POWER HARMONIC AGGREGATION OPERATOR WITH TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBERS FOR SOLVING MAGDM PROBLEMS

S. DAS AND D. GUHA

Abstract. Trapezoidal intuitionistic fuzzy numbers (TrIFNs) express abundant and flexible information in a suitable manner and are very useful to depict the decision information in the procedure of decision making. In this paper, some new aggregation operators, such as, trapezoidal intuitionistic fuzzy weighted power harmonic mean (TrIFWPHM) operator, trapezoidal intuitionistic fuzzy ordered weighted power harmonic mean (TrIFOWPHM) operator, trapezoidal intuitionistic fuzzy induced ordered weighted power harmonic mean (TrIFIOWPHM) operator and trapezoidal intuitionistic fuzzy hybrid power harmonic mean (TrIFhPHM) operator are introduced to aggregate the decision information. The desirable properties of these operators are presented in detail. A prominent characteristic of these operators is that, the aggregated value by using these operators is also a TrIFN. It is observed that the proposed TrIFWPHM operator is the generalization of trapezoidal intuitionistic fuzzy weighted harmonic mean (TrIFWHM) operator, trapezoidal intuitionistic fuzzy weighted arithmetic mean (TrIFWAM) operator, trapezoidal intuitionistic fuzzy weighted geometric mean (TrIFWGM) operator and trapezoidal intuitionistic fuzzy weighted quadratic mean (TrIFWQM) operator, *i.e.,* we can easily reduce the TrIFWPHM operator to TrIFWHM, TrIFWGM, TrIFWAM and TrIFWQM operators, depending upon the decision situation. Further, we develop an approach to multi-attribute group decision making (MAGDM) problem on the basis of the proposed aggregation operators. Finally, the effectiveness and applicability of our proposed MAGDM model, as well as comparison analysis with other approaches are illustrated with a practical example.

1. Introduction

Multi-attribute group decision making (MAGDM) is the process of selecting the beat alternative from a set of predefined alternatives which are assessed by a group of experts based on multiple attributes. In general, MAGDM process consists of four stages, which may be represented as information gathering, mathematical modeling, simulation and decision/action followed in a sequence. In this process, due to lack or abundance of information, subjective estimation or vagueness and incomplete knowledge about the complex systems experts' preferences may not be assessed with both precision and certainty. This uncertainty (i.e., hesitation)

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related to human cognitive processes viz., thinking, reasoning, etc., present in experts' preference values can be suitably captured by intuitionistic fuzzy sets (IFSs). IFS, introduced by Atanassov [1], is characterized by a membership function and a non-membership function, and, thus, can describe uncertain information in a more meaningful way than Zadeh's [65] fuzzy set, which is characterized by a membership function. Thus, IFS theory provides a very strong framework to handle uncertainty of real-world decision situations. Eventually, in less than three decades since its first appearance, IFS theory is investigated by many authors [3, 34, 15] and are used for decision making [6, 11, 41, 43, 44], medical diagnosis [16, 22], pattern recognition [31, 17], to name a few. Authors have also devoted considerable attention to the generalization of IFS theory. Atanassov and Gargov [2] proposed the concept of interval-valued intuitionistic fuzzy set, which is characterized by a membership function, a non-membership function, and a hesitancy function whose values are intervals. It is worthwhile to mention that, the research concentrated on finite universe of discourse only. In view of this, from the concept of IFSs, intuitionistic fuzzy numbers (IFNs) was defined [7, 36, 37, 38, 39] with the universe of discourse as real line. Basically, the concept of IFNs can be viewed as an extensive approach of conventional IFS which is based on discrete sets. In information integration process, discrete sets may loss partial information [64] while continuous sets maintain the integrity of information and, thus, are more capable to model incomplete and abundant information than discrete sets. With this view, extending the concept of discrete sets to continuous sets, IFNs are defined which can more suitably model imprecise data involved in real-world decision making problems.

In recent times, IFNs have received increasing [7, 19, 40, 45] attention because of their ability to handle imprecise and abundant information in decision making situation. In order to get the decision result in MAGDM problem, where experts' opinions are modeled as IFNs, an important step is the aggregation of IFNs and in this step aggregation or fusion of information by suitable aggregation operator plays an important role. Aggregation operators are used to calculate the overall performance of all the alternatives over the attributes. Aggregation operators are widely useful to decision making problem [48, 52, 56], neural networks [58], fuzzy logic controller [57, 59] and many other fields concerning the fusion of a collection of information granules. In the literature, several kinds of aggregation operator, such as, ordered weighted averaging operator [60], ordered weighted geometric operator [12, 51], power average [61], power geometric [55], Choquet integral [13, 18], Bonferroni mean [63], linguistic approach based aggregation operators [26], Shapley function based aggregation operators [27, 28, 29, 30] are introduced.

Recently, some aggregation operators are developed to aggregate IFNs. Wu and Cao [50] defined same families of geometric aggregation operators to aggregate trapezoidal IFNs (TrIFNs). Wang *et al.* [47] defined AND and OR operators for triangular IFNs (TIFNs) and applied them in fault tree analysis. Wan and Dong [35] defined ordered weighted aggregation operator and hybrid aggregation operator for TrIFNs and applied to MAGDM problem. Jianqiang and Zhong [21] developed the intuitionistic trapezoidal weighted arithmetic average operator and the intuitionistic trapezoidal weighted geometric average operator. In 2013, a power average operator of TrIFNs was introduced by Wan [42] and also an application to MAGDM was discussed. In 2015, Wan and Dong [46] developed power geometric operators for TrIFNs and applied it to MAGDM problem.

However, most of the existing aggregation operators of TrIFNs, do not take into account the situation where extreme outlier information exist in the data set being fused. In fact, handling the outlier data suitably to ultimate aggregation results, is very important for some real-life decision problem. Considering the expressed feature of the data set, in the existing literature, harmonic mean [20] is generally considered as a fusion technique as harmonic mean gives less significance to high-value outliers [25]. With this view, in order to handle uncertain data Xu [54] proposed fuzzy weighted harmonic mean operator, fuzzy ordered weighted harmonic operator and fuzzy hybrid harmonic operator for fuzzy numbers. Park *et al.* [32] developed some harmonic operators in the linguistic environment, such as, 2-tuple linguistic harmonic operator, 2-tuple linguistic weighted harmonic operator, 2-tuple linguistic ordered weighted harmonic operator, and 2-tuple linguistic hybrid harmonic operator.

However, the above harmonic operators cannot directly be applied to the case where the aggregation arguments are IFNs, such as, TrIFNs, TIFNs. In other words, there is no such aggregation operator, based on operational laws of IFNs, which considers the outliers data in the aggregation process for the information of IFNs. Thus, to overcome these defeats, the aim of this paper is to introduce, some new aggregation operators for aggregating IFNs which can reduce the influence of outlier data in the decision results by giving less importance to the outliers and also work as a generalized mean. Particularly, we analyze the operations of TrIFNs and also present a generalization of harmonic mean for fusing TrIFNs information, which we will refer to as trapezoidal intuitionistic fuzzy weighted power harmonic mean (TrIFWPHM) operator. It is shown that the proposed TrIFWPHM operator is the generalization of trapezoidal intuitionistic fuzzy weighted harmonic mean (TrIFWHM) operator, trapezoidal intuitionistic fuzzy weighted arithmetic mean (TrIFWAM) operator, trapezoidal intuitionistic fuzzy weighted geometric mean (TrIFWGM) operator and trapezoidal intuitionistic fuzzy weighted quadratic mean (TrIFWQM) operator depending upon the decision situation. We also define other three power harmonic mean operators based on TrIFNs, namely, trapezoidal intuitionistic fuzzy ordered weighted power harmonic mean (TrIFOWPHM) operator, trapezoidal intuitionistic fuzzy induced ordered weighted power harmonic mean (TrIFIOWPHM) operator and trapezoidal intuitionistic fuzzy hybrid power harmonic mean (TrIFhPHM) operator. Then an approach to MAGDM problem in which the ratings of alternatives on the basis of attributes are expressed as IFNs, is developed based on the proposed operators of IFNs.

The rest of the paper is planned as follows: In Section 2, definition and operations related to IFNs are reviewed. In this section, we point out some errors in the definition of the arithmetic operations for TIFNs formulated by Wang *et al.*[47] with the help of some examples. A ranking process of IFNs based on centroid point is also described in this section. Section 3 describes the concept of basic aggregation operators. In Section 4, four kinds of power harmonic operators of

TrIFNs are developed and their desirable properties are studied. An application of the proposed aggregation operator is analyzed in Section 5. In Section 6, a numerical example is used to illustrate the effectiveness of the proposed operator and the sensitivity analysis and comparison analysis are also conducted in this section. The paper concludes in Section 7.

2. Preliminaries

2.1. Definition and Arithmetic Operations of IFNs.
We present here some definitions and concepts that will be required for our subsequent developments. We start by recalling the definition of TrIFN.

Definition 2.1. [50] A TrIFN $A = \langle[(a,b,c,d),\mu];[(a',b,c,d'),\nu]\rangle$ is a special IFS on the real line $\mathbb{R}$ and its membership and non-membership functions are respectively defined as follows:

$$\mu_A(x) = \begin{cases} \dfrac{(x-a)\mu}{(b-a)}, & \text{for } \; a \le x \le b, \\ \mu, & \text{for } \; b \le x \le c, \\ \dfrac{(d-x)\mu}{(d-c)}, & \text{for } c \le x \le d, \\ 0, & \text{for } \; x \le a, x \ge d. \end{cases} \tag{1}$$

and

$$\nu_A(x) = \begin{cases} \dfrac{(b-x)+(x-a')\nu}{(b-a')}, & \text{for } \; a' \le x \le b, \\ \nu, & \text{for } \; b \le x \le c, \\ \dfrac{(x-c)+(d'-x)\nu}{(d'-c)}, & \text{for } c \le x \le d', \\ 1, & \text{for } \; x \le a', x \ge d'. \end{cases} \tag{2}$$

where μ and ν represent the maximum degree of membership and the minimum degree of non-membership of an element $x \in A$, respectively, such that they satisfy the conditions: $0 \le \mu,\nu \le 1$; $0 \le \mu+\nu \le 1$. The function $\Pi_A(x) = 1-\mu_A(x)-\nu_A(x)$ is called hesitancy or indeterminacy of an element x belonging to A.

For sake of simplicity and without loss of generality, throughout this paper we have considered $a = a'$ and $d = d'$. Symbolically, then TrIFN can be represented as $A = [(a,b,c,d);\mu,\nu]$. If $b = c$ then TrIFN transform to a TIFN.

For the purpose of understanding, we provide the following interpretation of TrIFN in the context of modeling decision maker's opinion. If a TrIFN $A = [(a,b,c,d);\mu,\nu]$ represents an expert's opinion, then the parameters μ and ν express the corresponding degrees of satisfaction and dissatisfaction of the expert, respectively. In the context of MAGDM, a TrIFN A allows one to simulate human cognitive processes, viz., thinking, reasoning etc. in a more suitable manner than IFSs representation. In many practical cases, it is also observed that the decision maker may not be able to get proper decision results based on the imprecise information quantified by IFSs. In this situation, remodeling available decision information is required and IFNs provide a more appropriate tool to model such uncertain information. For example, a TrIFN $A = [(a, b, c, d); \mu, \nu]$ may represent an imprecise quantity 'approximately A', which is approximately equal to A. The imprecise quantity 'approximately A' is expressed using any value between a and d with different degrees of satisfaction and dissatisfaction of the expert. In other words, the most possible value occurs in between b and c with the satisfaction level μ and the dissatisfaction level ν; the pessimistic value is a with the satisfaction level 0 and the dissatisfaction level 1; the optimistic value is d with the satisfaction level 0 and the dissatisfaction level 1; other values are in (a, d) with satisfaction degree $\mu_A(x)$ and the dissatisfaction degree $\nu_A(x)$ defined in equations (1) and (2), respectively.

Definition 2.2. [46] Let $A = [(a, b, c, d); \mu, \nu]$ be a TrIFN. If $a \geq 0$ and one of the four values a, b, c and d is not equal to zero, then the TrIFN A is called positive TrIFN.

