \in U, v \in V $. Thus we obtain $(u \in U) $:

$$A^*(u) = \sup_{v \in V} \{ \alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \vee B^*(v)] \} $$

$$= \sup_{v \in V} \{ \alpha \rightarrow_2 [(1 - (A(u) \rightarrow_1 B(v)) \vee B^*(v)] \} $$

$$= \sup_{v \in V} \{ (1 - \alpha) \vee (1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v) \} $$

$$= \sup_{v \in V} \{ (1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v) \} \vee (1 - \alpha) \} .$$

(v) Suppose that $\rightarrow_2$ takes $I_{GR}$. We get from Example 3.7 that $T_{GR}$ is the function residual to $I_{GR}$. It follows from (15) that we get $1 > B^*(v)$, $A(u) \rightarrow_1 B(v) > 0 $ hold for any $u \in U, v \in V $. So we have $(u \in U) $:

$$A^*(u) = \sup_{v \in V} \{ T_{GR}''((B^*(v))', T_{GR}(A(u) \rightarrow_1 B(v), \alpha)) \} $$

$$= \sup_{v \in V} \{ 1 - T_{GR}''((B^*(v))', A(u) \rightarrow_1 B(v)) \} $$

$$= \sup_{v \in V} \{ 1 - (B^*(v))' \} = \sup_{v \in V} \{ B^*(v) \} .$$

**Theorem 4.18.** If $\rightarrow_2 \in \{ I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_R, I_{Ya}, I_{EP}, I_{RR} \} $, $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1], (15)$ holds, then the condition, which the infimum $A^*$ of DFI-solutions of FMT is the minimum, is as follows, respectively (for any $u \in U, v \in V $):

(i) Let $\rightarrow_2$ take $I_L$, $\sup_{v \in V} \{ B^*(v) - (A(u) \rightarrow_1 B(v)) \} > B^*(v) - (A(u) \rightarrow_1 B(v))$ holds;

(ii) Let $\rightarrow_2$ take $I_G$, $\sup_{v \in V} \{ B^*(v) \} > B^*(v)$ holds;
(iii) Let \( \rightarrow \) take \( I_{Go} \), \( \sup_{v \in V} \{B^*(v)/(A(u) \rightarrow_1 B(v))\} > B^*(v)/(A(u) \rightarrow_1 B(v)) \) holds;

(iv) Let \( \rightarrow \) take \( I_{FD} \), \( \sup_{v \in V}\{(1 - (A(u) \rightarrow_1 B(v))) \lor B^*(v)\} > [(1 - (A(u) \rightarrow_1 B(v))) \lor B^*(v)] \lor (1 - \alpha) \) holds;

(v) Let \( \rightarrow \) take \( I_{GR} \), \( \sup_{v \in V}\{B^*(v)\} \lor \) holds;

(vi) Let \( \rightarrow \) take \( I_{R} \), \( \sup_{v \in V}\{\alpha'/(\alpha/(A(u) \rightarrow_1 B(v))) \lor (B^*(v)^*)\} > \alpha'/[(A(u) \rightarrow_1 B(v))) \lor (B^*(v)^*)] \) holds;

(vii) Let \( \rightarrow \) take \( I_{Ya} \), \( \sup_{v \in V}\{1 - (\sqrt{1 - B^*(v)}) - (1 - (A(u) \rightarrow_1 B(v))) - (1 - \alpha)\} > 1 - (\sqrt{1 - B^*(v)}) - (1 - (A(u) \rightarrow_1 B(v))) - (1 - \alpha)^2 \) holds;

(viii) Let \( \rightarrow \) take \( I_{EP} \), \( \sup_{v \in V}\{2\phi_{EP}(u,v) - \alpha \times \phi_{EP}(u,v)\} > [\alpha + \phi_{EP}(u,v)] - \alpha \times \phi_{EP}(u,v) \) holds;

(ix) Let \( \rightarrow \) take \( I_{RR} \), \( \sup_{v \in V}\{\alpha'/(\alpha/(A(u) \rightarrow_1 B(v))) \lor (B^*(v)^*) \lor B^*(v)^*\} > \alpha'/(\alpha/(A(u) \rightarrow_1 B(v))) \lor (B^*(v)^*) \lor B^*(v)^* \) holds.

Proof. Note that if the infimum \( A^* \) of DFI-solutions of FMT is a DFI-solution of FMT, then \( A^* \) is the minimum of DFI-solutions of FMT. Thus it is enough to prove that \( A^* \) is a DFI-solution of FMT (i.e., \( A^* \) should make (8) hold for any \( u \in U, v \in V \)). We still prove the cases of \( I_{FD}, I_{GR} \) as examples.