2.2. **A Note on Arithmetic Operations of IFNs.** In 2006, Shu *et al.* [33] defined four arithmetic operations for TIFNs to develop an algorithm for intuitionistic fuzzy fault-tree analysis. Li [23] pointed out and corrected some errors of the arithmetic operations of TIFNs defined by Shu *et al.* [33]. The flexibility and applicability of the arithmetic operations proposed by Li [23], was analyzed in [42, 46]. Again in 2013, Wang *et al.* [47] formulated a new definition for the arithmetic operations of TIFNs and applied them to fault-tree analysis of a printed circuit board assembly system. Here we first analyze the arithmetic operations of TIFNs, defined by Wang *et al.* [47], which are given below.

Definition 2.3. Let $A = [(a_1, b_1, c_1); \mu_1, \nu_1]$ and $B = [(a_2, b_2, c_2); \mu_2, \nu_2]$ be two TIFNs. Then

1) $A + B = \left[(a_1 + a_2, b_1 + b_2, c_1 + c_2); \frac{\|A\|\mu_1 + \|B\|\mu_2}{\|A\| + \|B\|}, \frac{\|A\|\nu_1 + \|B\|\nu_2}{\|A\| + \|B\|}\right]$
2) $A - B = \left[(a_1 - c_2, b_1 - b_2, c_1 - a_2); \frac{\|A\|\mu_1 + \|B\|\mu_2}{\|A\| + \|B\|}, \frac{\|A\|\nu_1 + \|B\|\nu_2}{\|A\| + \|B\|}\right]$

For $A > 0$ and $B > 0$,

3) $A \times B = [(a_1 a_2, b_1 b_2, c_1 c_2); \mu_1 \mu_2, \nu_1 + \nu_2 - \nu_1 \nu_2]$
4) $\frac{A}{B} = [(\frac{a_1}{c_2}, \frac{b_1}{b_2}, \frac{c_1}{a_2}); \mu_1 \mu_2, \nu_1 + \nu_2 - \nu_1 \nu_2]$

where $||A|| = \frac{|a_1| + 2|b_1| + |c_1|}{4}$ and $||B|| = \frac{|a_2| + 2|b_2| + |c_2|}{4}$.

After analyzing the Definition 2.3, it is observed that, the addition and subtraction operations given in [47] have certain demerits. To support it, we have provided sets of examples in Table 1 to Table 3 as given below.

Five different example sets (Set-1 to Set-5) consisting of two TIFNs, A_i and B_i, are provided in Table 1. The addition between the numbers of each set are calculated by using Definition 2.3 and the output for each set is $(A_i + B_i) = [(6, 9, 12); 0.333, 0.333]$. The table shows that the output for each set is identical, although the sets themselves are not identical, as the degree of membership and

	Example sets	Results
Set-1:	$A_1 = [(2, 3, 4); 0.6, 0.4]$ $B_1 = [(4, 6, 8); 0.2, 0.3]$	$A_1 + B_1 =$ $[(6, 9, 12); 0.333, 0.333]$
Set-2:	$A_2 = [(2, 3, 4); 0.7, 0.3]$ $B_2 = [(4, 6, 8); 0.15, 0.35]$	$A_2 + B_2 =$ $[(6, 9, 12); 0.333, 0.333]$
Set-3:	$A_3 = [(2, 3, 4); 0.5, 0.5]$ $B_3 = [(4, 6, 8); 0.25, 0.25]$	$A_3 + B_3 =$ $[(6, 9, 12); 0.333, 0.333]$
Set-4:	$A_4 = [(2, 3, 4); 0.8, 0.2]$ $B_4 = [(4, 6, 8); 0.1, 0.4]$	$A_4 + B_4 =$ $[(6, 9, 12); 0.333, 0.333]$
Set-5:	$A_5 = [(2, 3, 4); 0.4, 0.6]$ $B_5 = [(4, 6, 8); 0.3, 0.2]$	$A_5 + B_5 =$ $[(6, 9, 12); 0.333, 0.333]$

Table 1. Example Sets of TIFNs (Set-1 to Set-5)

non-membership of each TIFN in each set is different. Hence, the obtained result is not satisfactory.

	Example sets	Results
Set-6:	$A_1 = [(1, 2, 3); 0.6, 0.3]$ $B_1 = [(2, 3, 4); 0.5, 0.25]$	$A_1 + B_1 =$ $[(3, 5, 7); 0.54, 0.27]$
Set-7:	$A_2 = [(0.5, 1, 1.5); 0.6, 0.3]$ $B_2 = [(1, 1.5, 2); 0.5, 0.25]$	$A_2 + B_2 =$ $[(1.5, 2.5, 3.5); 0.54, 0.27]$
Set-8:	$A_3 = [(5, 6, 7); 0.6, 0.3]$ $B_3 = [(8, 9, 10); 0.5, 0.25]$	$A_3 + B_3 =$ $[(13, 15, 17); 0.54, 0.27]$
Set-9:	$A_4 = [(3, 4, 5); 0.6, 0.3]$ $B_4 = [(5, 6, 7); 0.5, 0.25]$	$A_4 + B_4 =$ $[(8, 10, 12); 0.54, 0.27]$
Set-10:	$A_5 = [(7, 8, 9); 0.6, 0.3]$ $B_5 = [(11, 12, 13); 0.5, 0.25]$	$A_5 + B_5 =$ $[(18, 20, 22); 0.54, 0.27]$

Table 2. Example Sets of TIFNs (Set-6 to Set-10)

Further, in Table 2, five example sets (Set-6 to Set-10) consisting of two TIFNs, A_i and B_i, with different values of triplet 'a, b and c' are provided. The degree of membership and non-membership of all A_is and B_is are equal. After employing the addition operator given in Definition 2.3, we observe that membership (and non-membership) values of all the output ($i.e.$, $A_i + B_i$) for each set are identical. That is, the triplet 'a, b and c' does not affect on the computation of membership (and non-membership) degree of the output.

Again in Table 3, another five sets of examples (Set-11 to Set-15) of the same pattern as provided in Table-2, are considered. After applying the addition operation given in Definition 2.3, we observe that, membership (and non-membership) values of all the output ($i.e.$, $A_i + B_i$) are different. That is, the triplet 'a, b and c' affects on the computation of the membership (and non-membership) values of the output. Similar results will be obtained in the case of subtraction operation.

The above observations indicate that the two arithmetic operations, namely, addition and subtraction defined by Wang *et al.* [47] provide inconsistent results. With this view, in this paper, we have followed the arithmetic operations of TrIFNs, proposed by Li [24], which are given below.

	Example sets	Results
Set-11:	$A_1 = [(1, 2, 3); 0.6, 0.3]$ $B_1 = [(1, 2.5, 3); 0.5, 0.25]$	$A_1 + B_1 =$ $[(2, 4.5, 6); 0.5471, 0.1559]$
Set-12:	$A_2 = [(2, 3, 4); 0.6, 0.3]$ $B_2 = [(3.5, 4, 6); 0.5, 0.25]$	$A_2 + B_2 =$ $[(5.5, 7, 10); 0.5401, 0.2703]$
Set-13:	$A_3 = [(3, 4, 5); 0.6, 0.3]$ $B_3 = [(4, 5, 6); 0.5, 0.25]$	$A_3 + B_3 =$ $[(7, 9, 11); 0.5444, 0.2722]$
Set-14:	$A_4 = [(5, 6, 7); 0.6, 0.3]$ $B_4 = [(5, 7, 8); 0.5, 0.25]$	$A_4 + B_4 =$ $[(10, 13, 15); 0.5474, 0.2735]$
Set-15:	$A_5 = [(7, 8, 9); 0.6, 0.3]$ $B_5 = [(10, 12, 13); 0.5, 0.25]$	$A_5 + B_5 =$ $[(17, 19, 21); 0.5405, 0.2703]$

TABLE 3. Example Sets of TIFNs (Set-11 to Set-15)

Definition 2.4. [24] Let $A_1 = [(a_1, b_1, c_1, d_1); \mu_1, \nu_1]$ and $A_2 = [(a_2, b_2, c_2, d_2); \mu_2, \nu_2]$ be two positive TrIFNs and $\alpha(\geq 0)$ be a scalar, then

(1) $A_1 + A_2 = [(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); min(\mu_1, \mu_2), max(\nu_1, \nu_2)]$

(2) $$A_1 A_2 = \begin{cases} [(a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2); min(\mu_1, \mu_2), max(\nu_1, \nu_2)], & \text{for } a_1 > 0 \text{ and } a_2 > 0 \\ [(a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2); min(\mu_1, \mu_2), max(\nu_1, \nu_2)], & \text{for } d_1 < 0 \text{ and } a_2 > 0 \\ [(d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2); min(\mu_1, \mu_2), max(\nu_1, \nu_2)], & \text{for } d_1 < 0 \text{ and } d_2 < 0 \end{cases}$$

(3) $$\alpha A_1 = \begin{cases} [(\alpha a_1, \alpha b_1, \alpha c_1, \alpha d_1); \mu_1, \nu_1)], & \text{for } \alpha > 0, \\ [(\alpha d_1, \alpha c_1, \alpha b_1, \alpha a_1); \mu_1, \nu_1], & \text{for } \alpha < 0 \end{cases}$$

(4) $$A_1^\alpha = \begin{cases} [(a_1^\alpha, b_1^\alpha, c_1^\alpha, d_1^\alpha); \mu_1, \nu_1], & \text{for } \alpha > 0, \\ [(d_1^\alpha, c_1^\alpha, b_1^\alpha, a_1^\alpha); \mu_1, \nu_1], & \text{for } \alpha < 0 \end{cases}$$

(5) $\frac{1}{A_1} = [(\frac{1}{d_1}, \frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}); \mu_1, \nu_1]$

2.3. **Ranking of IFN.** In comparing IFNs, researchers have introduced various ranking methods. One of such methods that are employed in this study is a centroid based ranking method of TrIFNs [14]. We present below the method to compute the centroid point of a TrIFN by using the following steps.

Step I: *Computation of X coordinate of the centroid point.*
Let $A = [(a, b, c, d); \mu, \nu]$ be a TrIFN, which is shown in Fig.1. Let $f_A^L : [a, b] \to [0, \mu]$, $f_A^R : [c, d] \to [0, \mu]$ be the left and right parts of the membership function μ_A and $g_A^L : [a, b] \to [0, \nu]$, $g_A^R : [c, d] \to [0, \nu]$ are left and right parts of non-membership function ν_A of TrIFN A. The membership and non-membership functions are defined in the equations (1) and (2), respectively. Functions $f_A^L(x), f_A^R(x), g_A^L(x)$ and $g_A^R(x)$ can be analytically expressed by utilizing equations (1) and (2) as follows:

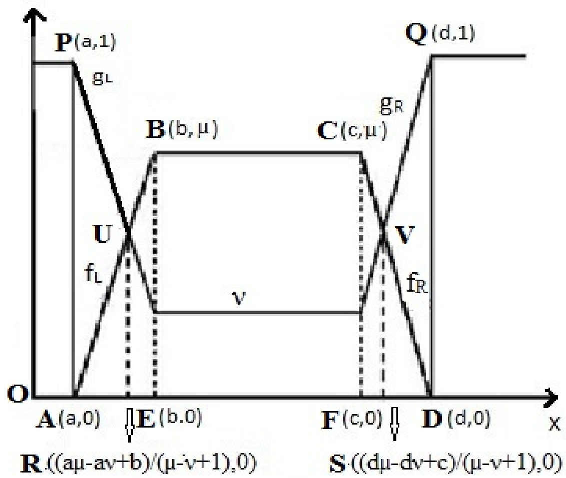

FIGURE 1. TrIFN A

$f_A^L(x) = \frac{\mu(x-a)}{b-a},\ for\ a \leq x \leq b;\ g_A^L(x) = \frac{(x-b)+\nu(a-x)}{a-b},\ for\ a \leq x \leq b;$

$f_A^R(x) = \frac{\mu(d-x)}{d-c},\ for\ c \leq x \leq d;\ g_A^R(x) = \frac{(x-c)+\nu(d-x)}{d-c},\ for\ c \leq x \leq d.$

In order to determine the centroid point (X_A, Y_A) of TrIFN A, the area under the membership and non-membership functions is considered together. To determine the X coordinate, first of all, the whole TrIFN is split into five rectangles (Fig.1): **ARUP, REBU, EFCB, FSVC** and **SDQV**, where the coordinates of the corner points of rectangles are computed as follows:

$\mathbf{A} : (a, 0),\ \mathbf{B} : (b, \mu), \mathbf{C} : (c, \mu), \mathbf{D} : (d, 0),\ \mathbf{E} : (b, 0),$
$\mathbf{F} : (c, 0),\ \mathbf{Q} : (d, 1), \mathbf{R} : (\frac{a\mu-a\nu+b}{\mu-\nu+1}, 0),\ \mathbf{S} : (\frac{d\mu-d\nu+c}{\mu-\nu+1}, 0),$
$\mathbf{P} : (a, 1), \mathbf{U} :(\frac{a\mu-a\nu+b}{\mu-\nu+1}, \frac{\mu}{\mu-\nu+1}),\ \mathbf{V} : (\frac{d\mu-d\nu+c}{\mu-\nu+1}, \frac{\mu}{\mu-\nu+1}).$

Then the X coordinate (X_A) of the centroid point of TrIFN A can be computed by using the following formula.

$$X_A = \frac{\int\limits_a^{R_x} x g_A^L dx + \int\limits_{R_x}^{b} x f_A^L dx + \int\limits_b^c x\mu dx + \int\limits_c^{S_x} x f_A^R dx + \int\limits_{S_x}^{d} x g_A^R dx}{\int\limits_a^{R_x} g_A^L dx + \int\limits_{R_x}^{b} f_A^L dx + \int\limits_b^c \mu dx + \int\limits_c^{S_x} f_A^R dx + \int\limits_{S_x}^{d} g_A^R dx} \tag{3}$$

where R_x and S_x are X coordinate of the points **R** and **S** respectively.

Step II: *Computation of Y coordinate of the centroid point.*
To determine the Y coordinate of TrIFN, the inverse functions of left and right parts of the membership and non-membership functions of TrIFN A are considered. The

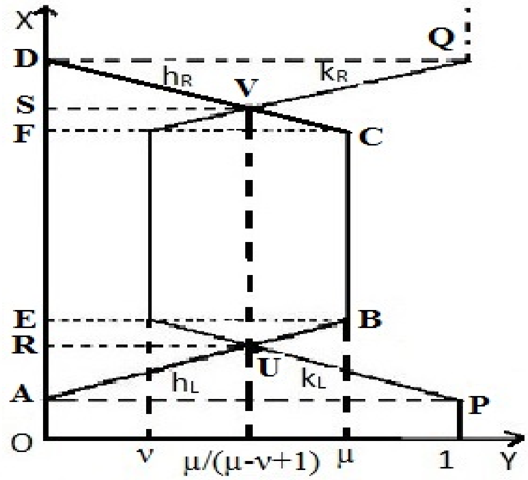

Figure 2. Inverse of TrIFN A

whole area is split into three geometric areas (Fig.2): **ABCD, DVQ** and **APU**, where the coordinates of the corner points of geometric areas are provided in Step I. As f_A^L, f_A^R, g_A^L and g_A^R are strictly monotonic and continuous functions, their inverse functions should exist and also be continuous and strictly monotonic. Let the inverse functions of f_A^L and f_A^R be $h_A^L : [0,\mu] \to [a,b]$ and $h_A^R : [0,\mu] \to [c,d]$, respectively. Again, $k_A^L : [0,\nu] \to [a,b]$ and $k_A^R : [0,\nu] \to [c,d]$ be the inverse functions of g_A^L and g_A^R respectively. The inverse functions $h_A^L(y), h_A^R(y), k_A^L(y)$ and $k_A^R(y)$ (by using equations (1) and (2)) can be analytically expressed as follows:

$$h_A^L(y) = a + \frac{(b-a)y}{\mu},\ for\ 0 \le y \le \mu;\ k_A^L(y) = \frac{(a-b)y+(b-a\nu)}{1-\nu},\ for\ \nu \le y \le 1;$$

$$h_A^R(y) = d - \frac{(d-c)y}{\mu},\ for\ 0 \le y \le \mu;\ k_A^R(y) = \frac{(d-c)y+(c-d\nu)}{1-\nu},\ for\ \nu \le y \le 1.$$

Then the Y coordinate (Y_A) of the centroid point of TrIFN A can be computed by using the following formula.

$$Y_A = \frac{\int_0^{\mu} y(h_A^R - h_A^L)dy + [\int_0^1 yd.dy - \int_0^{V_y} yh_A^R dy - \int_{V_y}^1 yk_A^R dy] + [\int_0^{U_y} yh_A^L dy + \int_{U_y}^1 yk_A^L dy - \int_0^1 aydy]}{\int_0^{\mu} (h_A^R - h_A^L)dy + [\int_0^1 d.dy - \int_0^{V_y} h_A^R dy - \int_{V_y}^1 k_A^R dy] + [\int_0^{U_y} h_A^L dy + \int_{U_y}^1 k_A^L dy - \int_0^1 ady]} \qquad (4)$$

where U_y and V_y are the Y coordinates of the points **U** and **V** respectively.

The rationality of the centroid formulae (3) and (4) were justified in [14]. Now, the ranking may be done in the following way:

For any two different TrIFNs A and B with centroid points (X_A, Y_A) and (X_B, Y_B) respectively, we have

(a)If $X_A > X_B$, then $A > B$;
(b)If $X_A < X_B$, then $A < B$;
(c)If $X_A = X_B$, then
if $Y_A > Y_B$, then $A > B$;
else if $Y_A < Y_B$, then $A < B$;
else $Y_A = Y_B$, then $A = B$.

We rank TrIFNs A and B based on their X's values. If their X's values are equal then the attention is given to the Y's values.

3. Basic Aggregation Operators

In this section, we review some basic aggregation techniques and some of their fundamental characteristics. We start by recalling the concept of a generating function.

We recall that a generating function $g : [0, 1] \to [-\infty, \infty]$ is a univariate continuous strictly monotone function. Generating function g is invertible but g is not necessarily a bijection function.

Definition 3.1. [5] Let $\{x_1, x_2, ..., x_n\}$ be a collection of real numbers and $w = (w_1, w_2, .., w_n)^T$ be the weight vector of x_i and g be a generating function. Then the weighted quasi-arithmetic mean is

$$M(x_1, x_2, ..., x_n) = g^{-1}\Big(\sum_{j=1}^{n} w_j g(x_j)\Big)$$

Note: It is to be noted that, the weighted power harmonic means are a subclass of weighted quasi-arithmetic means with the following generating function

$$g(t) = \begin{cases} \dfrac{1}{t^r}, & \text{for } r \neq 0, \\ log(t), & \text{for } r = 0, \end{cases}$$

So, definition of weighted power harmonic mean can be given as follows:

Definition 3.2. Let $\{x_1, x_2, ..., x_n\}$ be a collection of real numbers and $w = (w_1, w_2, .., w_n)^T$ be the weight vector of x_i and $r \in \mathbb{R}$. Then the weighted power harmonic (WPH) mean operator is defined as follows:

$$WPH(x_1, x_2, ..., x_n) = \begin{cases} \dfrac{1}{\Big(\sum\limits_{j=1}^{n} \dfrac{w_j}{x_j^r}\Big)^{\frac{1}{r}}}, & \text{for } r \neq 0, \\ \prod\limits_{j=1}^{n} x_j^{w_j}, & \text{for } r = 0, \end{cases} \tag{5}$$

In the following we consider some special cases of power harmonic mean operator.

- If $r = \infty$ then $WPH(x_1, x_2, ..., x_n) = min\{x_1, x_2, ..., x_n\}$

- If $r = -\infty$ then $WPH(x_1, x_2, ..., x_n) = max\{x_1, x_2, ..., x_n\}$
- If $r = 1$ then $WPH(x_1, x_2, ..., x_n)$=weighted harmonic mean=$\frac{1}{\sum_j^n \frac{w_j}{x_j}}$
- If $r = 0$ then $WPH(x_1, x_2, ..., x_n)$=weighted geometric mean=$\prod_j^n x_j^{w_j}$
- If $r = -1$ then $WPH(x_1, x_2, ..., x_n)$=weighted arithmetic mean=$\sum_j^n w_j x_j$
- If $r = -2$ then $WPH(x_1, x_2, ..., x_n)$=weighted quadratic mean=$\sqrt{\sum_j^n w_j x_j^2}$

4. Four Kinds of Power Harmonic Operators of TrIFNs

In this section, weighted power harmonic mean operator is defined under intutionistic fuzzy environment. Let Ω be the set of TrIFNs.

Definition 4.1. Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs. A trapezoidal intuitionistic fuzzy weighted power harmonic mean (TrIFWPHM) operator is a mapping TrIFWPHM : $\Omega^n \to \Omega$ which is defined by

$$TrIFWPHM(A_1, A_2, ..., A_n) = \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{A_j^r}\Big)^{\frac{1}{r}}} \tag{6}$$

where $w = (w_1, w_2, .., w_n)^T$ be the weight vector of $A_j(j = 1, .., n), w_j > 0, \sum_{j=1}^{n} w_j = 1$ and $r \in \mathbb{R}^*(= \mathbb{R} \setminus 0)$.

Theorem 4.2. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs and $w = (w_1, w_2, .., w_n)^T$ is the weight vector of $A_j(j = 1, .., n), w_j > 0$, $\sum_{j=1}^{n} w_j = 1$ and $r \in \mathbb{R}^*$. Then, the aggregated value by utilizing the operator TrIFWPHM, is also a TrIFN and*

$$\begin{aligned} &TrIFWPHM(A_1, A_2, ..., A_n) \\ &= \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}}\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big] \end{aligned} \tag{7}$$

Proof. By the operation laws defined in Definition 2.4, we have,

$$A_1^r = \begin{cases} [(a_1^r, b_1^r, c_1^r, d_1^r); \mu_1, \nu_1], & \text{for } r > 0, \\ [(d_1^r, c_1^r, b_1^r, a_1^r); \mu_1, \nu_1], & \text{for } r < 0 \end{cases}$$

Consider $r > 0$. This theorem can be proved by mathematical inductions.

For n=2,

$$\frac{1}{\frac{w_1}{A_1^r}+\frac{w_2}{A_2^r}} = \frac{1}{\frac{w_1}{[(a_1^r,b_1^r,c_1^r,d_1^r);\mu_1,\nu_1]}+\frac{w_2}{[(a_2^r,b_2^r,c_2^r,d_2^r);\mu_2,\nu_2]}}$$
$$= \frac{1}{w_1[(\frac{1}{d_1^r},\frac{1}{c_1^r},\frac{1}{b_1^r},\frac{1}{a_1^r});\mu_1,\nu_1]+w_2[(\frac{1}{d_2^r},\frac{1}{c_2^r},\frac{1}{b_2^r},\frac{1}{a_2^r});\mu_2,\nu_2]}$$
$$= \frac{1}{[(\frac{w_1}{d_1^r}+\frac{w_2}{d_2^r},\frac{w_1}{c_1^r}+\frac{w_2}{c_2^r},\frac{w_1}{b_1^r}+\frac{w_2}{b_2^r},\frac{w_1}{a_1^r}+\frac{w_2}{a_2^r});\min\{\mu_1,\mu_2\},\max\{\nu_1,\nu_2\}]} \tag{8}$$
$$= \Big[\Big(\frac{1}{\frac{w_1}{a_1^r}+\frac{w_2}{a_2^r}},\frac{1}{\frac{w_1}{b_1^r}+\frac{w_2}{b_2^r}},\frac{1}{\frac{w_1}{c_1^r}+\frac{w_2}{c_2^r}},\frac{1}{\frac{w_1}{d_1^r}+\frac{w_2}{d_2^r}}\Big);\min\{\mu_1,\mu_2\},\max\{\nu_1,\nu_2\}\Big]$$

Therefore, $TrIFWPHM(A_1,A_2) = \frac{1}{\left(\frac{w_1}{A_1^r}+\frac{w_2}{A_2^r}\right)^{\frac{1}{r}}}$

$$= \Big[\Big(\frac{1}{(\frac{w_1}{a_1^r}+\frac{w_2}{a_2^r})^{\frac{1}{r}}},\frac{1}{(\frac{w_1}{b_1^r}+\frac{w_2}{b_2^r})^{\frac{1}{r}}},\frac{1}{(\frac{w_1}{c_1^r}+\frac{w_2}{c_2^r})^{\frac{1}{r}}},\frac{1}{(\frac{w_1}{d_1^r}+\frac{w_2}{d_2^r})^{\frac{1}{r}}}\Big);\min\{\mu_1,\mu_2\},\max\{\nu_1,\nu_2\}\Big]$$
$$= \Big[\Big(\frac{1}{(\sum_{j=1}^{2}\frac{w_j}{a_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{2}\frac{w_j}{b_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{2}\frac{w_j}{c_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{2}\frac{w_j}{d_j^r})^{\frac{1}{r}}}\Big);\min_j\{\mu_j\},\max_j\{\nu_j\}\Big]$$

So, equation (7) is true for n=2.