(iv) Let \( \rightarrow \) take \( I_{FD} \). It follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is \( A^*(u) = \sup_{v \in V}\{(1 - (A(u) \rightarrow_1 B(v))) \lor B^*(v)\} \lor (1 - \alpha) \). It follows from (15) that the following formula holds for any \( u \in U, v \in V \):

\[
1 > B^*(v), \ A(u) \rightarrow_1 B(v) > B^*(v), \ [1 - (A(u) \rightarrow_1 B(v))] \lor B^*(v) < \alpha. \tag{22}
\]

By the condition given in (iv), we obtain that \( A^*(u) > [(1 - (A(u) \rightarrow_1 B(v))) \lor B^*(v)] \lor (1 - \alpha) \), so the following formula holds for any \( u \in U, v \in V \):

\[
A^*(u) > B^*(v), \ A(u) \rightarrow_1 B(v) > 1 - A^*(u), \ \alpha > 1 - A^*(u). \tag{23}
\]

It follows from (22) and (23) that we have:

\[
(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) = (A(u) \rightarrow_1 B(v)) \rightarrow_2 [(1 - A^*(u)) \lor B^*(v)]
\]

\[
= [1 - (A(u) \rightarrow_1 B(v))] \lor [(1 - A^*(u)) \lor B^*(v)] < \alpha.
\]

Thus \( A^* \) makes (8) hold for any \( u \in U, v \in V \), and hence it is a DFI-solution of FMT.

(v) Let \( \rightarrow \) take \( I_{GR} \). We get from Proposition 4.17 that the infimum of DFI-solutions of FMT is \( A^*(u) = \sup_{v \in V}\{B^*(v)\} \). It follows from (15) that \( 1 > B^*(v), \ A(u) \rightarrow_1 B(v) > 0 \) hold for any \( u \in U, v \in V \). By the condition given in (v), we obtain that \( A^*(u) = \sup_{v \in V}\{B^*(v)\} > B^*(v) \) holds for any \( u \in U, v \in V \). Thus

\[
(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) = (A(u) \rightarrow_1 B(v)) \rightarrow_2 0 = 0 < \alpha.
\]

So \( A^* \) is a DFI-solution of FMT. \( \square \)
Example 4.19. Let shows some results of the algorithm for FMT from Proposition 4.17 and Theorem Proposition 4.8, Proposition 4.9 and Theorem 4.11. Moreover, Table 4 and Table 5 conclude some results of the algorithm for FMT from Theorem 4.6, (C4), (C11) and (C12).

\[
\sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\} = \text{A fuzzy implication satisfying (C4), (C11) and (C12)}
\]

\[
\sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\} = \text{A fuzzy implication satisfying (C4), (C6), (C11) and (C12)}
\]

\[
\sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\} = \text{An R-implication from from left-continuous t-norm}
\]

\[
\sup_{v \in V} \{T'(B^*(v)), T(A(u) \rightarrow_1 B(v), \alpha)\} = \text{A fuzzy implication satisfying (C11), (C12) and (C13)}
\]

### Table 3. Some Results (I) of the DFI-restriction Inference Algorithm for FMT

<table>
<thead>
<tr>
<th>(\rightarrow_2)</th>
<th>The infimum (A^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_L)</td>
<td>(\sup_{v \in V} {B^*(v) - (A(u) \rightarrow_1 B(v)) + 2 - \alpha})</td>
</tr>
<tr>
<td>(I_G)</td>
<td>(\sup_{v \in V} {B^*(v)})</td>
</tr>
<tr>
<td>(I_{Go})</td>
<td>(\sup_{v \in V} {B^*(v)/[\alpha \times (A(u) \rightarrow_1 B(v))]})</td>
</tr>
<tr>
<td>(I_{FD})</td>
<td>(\sup_{v \in V} {(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)] \vee (1 - \alpha)})</td>
</tr>
<tr>
<td>(I_{GR})</td>
<td>(A^<em>(u) = \sup_{v \in V} {B^</em>(v)})</td>
</tr>
<tr>
<td>(I_R)</td>
<td>(\sup_{v \in V} {\alpha'/[(A(u) \rightarrow_1 B(v)) \times \langle B^*(v)\rangle]})</td>
</tr>
<tr>
<td>(I_{Ya})</td>
<td>(\sup_{v \in V} {1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - \alpha} - \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2})</td>
</tr>
<tr>
<td>(I_{EP})</td>
<td>(\sup_{v \in V} {[2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]/[\alpha + \phi_{EP}(u, v) \alpha \times \phi_{EP}(u, v)]})</td>
</tr>
<tr>
<td>(I_{RR})</td>
<td>(\sup_{v \in V} {\alpha'/[(A(u) \rightarrow_1 B(v)) \times \langle B^<em>(v)\rangle] \vee (A(u) \rightarrow_1 B(v)) \vee B^</em>(v)})</td>
</tr>
</tbody>
</table>