Assume that (7) holds for n=k, *i.e.*,

$$TrIFWPHM(A_1,A_2,...,A_k)$$
$$= \Big[\Big(\frac{1}{(\sum_{j=1}^{k}\frac{w_j}{a_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{k}\frac{w_j}{b_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{k}\frac{w_j}{c_j^r})^{\frac{1}{r}}},\frac{1}{(\sum_{j=1}^{k}\frac{w_j}{d_j^r})^{\frac{1}{r}}}\Big);\min_j\{\mu_j\},\max_j\{\nu_j\}\Big]$$

Now for n=k+1,

$$TrIFWPHM(A_1,A_2,...,A_k,A_{k+1}) = \frac{1}{\Big(\sum_{j=1}^{k}\frac{w_j}{A_j^r}+\frac{w_{k+1}}{A_{k+1}^r}\Big)^{\frac{1}{r}}}$$

$$= \frac{1}{\Big[\Big(\Big(\sum_{j=1}^{k}\frac{w_j}{d_j^r}+\frac{w_{k+1}}{d_{k+1}^r}\Big)^{\frac{1}{r}},\Big(\sum_{j=1}^{k}\frac{w_j}{c_j^r}+\frac{w_{k+1}}{c_{k+1}^r}\Big)^{\frac{1}{r}},\Big(\sum_{j=1}^{k}\frac{w_j}{b_j^r}+\frac{w_{k+1}}{b_{k+1}^r}\Big)^{\frac{1}{r}},\Big(\sum_{j=1}^{k}\frac{w_j}{a_j^r}+\frac{w_{k+1}}{a_{k+1}^r}\Big)^{\frac{1}{r}}\Big);\alpha,\beta\Big]}$$

$\alpha = \min\{\mu_j,\mu_{k+1}\}, \beta = \max\{\nu_j,\nu_{k+1}\}$

(using equation (8) and operational laws)

$$= \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{k}\frac{w_j}{a_j^r}+\frac{w_{k+1}}{a_{k+1}^r}\Big)^{\frac{1}{r}}},\frac{1}{\Big(\sum_{j=1}^{k}\frac{w_j}{b_j^r}+\frac{w_{k+1}}{b_{k+1}^r}\Big)^{\frac{1}{r}}},\frac{1}{\Big(\sum_{j=1}^{k}\frac{w_j}{c_j^r}+\frac{w_{k+1}}{c_{k+1}^r}\Big)^{\frac{1}{r}}},$$
$$\frac{1}{\Big(\sum_{j=1}^{k}\frac{w_j}{d_j^r}+\frac{w_{k+1}}{d_{k+1}^r}\Big)^{\frac{1}{r}}}\Big);\min\{\mu_j,\mu_{k+1}\},\max\{\nu_j,\nu_{k+1}\}\Big]$$

$$= \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{k+1} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{k+1} \frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{k+1} \frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{k+1} \frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}} \Big); \min_j\{\mu_j\}, \max_j\{\nu_j\} \Big]$$

Hence, by the mathematical induction, equation (7) holds for all n.

Similarly, for $r < 0$, theorem can be proved easily. □

Now, we investigate some desirable properties of the TrIFWPHM operator.

P1) **Idempotency**: If all A_j are equal, *i.e.*, $A_j = A = [(a, b, c, d); \mu, \nu]$, $\forall j (j = 1, 2, ..., n)$. Then

$$TrIFWPHM(A_1, A_2, ..., A_n) = TrIFWPHM(A, A, ..., A) = A.$$

Proof. We have, $TrIFWPHM(A_1, A_2, ..., A_n)$

$$= \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}} \Big); \min_j\{\mu_j\}, \max_j\{\nu_j\} \Big]$$

$$= \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{n} w_j\Big)^{\frac{1}{r}}/a}, \frac{1}{\Big(\sum_{j=1}^{n} w_j\Big)^{\frac{1}{r}}/b}, \frac{1}{\Big(\sum_{j=1}^{n} w_j\Big)^{\frac{1}{r}}/c}, \frac{1}{\Big(\sum_{j=1}^{n} w_j\Big)^{\frac{1}{r}}/d} \Big); \min_j\{\mu_j\}, \max_j\{\nu_j\} \Big]$$

(as $A_j = A = [(a, b, c, d); \mu, \nu]$, $\forall j (j = 1, 2, ..., n)$)

$= [(a, b, c, d); \mu, \nu] (\because \sum_{j=1}^{n} w_j = 1, \min\{\mu, \mu, ..., \mu\} = \mu, \max\{\nu, \nu, ..., \nu\} = \nu)$

=A □

P2) **Monotonicity**: Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ and $A_j^{'} = [(a_j^{'}, b_j^{'}, c_j^{'}, d_j^{'}); \mu_j^{'}, \nu_j^{'}]$ be two collections of TrIFNs. If $a_j \leq a_j^{'}$, $b_j \leq b_j^{'}$, $c_j \leq c_j^{'}$, $d_j \leq d_j^{'}$, $\mu_j \leq \mu_j^{'}$ and $\nu_j \geq \nu_j^{'}$, $\forall j (j = 1, 2, ..., n)$, then

$$TrIFWPHM(A_1, A_2, ..., A_n) \leq TrIFWPHM(A_1^{'}, A_2^{'}, ..., A_n^{'}).$$

Proof. **Case-I**: For $r > 0$.

Since $a_j \leq a_j^{'} \Rightarrow a_j^r \leq a_j^{'r}$, $\forall j$.

$\Rightarrow \frac{w_j}{a_j^r} \geq \frac{w_j}{a_j^{'r}}$ $(\because all\ w_j > 0)$

$$\Rightarrow \Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n} \frac{w_j}{a_j^{'r}}\Big)^{\frac{1}{r}}$$

$$\Rightarrow \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^{'r}}\Big)^{\frac{1}{r}}}$$

Similarly, $\frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{b_j^{'r}}\Big)^{\frac{1}{r}}}$, $\frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{c_j^{'r}}\Big)^{\frac{1}{r}}}$
and $\frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{d_j^{'r}}\Big)^{\frac{1}{r}}}$

Since $\mu_j \leq \mu_j^{'},\ \forall j \Rightarrow \min\{\mu_j\} \leq \min\{\mu_j^{'}\},\ \forall j$

and $\nu_j \geq \nu_j^{'},\ \forall j \Rightarrow \max\{\nu_j\} \geq \max\{\nu_j^{'}\},\ \forall j$

Hence,

$$\Big[\Big(\frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}}\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big]$$
$$\leq \Big[\Big(\frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{a_j^{'r}}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{b_j^{'r}}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{c_j^{'r}}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{d_j^{'r}}\Big)^{\frac{1}{r}}}\Big); \min_j\{\mu_j^{'}\}, \max_j\{\nu_j^{'}\}\Big]$$

$$\Rightarrow TrIFWPHM(A_1, A_2, ..., A_n) \leq TrIFWPHM(A_1^{'}, A_2^{'}, ..., A_n^{'}).$$

Case-II: For $r < 0$.

Since $a_j \leq a_j^{'} \Rightarrow a_j^r \geq a_j^{'r}$, $\forall j$.

$$\Rightarrow \tfrac{w_j}{a_j^r} \leq \tfrac{w_j}{a_j^{'r}}\ (\because all\ w_j > 0)$$
$$\Rightarrow \Big(\sum_{j=1}^{n}\frac{w_j}{a_j^r}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n}\frac{w_j}{a_j^{'r}}\Big)^{\frac{1}{r}}$$
$$\Rightarrow \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum\limits_{j=1}^{n}\frac{w_j}{a_j^{'r}}\Big)^{\frac{1}{r}}}$$

So, in the same way (as Case-I), it can be proved that

$$TrIFWPHM(A_1, A_2, ..., A_n) \leq TrIFWPHM(A_1^{'}, A_2^{'}, ..., A_n^{'}).$$ □

P3) **Boundedness**: Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ $(j = 1, 2, .., n)$ be a collection of TrIFNs and let

$$A^- = [(\min_j\{a_j\}, \min_j\{b_j\}, \min_j\{c_j\}, \min_j\{d_j\}); \min_j\{\mu_j\}, \max_j\{\nu_j\}]$$
$$A^+ = [(\max_j\{a_j\}, \max_j\{b_j\}, \max_j\{c_j\}, \max_j\{d_j\}); \max_j\{\mu_j\}, \min_j\{\nu_j\}].$$

Then $A^- \leq TrIFWPHM(A_1, A_2, ..., A_n) \leq A^+$.

Proof. **Case-I**: For $r > 0$.

Since $\min\{a_j\} \leq a_j \leq \max\{a_j\}$, $\forall j$

$$\Rightarrow \frac{w_j}{\min\{a_j\}} \geq \frac{w_j}{a_j} \geq \frac{w_j}{\max\{a_j\}} \ (\because all\ w_j > 0)$$
$$\Rightarrow \Big(\sum_{j=1}^{n} \frac{w_j}{\min\{a_j^r\}}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n} \frac{w_j}{\max\{a_j^r\}}\Big)^{\frac{1}{r}}$$
$$\Rightarrow \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\min\{a_j^r\}}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\max\{a_j^r\}}\Big)^{\frac{1}{r}}}$$

Similarly, $\frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\min\{b_j^r\}}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{b_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\max\{b_j^r\}}\Big)^{\frac{1}{r}}}$,

$$\frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\min\{c_j^r\}}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{c_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\max\{c_j^r\}}\Big)^{\frac{1}{r}}}$$

and $\frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\min\{d_j^r\}}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{d_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\max\{d_j^r\}}\Big)^{\frac{1}{r}}}$

Again, $\min\{\mu_j\} \leq \mu_j \leq \max\{\mu_j\}\ \forall j$

and $\min\{\nu_j\} \leq \nu_j \leq \max\{\nu_j\}\ \forall j$

By the monotonicity and idempotency properties,

$TrIFWPHM(A^-) \leq TrIFWPHM(A_1, A_2, ..., A_n) \leq TrIFWPHM(A^+)$
$\Rightarrow A^- \leq TrIFWPHM(A_1, A_2, ..., A_n) \leq A^+$

Case-II: For $r < 0$.

Since $\min\{a_j\} \leq a_j \leq \max\{a_j\}\ \forall j$

$\Rightarrow \min\{a_j^r\} \geq a_j^r \geq \max\{a_j^r\}$ (as $r < 0$)

$$\Rightarrow \frac{w_j}{\min\{a_j^r\}} \leq \frac{w_j}{a_j^r} \leq \frac{w_j}{\max\{a_j^r\}} \ (\because all\ w_j > 0)$$
$$\Rightarrow \Big(\sum_{j=1}^{n} \frac{w_j}{\min\{a_j^r\}}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}} \geq \Big(\sum_{j=1}^{n} \frac{w_j}{\max\{a_j^r\}}\Big)^{\frac{1}{r}}$$
$$\Rightarrow \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\min\{a_j^r\}}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{a_j^r}\Big)^{\frac{1}{r}}} \leq \frac{1}{\Big(\sum_{j=1}^{n} \frac{w_j}{\max\{a_j^r\}}\Big)^{\frac{1}{r}}}$$

So, in a way similar to Case-I, it can be proved that

$$A^{-} \leq TrIFWPHM(A_1, A_2, ..., A_n) \leq A^{+} \qquad \square$$

Remark 4.3. The operator TrIFWPHM defined in equation (7) is a generalized average operator. The generalization property of TrIFWPHM operator can be described by taking different values of the parameter 'r' [5]. In the following list, let us consider some special cases.

- When r=1, the operator TrIFWPHM reduces to TrIFWHM operator which is defined as follows:
$$TrIFWHM(A_1, A_2, ..., A_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{A_j}} = \Big[\Big(\frac{1}{\sum_{j=1}^{n} \frac{w_j}{a_j}}, \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}}, \frac{1}{\sum_{j=1}^{n} \frac{w_j}{c_j}}, \frac{1}{\sum_{j=1}^{n} \frac{w_j}{d_j}}\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big] \qquad (9)$$
- When r=0, the operator TrIFWPHM reduces to TrIFWGM operator which is defined as follows:
$$TrIFWGM(A_1, A_2, ..., A_n) = \prod_{j=1}^{n} A_j^{w_j} = \Big[\Big(\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}, \prod_{j=1}^{n} d_j^{w_j}\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big] \qquad (10)$$
- When r=-1, the operator TrIFWPHM reduces to TrIFWAM operator which is defined as follows:
$$TrIFWAM(A_1, A_2, ..., A_n) = \sum_{j=1}^{n} w_j A_j = \Big[\Big(\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big] \qquad (11)$$
- When r=-2, the operator TrIFWPHM reduces to TrIFWQM operator which is defined as follows:
$$TrIFWQM(A_1, A_2, ..., A_n) = \sqrt{\sum_{j}^{n} w_j A_j^2} = \Big[\Big(\sqrt{\sum_{j}^{n} w_j a_j^2}, \sqrt{\sum_{j}^{n} w_j b_j^2}, \sqrt{\sum_{j}^{n} w_j c_j^2}, \sqrt{\sum_{j}^{n} w_j d_j^2}\Big); \min_j\{\mu_j\}, \max_j\{\nu_j\}\Big] \qquad (12)$$

Based on Definition 2.4 and ordered weighted harmonic mean operator [10, 62], in the following, we introduce trapezoidal intuitionistic fuzzy ordered weighted power harmonic mean (TrIFOWPHM) operator.

Definition 4.4. Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs. A trapezoidal intuitionistic fuzzy ordered weighted power harmonic mean (TrIFOWPHM) operator is a mapping TrIFOWPHM : $\Omega^n \rightarrow \Omega$ which is defined by

$$TrIFOWPHM(A_1, A_2, ..., A_n) = \frac{1}{\Big(\sum_{j=1}^{n} \frac{\omega_j}{A_{\rho(j)}^r}\Big)^{\frac{1}{r}}} \qquad (13)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in \mathbb{R}^*$. $(\rho_1, \rho_2, ..., \rho_n)$ is a permutation of $(1, 2, ..., n)$ and $A_{\rho(j)} \leq A_{\rho(j-1)}, \ \forall j = 2, 3, ..., n$.

Similar to Theorem 4.2, we have the following results.

Theorem 4.5. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs, then their aggregated value by using the TrIFOWPHM operator is also a TrIFN, i.e.,*

$$TrIFOWPHM(A_1, A_2, ..., A_n)$$
$$= \Big[\Big(\frac{1}{\big(\sum_{j=1}^{n} \frac{\omega_j}{a^r_{\rho(j)}}\big)^{\frac{1}{r}}}, \frac{1}{\big(\sum_{j=1}^{n} \frac{\omega_j}{b^r_{\rho(j)}}\big)^{\frac{1}{r}}}, \frac{1}{\big(\sum_{j=1}^{n} \frac{\omega_j}{c^r_{\rho(j)}}\big)^{\frac{1}{r}}}, \frac{1}{\big(\sum_{j=1}^{n} \frac{\omega_j}{d^r_{\rho(j)}}\big)^{\frac{1}{r}}}\Big); \min_j\{\mu_{\rho(j)}\}, \max_j\{\nu_{\rho(j)}\}\Big] \quad (14)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in \mathbb{R}^*$. $(\rho_1, \rho_2, ..., \rho_n)$ is a permutation of (1,2,...,n) and $A_{\rho(j)} \leq A_{\rho(j-1)}, \ j = 2, 3, ..., n$.

The desirable properties of TrIFOWPHM operator are given below.

Theorem 4.6. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ be a collection of TrIFNs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in R^*$. Then we have,*

(1) **Idempotency***: If all A_j are equal, i.e., $A_j = A = [(a, b, c, d); \mu, \nu], \ \forall j(j = 1, 2, ..., n)$. Then*

$$TrIFOWPHM(A_1, A_2, ..., A_n) = TrIFOWPHM(A, A, ..., A) = A.$$

(2) **Monotonicity***: Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ and $A'_j = [(a'_j, b'_j, c'_j, d'_j); \mu'_j, \nu'_j]$ be two collections of TrIFNs. If $a_j \leq a'_j$, $b_j \leq b'_j$, $c_j \leq c'_j$, $d_j \leq d'_j$, $\mu_j \leq \mu'_j$ and $\nu_j \geq \nu'_j$, $\forall j$. Then*

$$TrIFOWPHM(A_1, A_2, ..., A_n) \leq TrIFOWPHM(A'_1, A'_2, ..., A'_n).$$

(3) **Boundedness***: Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ $(j = 1, 2, .., n)$ be a collection of TrIFNs and let*

$$A^- = [(\min_j\{a_j\}, \min_j\{b_j\}, \min_j\{c_j\}, \min_j\{d_j\}); \min_j\{\mu_j\}, \max_j\{\nu_j\}]$$
$$A^+ = [(\max_j\{a_j\}, \max_j\{b_j\}, \max_j\{c_j\}, \max_j\{d_j\}); \max_j\{\mu_j\}, \min_j\{\nu_j\}].$$

Then $A^{-} \leq TrIOFWPHM(A_1, A_2, ..., A_n) \leq A^{+}$.

(4) **Commutativity**: *If* $\{A_1^{'}, A_2^{'}, ..., A_n^{'}\}$ *is any permutation of* $\{A_1, A_2, ..., A_n\}$. *Then*

$$TrIFOWPHM(A_1, A_2, ..., A_n) = TrIFOWPHM(A_1^{'}, A_2^{'}, ..., A_n^{'})$$

Remark 4.7. The operator TrIFWPHM (defined in the equation (7)) does not have commutative property, but the operator TrIFOWPHM (defined in the equation (14)) has commutative property. Furthermore, in the following, depending on the weight vector ω, some special cases of the TrIFOWPHM operator are analyzed.

- If the weight vector $\omega = (1, 0, ..., 0)^T$, then $TrIFOWPHM(A_1, A_2, ..., A_n) = \max_j\{A_j\}$
- If the weight vector $\omega = (0, 0, ..., 1)^T$, then $TrIFOWPHM(A_1, A_2, ..., A_n) = \min_j\{A_j\}$
- If $\omega_j = 1$ and $\omega_i = 0, i \neq j$, then $TrIFOWPHM(A_1, A_2, ..., A_n) = A_{\rho(j)}$, where, $A_{\rho(j)}$ is the j^{th} largest TrIFN $A_j (j = 1, 2, .., n)$
- If the weight vector $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then TrIFOWPHM operator reduces to TrIFPHM operator.

Remark 4.8. Depending on different values of the parameter 'r', some special cases of the TrIFOWPHM operator are given below.

- When r=1, the operator TrIFOWPHM reduces to trapezoidal intuitionistic fuzzy ordered weighted harmonic mean (TrIFOWHM) operator.
- When r=0, the operator TrIFOWPHM reduces to trapezoidal intuitionistic fuzzy ordered weighted geometric mean (TrIFOWGM) operator.
- When r=-1, the operator TrIFOWPHM reduces to trapezoidal intuitionistic fuzzy ordered weighted arithmetic mean (TrIFOWAM) operator.
- When r=-2, the operator TrIFOWPHM reduces to trapezoidal intuitionistic fuzzy ordered weighted quadratic mean (TrIFOWQM) operator.

In the following, we develop the trapezoidal intuitionistic fuzzy induced ordered weighted power harmonic mean (TrIFIOWPHM) operator which is defined as follows.

Definition 4.9. Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs. A trapezoidal intuitionistic fuzzy induced ordered weighted power harmonic mean (TrIFIOWPHM) operator is defined by

$$TrIFIOWPHM((U_1, A_1), (U_2, A_2), ..., (U_n, A_n)) = \frac{1}{\left(\sum_{j=1}^{n} \frac{\omega_j}{B_j^r}\right)^{\frac{1}{r}}} \tag{15}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in R^*$. $B_j = [(e_j, f_j, g_j, h_j); \mu_j^{'}, \nu_j^{'}]$ is the A_i value of the TrIFOWPHM pair

(U_i, A_i) having the j^{th} largest U_i. In (U_i, A_i), U_i is referred as the order including variable and A_i is referred as the argument variable.

Similar to Theorem 4.2, we have the following results.

Theorem 4.10. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs, then their aggregated value by using the TrIFIOWPHM operator is also a TrIFN, i.e.,*

$$TrIFIOWPHM(A_1, A_2, ..., A_n) = \Big[\Big(\frac{1}{\Big(\sum_{j=1}^{n}\frac{\omega_j}{e_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n}\frac{\omega_j}{f_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n}\frac{\omega_j}{g_j^r}\Big)^{\frac{1}{r}}}, \frac{1}{\Big(\sum_{j=1}^{n}\frac{\omega_j}{h_j^r}\Big)^{\frac{1}{r}}}\Big); \min_j\{\mu_j'\}, \max_j\{\nu_j'\}\Big] \tag{16}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in \mathbb{R}^*$.

The TrIFIOWPHM operator has the following properties.

Theorem 4.11. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ be a collection of TrIFNs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $A_j(j = 1, ..., n)$ and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$ and $r \in \mathbb{R}^*$. Then we have,*

(1) **Commutativity***: If $((U_1, A_1'), (U_2, A_2'), ..., (U_n, A_n'))$ is any permutation of $((U_1, A_1), (U_2, A_2), ..., (U_n, A_n))$, then*
$TrIFIOWPHM((U_1, A_1), (U_2, A_2), ..., (U_n, A_n))$
$= TrIFIOWPHM((U_1, A_1'), (U_2, A_2'), ..., (U_n, A_n'))$

(2) **Idempotency***: Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j]$ and $A = [(a, b, c, d); \mu, \nu]$. If $A_j = A$, $\forall j$, then*
$TrIFIOWPHM((U_1, A_1), (U_2, A_2), ..., (U_n, A_n)) = A.$

(3) **Monotonicity***: If $A_j \leq A_j'$, $\forall j$, then*
$TrIFIOWPHM((U_1, A_1), (U_2, A_2), ..., (U_n, A_n))$
$\leq TrIFIOWPHM((U_1, A_1'), (U_2, A_2'), ..., (U_n, A_n')).$

Remark 4.12. Depending on different values of the parameter 'r', some special cases of the TrIFIOWPHM operator are given below.

- When r=1, the operator TrIFIOWPHM reduces to trapezoidal intuitionistic fuzzy induced ordered weighted harmonic mean (TrIFIOWHM) operator.
- When r=0, the operator TrIFIOWPHM reduces to trapezoidal intuitionistic fuzzy induced ordered weighted geometric mean (TrIFIOWGM) operator.
- When r=-1, the operator TrIFIOWPHM reduces to trapezoidal intuitionistic fuzzy induced ordered weighted arithmetic mean (TrIFIOWAM) operator.

- When r=-2, the operator TrIFIOWPHM reduces to trapezoidal intuitionistic fuzzy induced ordered weighted quadratic mean (TrIFIOWQM) operator.