### Table 4. Some Results (II) of the DFI-restriction Inference Algorithm for FMT

Table 3 concludes some results of the algorithm for FMT from Theorem 4.6, Proposition 4.8, Proposition 4.9 and Theorem 4.11. Moreover, Table 4 and Table 5 shows some results of the algorithm for FMT from Proposition 4.17 and Theorem 4.18.

**Example 4.19.** Let \(U = V = [0, 1]\), \(A(u) = (4 - u)/4\), \(B(v) = (3 + v)/4\), \(B^*(v) = (1 - v)/4\) and \(\alpha = 1/2\), where \(u \in U, v \in V\). Suppose that \(\rightarrow_2 = I_R, \rightarrow_1 = I_L\) in the DFI-restriction inference algorithm for FMT. We now compute the infimum of DFI-solutions of FMT.

\[
A(u) \rightarrow_1 B(v) = I_L(A(u), B(v)) = \begin{cases} 
1 - \frac{4 - u}{4} + \frac{3 + v}{4}, & \text{if } \frac{4 - u}{4} > \frac{3 + v}{4} \\
1, & \text{if } \frac{4 - u}{4} \leq \frac{3 + v}{4}
\end{cases}
\]

\[
= \begin{cases} 
\frac{3 + u + v}{4}, & \text{if } u + v < 1 \\
1, & \text{if } u + v \geq 1
\end{cases}
\]
Some Results (III) of the DFI-restriction Inference

Table 5. Some Results (III) of the DFI-restriction Inference

Algorithm for FMT

For (15), we have

\[
\begin{align*}
(A(u) \rightarrow B(v)) & \rightarrow_2 (1 \rightarrow_2 B^*(v)) \\
& = (A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v) \\
& = \left\{ \begin{array}{ll}
1 - \frac{3u+3v}{4} + \frac{3+u+v}{4} \times \frac{1}{1-u}, & \text{if } u + v < 1 \\
\frac{1}{1-u}, & \text{if } u + v \geq 1
\end{array} \right.
\end{align*}
\]

thus (15) holds. Then it follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is as follows (u ∈ U):

\[
A^*(u) = \sup_{v \in X} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))] \}
\]

\[
= \sup_{v \in X} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))] \mid u + v < 1 \}
\]

\[
\vee \sup_{v \in X} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))] \mid u + v \geq 1 \}
\]

\[
= \sup_{v \in [0,1]} \left\{ \frac{1}{2} \left\lfloor \frac{3+u+v}{4} \times \frac{3+v}{4} \right\rfloor \mid u + v < 1 \right\}
\]

\[
\vee \sup_{v \in [0,1]} \left\{ \frac{1}{2} \left\lfloor \frac{3+v}{4} \right\rfloor \mid u + v \geq 1 \right\}
\]

\[
= \sup_{v \in [0,1]} \left\{ \frac{8}{(3+u+v) \times (3+v)} \mid u + v < 1 \right\} \vee \sup_{v \in [0,1]} \left\{ \frac{2}{3+v} \mid u + v \geq 1 \right\}.
\]
(i) Suppose \( u = 1 \), then \( \{ v \in [0, 1] \mid u + v < 1 \} = \emptyset \), and \( 0 \in \{ v \in [0, 1] \mid u + v \geq 1 \} \). Taking into account that \( \frac{2}{3 + v} \) is decreasing w.r.t. \( v \), we get

\[
A^*(u) = (\sup \emptyset) \lor \frac{2}{3} = 0 \lor \frac{2}{3} = \frac{2}{3} = \frac{8}{9 + 3u}.
\]

(ii) Suppose \( 0 \leq u < 1 \), then \( 0 \in \{ v \in [0, 1] \mid u + v < 1 \} \). Since \( \frac{8}{3 + u + v} \times (3 + v) = \frac{8}{3 + u} \) are decreasing w.r.t. \( v \), we have

\[
A^*(u) = \frac{8}{3 + u} \lor \frac{2}{3 + 1 - u} = \frac{8}{9 + 3u} \lor \frac{2}{4 - u} = \frac{8}{9 + 3u},
\]

where \( \frac{8}{9 + 3u} > \frac{2}{4 - u} \) since \( 0 \leq u < 1 \).