It is worth noticing that, the TrIFWPHM operator gives the importance of each argument and the TrIFOWPHM operator gives the importance of the ordered position of each argument. To avoid this limitation, we introduce a new operator, trapezoidal intuitionistic fuzzy hybrid power harmonic mean (TrIFhPHM) operator which emphasizes importance of both the given arguments and the ordered positions of arguments.

Definition 4.13. Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs. A trapezoidal intuitionistic fuzzy hybrid power harmonic mean (TrIFhPHM) operator is a mapping TrIFhPHM : $\Omega^n \to \Omega$ which is defined by

$$TrIFhPHM(A_1, A_2, ..., A_n) = \frac{1}{\Big(\sum_{j=1}^{n} \frac{\omega_j}{A'^r_{\rho(j)}}\Big)^{\frac{1}{r}}} \tag{17}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$, $r \in \mathbb{R}^*$ and $A'_{\rho(j)}$ is the j^{th} largest of the weighted TrIFNs $A'_j(A'_j = nw_jA_j, j = 1, 2, ..., n))$. Here n is called balancing coefficient and $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of A_j $(j = 1, 2, ..., n)$ and $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$ and $(\rho_1, \rho_2, ..., \rho_n)$ is a permutation of $(1, 2, ..., n)$ and $A'_{\rho(j)} \le A'_{\rho(j-1)}$, $\forall j = 2, 3, ..., n$.

Similar to Theorem 4.2, we have the following results.

Theorem 4.14. *Let $A_j = [(a_j, b_j, c_j, d_j); \mu_j, \nu_j](j = 1, 2, ..., n)$ be a collection of TrIFNs, then their aggregated value using the TrIFhPHM operator is also a TrIFN, i.e.,*

$$TrIFhPHM(A_1, A_2, ..., A_n)$$

$$= \Big[\Big(\frac{1}{(\sum_{j=1}^{n} \frac{\omega_j}{a'^r_{\rho(j)}})^{\frac{1}{r}}}, \frac{1}{(\sum_{j=1}^{n} \frac{\omega_j}{b'^r_{\rho(j)}})^{\frac{1}{r}}}, \frac{1}{(\sum_{j=1}^{n} \frac{\omega_j}{c'^r_{\rho(j)}})^{\frac{1}{r}}}, \frac{1}{(\sum_{j=1}^{n} \frac{\omega_j}{d'^r_{\rho(j)}})^{\frac{1}{r}}}\Big); \min_j\{\mu'_{\rho(j)}\}, \max_j\{\nu'_{\rho(j)}\}\Big] \tag{18}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the associated weight vector and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$, $r \in \mathbb{R}^*$. $(\rho_1, \rho_2, ..., \rho_n)$ is a permutation of $(1, 2, ..., n)$ and $A'_{\rho(j)} \le A'_{\rho(j-1)}$, $\forall j = 2, 3, ..., n$.

Remark 4.15. Depending on the weight vectors w and ω, some special cases of the operator TrIFhPHM are given below.

- If the weight vector $w = (w_1, w_2, ..., w_n)^T$ approaches to $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the vector $(A'_{\rho(1)}, A'_{\rho(2)}, ..., A'_{\rho(n)})$ approaches to $(A_{\rho(1)}, A_{\rho(2)}, ..., A_{\rho(n)})$ and the operator TrIFhPHM reduces to TrIFOWPHM operator (14).

- If the weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ approaches to $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the vector $(A'_{\rho(1)}, A'_{\rho(2)}, ..., A'_{\rho(n)})$ approaches to $(A_1, A_2, ..., A_n)$ and the operator TrIFhPHM reduces to TrIFWPHM operator (7).

Remark 4.16. Depending on different values of the parameter 'r', some special cases of the TrIFhPHM operator are given below.

- When r=1, the operator TrIFhPHM reduces to trapezoidal intuitionistic fuzzy hybrid harmonic mean (TrIFhHM) operator.
- When r=0, the operator TrIFhPHM reduces to trapezoidal intuitionistic fuzzy hybrid geometric mean (TrIFhGM) operator.
- When r=-1, the operator TrIFhPHM reduces to trapezoidal intuitionistic fuzzy hybrid arithmetic mean (TrIFhAM) operator.
- When r=-2, the operator TrIFhPHM reduces to trapezoidal intuitionistic fuzzy hybrid quadratic mean (TrIFhQM) operator.

5. An Application of Proposed Aggregation Operator on MAGDM Problem

In this section, a MAGDM problem, where the decision information is quantified by TrIFNs, is presented based on the proposed operator. Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m\}$ be the set of m alternatives. Experts assess these alternatives on the basis of some attributes. Let $C = \{C_1, C_2, ..., C_n\}$ be the set of n attributes and the corresponding weights are $w = (w_1, w_2, ..., w_n)^T$, where $w_j \in [0, 1]$, $j = 1, 2, ..., n$ and $\sum_{j=1}^{n} w_j = 1$. Let $D = \{D_1, D_2, ..., D_k\}$ be the set of k experts whose weights are $\psi = (\psi_1, \psi_2, ..., \psi_k)^T$, where $\psi_t \in [0, 1]$, $t = 1, 2, ..., k$ and $\sum_{t=1}^{k} \psi_t = 1$.

Generally, attribute set C can be divided into two subsets P and Q in which P is the subset of benefit attributes and Q is the subset of cost attributes and $P \cup Q = C, P \cap Q = \phi$. The expert's aim is to select the alternative which satisfies the higher value for benefit attributes and lower value for cost attributes, *i.e.,* the expert's intention is to give the maximum rating to the alternative with maximum benefit and minimum cost. Assume that, ratings of all the alternatives $\mathcal{A}_i (i = 1, 2, .., m)$ with respect to attributes $C_j (j = 1, 2, ..., n)$ are given by TrIFNs $A_{ij}^t = [(a_{ij}^t, b_{ij}^t, c_{ij}^t, d_{ij}^t); \mu_{ij}^t, \nu_{ij}^t]$, provided by the expert $D_t (t = 1, 2, ..., k)$. Here μ_{ij}^t and ν_{ij}^t represent the corresponding degree of satisfaction and dissatisfaction of the expert's judgement. A MAGDM problem with TrIFN can be described as follows:

Step 1: *Construction of decision matrix for individual expert D_t.*
Let $J_t = (A_{ij}^t)_{m \times n} (t = 1, 2, ..., k)$ be the decision matrix provided by the expert D_t which can be represented as

$$J_t = (A_{ij}^t)_{m \times n} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} A_{11}^t & A_{12}^t & \cdots & A_{1n}^t \\ A_{21}^t & A_{22}^t & \cdots & A_{2n}^t \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1}^t & A_{m2}^t & \cdots & A_{mn}^t \end{pmatrix} \end{array}.$$

Step 2: *Computation of the normalized decision matrix.*

In order to measure all attributes in the same scale and to facilitate inter-attribute comparisons, the primary task is to normalize the decision matrix $J_t(t = 1, 2, ..., k)$. The normalized decision matrix can be computed by using the following formulae

$$v_{ij}^t = \begin{cases} [(\frac{a_{ij}^t}{a_j^{max}}, \frac{b_{ij}^t}{a_j^{max}}, \frac{c_{ij}^t}{a_j^{max}}, \frac{d_{ij}^t}{a_j^{max}}); \mu_{ij}^t, \nu_{ij}^t], & \text{for } i = 1, 2, ..., m, j \in P, \\ [(1 - \frac{a_{ij}^t}{a_j^{max}}, 1 - \frac{b_{ij}^t}{a_j^{max}}, 1 - \frac{c_{ij}^t}{a_j^{max}}, 1 - \frac{d_{ij}^t}{a_j^{max}}); \mu_{ij}^t, \nu_{ij}^t], & \text{for } i = 1, 2, ..., m, j \in Q, \end{cases} \quad (19)$$

where P and Q are sets of benefit and cost attributes and $a_j^{max} = max\{a_{ij}^t, b_{ij}^t, c_{ij}^t, d_{ij}^t\}$, $i = 1, 2, ..., m, j = 1, 2, ..., n$. After normalization, the decision matrix J_t transformed into normalized decision matrix

$$N_t = (v_{ij}^t)_{m\times n} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} v_{11}^t & v_{12}^t & \cdots & v_{1n}^t \\ v_{21}^t & v_{22}^t & \cdots & v_{2n}^t \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1}^t & v_{m2}^t & \cdots & v_{mn}^t \end{pmatrix} \end{array}$$

where v_{ij}^t can be written as: $v_{ij}^t = [(v_{ij}^{t1}, v_{ij}^{t2}, v_{ij}^{t3}, v_{ij}^{t4}); \mu_{ij}^t, \nu_{ij}^t]$.

Step 3: *Accumulation of all k experts' ratings for each alternative.*

To aggregate all experts' ratings for each alternative with respect to each attribute, the degree of importance ψ_t of the expert D_t and the TrIFWPHM operator are used and aggregated matrix can be represented as follows:

$$R = (R_{ij})_{m\times n} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mn} \end{pmatrix} \end{array}$$

where R_{ij} can be calculated as

$$R_{ij} = TrIFWPHM(v_{ij}^1, v_{ij}^2, ..., v_{ij}^k) \quad (20)$$

Depending on the situation, the value of the parameter 'r' can be specified.

Step 4: *Computation of the final ratings of the alternatives over all attributes.*

The final ratings of all the alternatives over all attributes are derived by utilizing TrIFWPHM operator with the weight vector $w = (w_1, w_2, ..., w_n)^T$ of attributes and the aggregated value can be represented as

$$S_i = TrIFWPHM(R_{i1}, R_{i2}, ..., R_{in}) \quad (21)$$

Depending on the situation, the value of the parameter 'r' can also be specified in this step.

Step 5: *Ranking of the alternatives.*
Determine the rank of each alternatives according to the ranking process for TrIFNs defined in Section 2.3.

Step 6: End.

6. An Illustrative Example

In this section, a numerical example is adapted from [50] to illustrate the application of the proposed operator. A company wants to select the most appropriate supplier for one of the most important elements in their manufacturing system. After pre-evaluation, four suppliers $\mathcal{A}_i (i = 1, 2, 3, 4)$ are selected for further evaluation. The decision making committee assesses four suppliers on the basis of the following four attributes, quality of product C_1, technology capability C_2, pollution control C_3 and environment management C_4. In the decision making committee, there are four experts D_1, D_2, D_3 and D_4. Let $\psi = (0.22, 0.20, 0.28, 0.30)^T$ be the weight vector of experts and $w = (0.20, 0.30, 0.35, 0.15)^T$ be the weight vector of attributes.

Step 1: The ratings of all the alternatives (suppliers) with respect to attributes provided by four experts are given in Table 4 to Table 7.

	Atributes			
Alternatives	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(1,2,3,4);0.7,0.2]$	$[(2,3,4,5);0.5,0.4]$	$[(2,4,5,6);0.6,0.4]$	$[(15,16,18,20);0.8,0.1]$
$\mathcal{A}_2$	$[(4,5,6,7);0.6,0.3]$	$[(16,17,19,20);0.8,0.2]$	$[(3,4,5,6);0.7,0.2]$	$[(4,5,6,7);0.6,0.3]$
$\mathcal{A}_3$	$[(2,4,6,8);0.5,0.4]$	$[(2,4,6,7);0.7,0.2]$	$[(17,18,19,20);0.6,0.3]$	$[(3,4,6,7);0.7,0.1]$
$\mathcal{A}_4$	$[(17,18,19,20);0.5,0.5]$	$[(4,6,7,8);0.6,0.3]$	$[(3,6,8,9);0.5,0.5]$	$[(2,5,7,8);0.8,0.1]$

Table 4. Decision matrix given by expert D_1

	Attributes			
Alternatives	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(1,3,5,6);0.6,0.4]$	$[(2,3,4,6);0.6,0.3]$	$[(2,3,4,5);0.6,0.4]$	$[(17,18,19,20);0.6,0.3]$
$\mathcal{A}_2$	$[(3,5,6,7);0.6,0.3]$	$[(15,17,19,20);0.7,0.2]$	$[(3,4,5,6);0.7,0.2]$	$[(4,5,6,7);0.6,0.4]$
$\mathcal{A}_3$	$[(15,16,17,20);0.9,0.1]$	$[(2,4,5,7);0.5,0.3]$	$[(2,5,6,8);0.7,0.2]$	$[(3,5,6,7);0.8,0.1]$
$\mathcal{A}_4$	$[(2,6,8,9);0.6,0.2]$	$[(4,5,6,7);0.8,0.1]$	$[(16,17,19,20);0.7,0.3]$	$[(2,5,7,8);0.5,0.4]$

Table 5. Decision Matrix Given by Expert D_2

	Attributes			
Alternatives	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(16,17,18,20);0.8,0.1]$	$[(4,5,6,7);0.5,0.4]$	$[(2,4,5,6);0.6,0.4]$	$[(3,4,6,7);0.7,0.2]$
$\mathcal{A}_2$	$[(3,5,6,7);0.6,0.2]$	$[(2,3,4,6);0.6,0.3]$	$[(3,4,5,6);0.7,0.2]$	$[(16,17,19,20);0.8,0.2]$
$\mathcal{A}_3$	$[(4,5,6,8);0.5,0.4]$	$[(1,2,3,4);0.7,0.2]$	$[(17,18,19,20);0.6,0.25]$	$[(3,4,5,6);0.7,0.1]$
$\mathcal{A}_4$	$[(2,4,5,7);0.7,0.2]$	$[(15,16,18,19);0.6,0.2]$	$[(4,5,6,7);0.5,0.4]$	$[(2,5,6,7);0.6,0.3]$

Table 6. Decision Matrix Given by Expert D_3

	Attributes			
Alternatives	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(3,4,6,7);0.5,0.5]$	$[(5,6,7,8);0.4,0.4]$	$[(15,17,18,20);0.5,0.4]$	$[(4,7,8,9);0.7,0.2]$
$\mathcal{A}_2$	$[(17,18,19,20);0.8,0.1]$	$[(1,2,3,4);0.6,0.3]$	$[(4,5,6,8);0.4,0.4]$	$[(2,3,4,5);0.9,0.1]$
$\mathcal{A}_3$	$[(4,5,6,7);0.5,0.4]$	$[(16,17,18,19);0.8,0.1]$	$[(1,2,3,4);0.7,0.2]$	$[(5,6,7,8);0.5,0.4]$
$\mathcal{A}_4$	$[(2,4,6,8);0.7,0.2]$	$[(3,5,7,9);0.5,0.3]$	$[(5,6,7,8);0.5,0.4]$	$[(16,18,19,20);0.6,0.2]$

Table 7. Decision Matrix Given by Expert D_4

Step 2: The aforementioned attributes are benefit attributes. So, by using the equation (19), the normalized TrIFN decision matrices are obtained, which are shown in Table 8 to Table 11.

Step 3: The overall experts' ratings of all the alternatives $\mathcal{A}_i$ are computed by using equation (20) with $r = 1$ and the weight vector $\psi = (0.22, 0.20, 0.28, 0.30)^T$ of experts and aggregated values are given in Table 12. Here we have used TrIFWHM (TrIFWPHM with $r = 1$) operator as all the decision matrices (Table 8 to Table 11) consist of extreme outliers. It is worthwhile to mention that, too large data relative to others exist in Table 8 to Table 11. Thus, we have used TrIFWHM operator as a fusion technique, to relieve the influence of too large (i.e., extreme outlier) data in final results.

Step 4: The collective overall ratings of all the alternatives $\mathcal{A}_i$ are computed by using equation (21) with $r = -1$ and the weight vector $w = (0.20, 0.30, 0.35, 0.15)^T$ of attributes and final aggregated values are given in Table 13. Here TrIFWAM (TrIFWPHM with $r = -1$) operator is used as the data set (provided in Table 12) being fused, do not possess any outlier data.

	Attributes			
Alts.	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(0.05, 0.1, 0.15, 0.2); 0.7, 0.2]$	$[(0.1, 0.15, 0.2, 0.25); 0.5, 0.4]$	$[(0.1, 0.2, 0.25, 0.3); 0.6, 0.4]$	$[(0.75, 0.8, 0.9, 1); 0.8, 0.1]$
$\mathcal{A}_2$	$[(0.2, 0.25, 0.3, 0.35); 0.6, 0.3]$	$[(0.8, 0.85, 0.95, 1); 0.8, 0.2]$	$[(0.15, 0.2, 0.25, 0.3); 0.7, 0.2]$	$[(0.2, 0.25, 0.3, 0.35); 0.6, 0.3]$
$\mathcal{A}_3$	$[(0.1, 0.2, 0.3, 0.4); 0.5, 0.4]$	$[(0.1, 0.2, 0.3, 0.35); 0.7, 0.2]$	$[(0.85, 0.9, 0.95, 1); 0.6, 0.3]$	$[(0.15, 0.2, 0.3, 0.35); 0.7, 0.1]$
$\mathcal{A}_4$	$[(0.85, 0.9, 0.95, 1); 0.5, 0.5]$	$[(0.2, 0.3, 0.35, 0.4); 0.6, 0.3]$	$[(0.15, 0.3, 0.4, 0.45); 0.5, 0.5]$	$[(0.1, 0.25, 0.35, 0.4); 0.8, 0.1]$

TABLE 8. Normalized Decision Matrix Given by Expert D_1

	Attributes			
Alts.	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(0.05, 0.15, 0.25, 0.3); 0.6, 0.4]$	$[(0.1, 0.15, 0.2, 0.3); 0.6, 0.3]$	$[(0.1, 0.15, 0.2, 0.25); 0.6, 0.4]$	$[(0.85, 0.90, 0.95, 1); 0.6, 0.3]$
$\mathcal{A}_2$	$[(0.15, 0.25, 0.3, 0.35); 0.6, 0.3]$	$[(0.75, 0.85, 0.95, 1); 0.7, 0.2]$	$[(0.15, 0.2, 0.25, 0.3); 0.7, 0.2]$	$[(0.2, 0.25, 0.3, 0.35); 0.6, 0.4]$
$\mathcal{A}_3$	$[(0.75, 0.8, 0.85, 1.0); 0.9, 0.1]$	$[(0.1, 0.2, 0.25, 0.35); 0.5, 0.3]$	$[(0.1, 0.25, 0.3, 0.4); 0.7, 0.2]$	$[(0.15, 0.25, 0.3, 0.35); 0.8, 0.1]$
$\mathcal{A}_4$	$[(0.1, 0.3, 0.4, 0.45); 0.6, 0.2]$	$[(0.2, 0.25, 0.3, 0.35); 0.8, 0.1]$	$[(0.8, 0.85, 0.95, 1); 0.7, 0.3]$	$[(0.1, 0.25, 0.35, 0.4); 0.5, 0.4]$

TABLE 9. Normalized Decision Matrix Given by Expert D_2

	Attributes			
Alts.	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(0.8, 0.85, 0.9, 1.0); 0.9, 0.1]$	$[(0.2, 0.25, 0.3, 0.35); 0.5, 0.4]$	$[(0.1, 0.2, 0.25, 0.3); 0.6, 0.4]$	$[(0.15, 0.2, 0.3, 0.35); 0.8, 0.1]$
$\mathcal{A}_2$	$[(0.15, 0.25, 0.3, 0.35); 0.6, 0.2]$	$[(0.1, 0.15, 0.2, 0.3); 0.6, 0.2]$	$[(0.15, 0.2, 0.25, 0.3); 0.7, 0.2]$	$[(0.8, 0.85, 0.95, 1.0); 0.8, 0.2]$
$\mathcal{A}_3$	$[(0.2, 0.25, 0.3, 0.4); 0.5, 0.4]$	$[(0.05, 0.1, 0.15, 0.2); 0.7, 0.2]$	$[(0.85, 0.9, 0.95, 1); 0.6, 0.25]$	$[(0.15, 0.2, 0.25, 0.3); 0.7, 0.1]$
$\mathcal{A}_4$	$[(0.1, 0.2, 0.25, 0.35); 0.7, 0.2]$	$[(0.75, 0.8, 0.9, 0.95); 0.6, 0.2]$	$[(0.2, 0.25, 0.3, 0.35); 0.5, 0.4]$	$[(0.1, 0.25, 0.3, 0.35); 0.6, 0.3]$

TABLE 10. Normalized Decision Matrix Given by Expert D_3

	Attributes			
Alts.	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(0.15, 0.2, 0.3, 0.35);$ $0.5, 0.5]$	$[(0.25, 0.3, 0.35, 0.4);$ $0.4, 0.4]$	$[(0.75, 0.85, 0.9, 1);$ $0.5, 0.4]$	$[(0.2, 0.35, 0.4, 0.45);$ $0.7, 0.2]$
$\mathcal{A}_2$	$[(0.85, 0.9, 0.95, 1);$ $0.8, 0.1]$	$[(0.05, 0.1, 0.15, 0.2);$ $0.6, 0.3]$	$[(0.2, 0.25, 0.3, 0.4);$ $0.7, 0.2]$	$[(0.1, 0.15, 0.2, 0.25);$ $0.9, 0.1]$
$\mathcal{A}_3$	$[(0.2, 0.25, 0.3, 0.35);$ $0.5, 0.4]$	$[(0.8, 0.85, 0.9, 0.95);$ $0.8, 0.1]$	$[(0.05, 0.1, 0.15, 0.2);$ $0.7, 0.2]$	$[(0.25, 0.3, 0.35, 0.4);$ $0.5, 0.4]$
$\mathcal{A}_4$	$[(0.1, 0.2, 0.3, 0.4);$ $0.7, 0.2]$	$[(0.15, 0.25, 0.35, 0.45);$ $0.5, 0.3]$	$[(0.25, 0.3, 0.35, 0.4);$ $0.6, 0.3]$	$[(0.8, 0.9, 0.95, 1);$ $0.6, 0.2]$

Table 11. Normalized Decision Matrix Given by Expert D_4

	Attributes			
Alts.	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	$[(0.093, 0.1865, 0.2795,$ $0.3444);0.5,0.5]$	$[(0.1471, 0.2033, 0.257,$ $0.3229);0.4,0.4]$	$[(0.1351, 0.2389, 0.300,$ $0.3614);0.5,0.4]$	$[(0.2576, 0.3631, 0.4677,$ $0.530);0.6,0.3]$
$\mathcal{A}_2$	$[(0.2149, 0.3191, 0.3775,$ $0.4348);0.6,0.3]$	$[(0.107, 0.1865, 0.2603,$ $0.3505);0.6,0.3]$	$[(0.1622, 0.2128, 0.2632,$ $0.3243);0.4,0.4]$	$[(0.1835, 0.2494, 0.313,$ $0.3731);0.6,0.4]$
$\mathcal{A}_3$	$[(0.1863, 0.2725, 0.3446,$ $0.4334);0.5,0.4]$	$[(.0983, 0.1904, 0.2679,$ $0.343);0.5,0.3]$	$[(0.1164, .2296, 0.3132,$ $0.40);0.6,0.3]$	$[(0.1705, 0.2326, 0.2961,$ $0.3468);0.5,0.4]$
$\mathcal{A}_4$	$[(0.1241, 0.2624, 0.3507,$ $0.4516);0.5,0.5]$	$[(0.2235, 0.3243, 0.4059,$ $0.4801);0.5,0.3]$	$[(0.2317, 0.3238, 0.392,$ $0.4467);0.5,0.5]$	$[(0.1356, 0.319, 0.4083,$ $0.4651);0.5,0.4]$

Table 12. Individual Overall Attribute Values

Alts.	Final aggregation	Centroid point	Ranking order
$\mathcal{A}_1$	$S_1 = [(0.1487, 0.2364, .3082, 0.3717); 0.4, 0.5]$	$X_{S_1} = 0.2570, Y_{S_1} = 0.313$	
$\mathcal{A}_2$	$S_2 = [(0.1594, 0.2317, .2992, 0.3616); 0.4, 0.4]$	$X_{S_2} = 0.2601, Y_{S_2} = 0.3375$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_3$
$\mathcal{A}_3$	$S_3 = [(0.1331, 0.2269, .3033, 0.3816); 0.5, 0.4]$	$X_{S_3} = 0.2552, Y_{S_1} = 0.3427$	
$\mathcal{A}_4$	$S_4 = [(0.1933, 0.3111, .3904, 0.4605); 0.5, 0.5]$	$X_{S_4} = 0.3222, Y_{S_4} = 0.3583$	

Table 13. Final Aggregated Values and Ranking Values

Step 5: Finally, the decision results are obtained by the ranking method described in Section 2.3 and presented in Table 13.