Together we achieve

\[
A^*(u) = \frac{8}{9 + 3u}, \quad u \in U.
\]

**Example 4.20.** Let \( U, V, A, B, B^*, \alpha \) be the same as in Example 4.19. Suppose that \( \rightarrow_2 = I_R, \rightarrow_1 = I_R \) in the DFI-restriction inference algorithm for FMT, which degenerates into the full implication restriction inference algorithm for FMT (taking \( I_R \)). We now compute the infimum of DFI-solutions of FMT.

\[
A(u) \rightarrow_1 B(v) = I_R(A(u), B(v)) = 1 - \frac{4 - u}{4} + \frac{4}{4} \times \frac{3 + v}{4} = \frac{12 + u + v(4 - u)}{16}.
\]

For (15), we have

\[
(A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_2 B^*(v))
\]

\[
= (A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)
\]

\[
= 1 - \frac{12 + u + v(4 - u)}{16} + \frac{12 + u + v(4 - u)}{16} \times \frac{1 - v}{4}
\]

\[
= \frac{7}{16} - \frac{3u + 3v(4 - u)}{64} - \frac{v(12 + u + v(4 - u))}{64} < \alpha,
\]

which means (15) holds. It follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is as follows \( (u \in U) \):

\[
A^*(u) = \sup_{v \in V} \{ \alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))'] \}
\]

\[
= \sup_{v \in [0, 1]} \left\{ \frac{1}{2} \left( \frac{12 + u + v(4 - u)}{16} \times \frac{3 + v}{4} \right) \right\}
\]

\[
= \sup_{v \in [0, 1]} \left\{ \frac{32}{[12 + u + v(4 - u)] \times (3 + v)} \right\}.
\]

Because \( \frac{32}{12 + u + v(4 - u)} \times (3 + v) \) is decreasing w.r.t. \( v \), we obtain

\[
A^*(u) = \frac{32}{(12 + u) \times 3} = \frac{32}{36 + 3u}.
\]
Remark 4.21. For the same $U, V, A, B, B^*, \alpha$ in Example 4.19 and Example 4.20, since

\[ 0 < u \leq 1 \Rightarrow \frac{8}{9 + 3u} < \frac{32}{36 + 3u}, \]

\[ u = 0 \Rightarrow \frac{8}{9 + 3u} = \frac{32}{36 + 3u}, \]

the infimum of DFI-solutions of FMT (from the DFI-restriction inference algorithm) in Example 4.19 is smaller than the one (from the full implication restriction inference algorithm) in Example 4.20. From the viewpoint of DFI-restriction inference principle for FMT (which seeks the smallest $A^*$ satisfying (8)), the DFI-restriction inference algorithm for FMT in Example 4.19 makes the inference closer, then it is better than the full implication restriction inference algorithm for FMT in Example 4.20.

Remark 4.22. Aiming at the set of fuzzy implications $\{I_L, I_G, I_G^0, I_FD, I_GR, I_R, I_Ya, I_EP, I_RR\}$, we get from the DFI-restriction inference algorithm that there are $9 \times 9 = 81$ kinds of specific fuzzy inference algorithms for FMT, in which $7 \times 9 + 1 = 64$ kinds of actual specific algorithms for FMT exist (noting that the expression is independent of $\rightarrow_1$ for the case $\rightarrow_2 \in \{I_G, I_GR\}$, by virtue of Table 4 and Table 5). But, from the full implication restriction inference algorithm, there are only 9 kinds of specific algorithms for FMT (including 8 kinds of actual specific algorithms for FMT), see Table 6. These imply that the DFI-restriction inference algorithm can obtain more and better specific algorithms for FMT (comparing with the full implication restriction inference algorithm), thus it is superior to the full implication restriction inference algorithm.

5. Conclusions

A new fuzzy inference strategy called the double fuzzy implications-based restriction inference algorithm (DFI-restriction inference algorithm for short) is put forward and investigated. First, new restriction inference principle is presented, which improves the full implication restriction inference principle; and then the definitions as well as existing conditions of solutions of the DFI-restriction inference algorithm are provided. Second, the supremum (or infimum) of related solutions is achieved, and the condition which the corresponding supremum is the maximum (or, the corresponding infimum is the minimum) is obtained for the FMP and FMT.
problems, respectively. Third, the optimal solutions of the DFI-restriction inference algorithm are accomplished for several specific fuzzy implications. Lastly, it is found that the DFI-restriction inference algorithm is more reasonable than the full implication restriction inference algorithm, since the former can make the inference closer, and generate more, better specific inference algorithms. Such works would be helpful to applications of fuzzy inference as well as fuzzy system.

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