According to the above result, we have, $\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_3$, *i.e.,* supplier $\mathcal{A}_4$ will be the first choice, $\mathcal{A}_2$ second, $\mathcal{A}_1$ third and $\mathcal{A}_3$ fourth. Hence, the most perfect supplier is $\mathcal{A}_4$.

There are some important observations in the results of the above problem depending on the values of the parameter 'r'', which we would like to present below. In the above computation, in equation (20), we have used TrIFWHM ($r = 1$) operator because some extreme outliers exist in the decision matrices provided in Table 8 to Table 11. On the other hand, in equation (21), to compute the group overall opinions of the alternatives, we have used TrIFWAM ($r = -1$) operator as there do not exit any outlier data in the data set (provided in Table 12) being fused. Thus, we have taken the values of the parameter r as 1 and -1 in equations (20) and (21), respectively. But if we take $r = -1$ (i.e., TrIFWAM operator), in both of the steps, then experts' overall opinions of the alternatives with respect to all attributes are obtained as follows:

$S_1 = [(0.2981, 0.3759, 0.4420, 0.5164); 0.4, 0.5]$
$S_2 = [(0.2964, 0.3621, 0.4281, 0.4915); 0.4, 0.4]$
$S_3 = [(0.3246, 0.3949, 0.4512, 0.5103); 0.5, 0.4]$
$S_4 = [(0.3071, 0.3920, 0.4597, 0.5121); 0.5, 0.5]$

Subsequently, the ranking order of alternatives is $\mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_4 > \mathcal{A}_2$. This is happening as TrIFWAM operator cannot relieve the influence of unfair data present in Table 8 to Table 11.

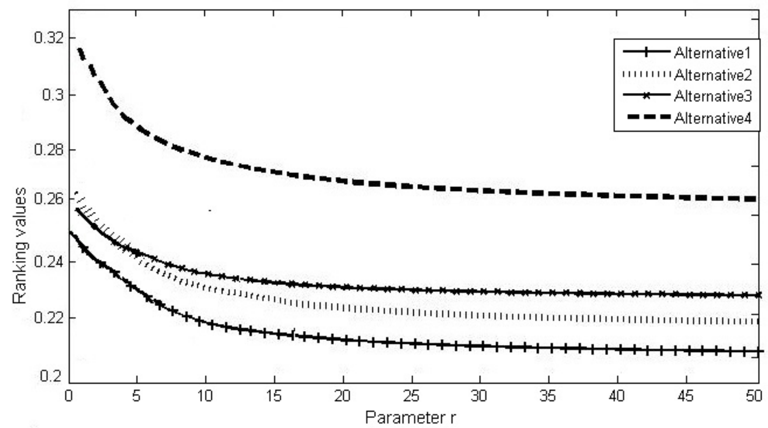

FIGURE 3. Variation of Ranking Values of Alternatives w.r.t. Parameter r

The proposed operator TrIFWPHM is a generalized average operator and the generalization property depends on the parameter 'r'. In general, r can take any real value. In equation (20), if we take different values of r in between 0 to 50 and in equation (21), if we fix r as -1, then the ranking values of all the alternatives are varied which is depicted in details in Fig.3. Subsequently, for any particular value of r, we may get the ranking result of all the alternatives from Fig.3. However, if $r \neq 1$, then the influence of outlier data cannot be computed in the aggregation process. Thus, from application point of view, we suggest the expert to take the values of r as 1 which allows us to handle outlier data suitably in ultimate aggregation results.

6.1. **Sensitivity Analysis.** Sensitivity analysis is the systematic investigation of some potential changes and errors of rating values and their effect on the variations of final ranking order [4]. We conduct a sensitivity analysis to explore the sensitivity of the ranking of alternatives by varying the membership and non-membership degrees in the rating values (TrIFNs). A slight variation of membership and non-membership degrees in the original rating values provided by experts is computed as follows:

$$A_{ij} = [(a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} + q.h, \nu_{ij} - q.h]$$

where $q = \frac{-\Delta v_j}{h}, ..., -1, 0, 1, ..., \frac{\Delta v_j}{h}$, h is the step length and the interval $[-\Delta v_j, \Delta v_j]$ $(j = 1, 2, 3, 4)$ represents the variation interval of membership and non-membership degrees with respect to four attributes.

The variation results are plotted in Fig.4 to Fig.7. From Fig.4 to Fig.7, it is observed that, when the variation values of membership and non-membership degrees

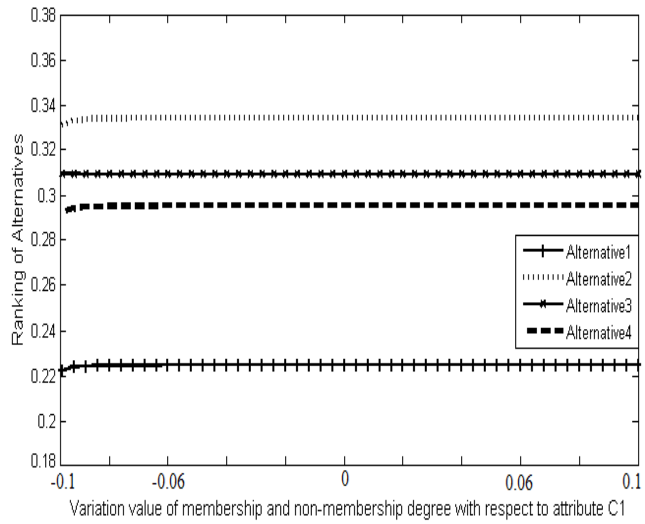

Figure 4. Ranking Order Sensitivity to the Membership and Non-membership Degrees with Respect to Attribute C1

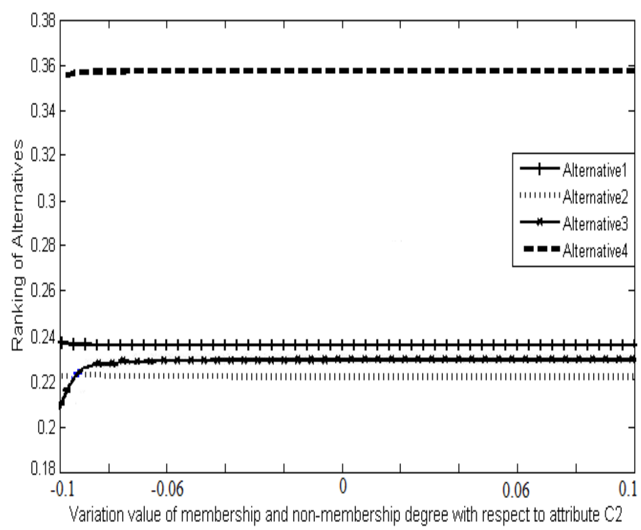

Figure 5. Ranking Order Sensitivity to the Membership and Non-membership Degrees with Respect to Attribute C2

with respect to three attributes C_1, C_3, C_4 changes from -0.1 to 0.1, the ranking order of four alternatives is remained constant. But with respect to attribute C_2, ranking order of alternatives $\mathcal{A}_2$ and $\mathcal{A}_3$ is changed at -0.09. It implicates that the

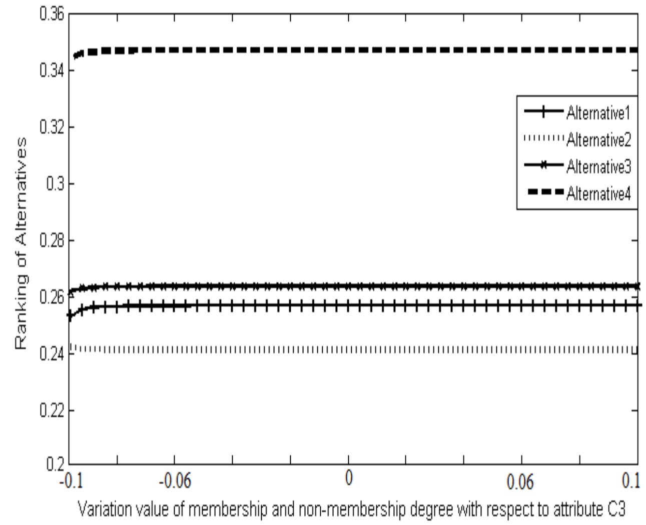

FIGURE 6. Ranking Order Sensitivity to the Membership and Non-membership Degrees with Respect to Attribute C3

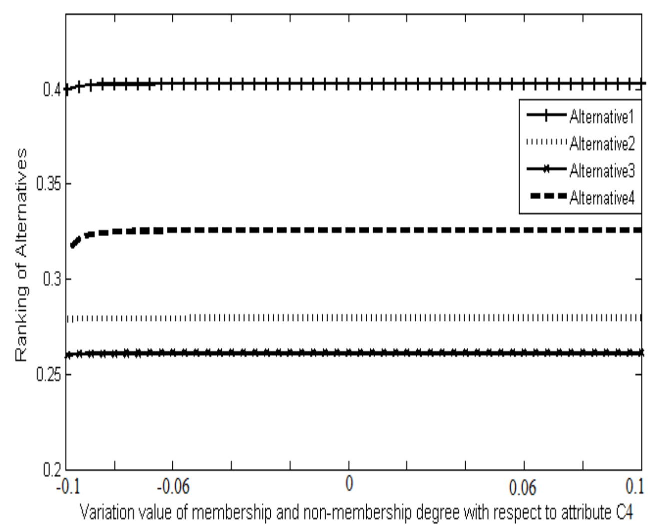

FIGURE 7. Ranking Order Sensitivity to the Membership and Non-membership Degrees with Respect to Attribute C4

alternatives $\mathcal{A}_1$ and $\mathcal{A}_4$ are not so sensitive to membership and non-membership degrees and alternatives $\mathcal{A}_2$ and $\mathcal{A}_3$ are more sensitive to membership and non-membership degrees than other two alternatives.

6.2. **Comparative Analysis.**

6.2.1. *Comparison of Performances with the Existing Aggregation Operators.* To further illustrate the effectiveness of the proposed operator, we solve the above supplier selection problem (described in Section 6) by using different operators, such as, weighted power average operator [42] of TrIFNs, weighted power geometric operator [46] of TrIFNs, trapezoidal intuitionistic fuzzy weighted geometric operator [50] and trapezoidal intuitionistic fuzzy weighted arithmetic operator [49]. In order to compare performance of TrIFWPHM operator with the aforementioned aggregation operators, we compute experts' over all ratings for alternatives, by using these aggregation operators and following the same steps as in the proposed decision making process, alternatives' final performances are calculated. Based on the alternatives' final performances, ranking order of the alternative in each cases are presented in Table 14.

Approach	Operator	Ranking results
Wan [42]	Weighted power average	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_3$
Wan and Dong [46]	Weighted power geometric	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_3$
Wu and Cao [50]	TrIFWGM	$\mathcal{A}_1 > \mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3$
Wei [49]	TrIFWAM	$\mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_2$
Proposed	TrIFWHM	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_1 > \mathcal{A}_3$

TABLE 14. The Ranking Order of Different Alternatives

It is clear from Table 14 that, the ranking order of the alternatives, by utilizing TrIFWGM [50] and TrIFWAM [49] operators, is quite different as these operators cannot relieve the influence of unfair data. On the other hand, the ranking results by utilizing the proposed TrIFWPHM operator, weighted power average operator [42] and weighted power geometric operator [46] of TrIFNs is the same. Both weighted power average operator [42] and weighted power geometric operator [46] of TrIFNs focus on assigning low weight to the unfair data by considering relationship among the input TrIFN arguments. They directly do not address outlier information among the TrIFN data sets. In this respect, our proposed TrIFWHM operator focus on directly the outlier data to relieve the influence of it to the ultimate aggregation results.

In the following section, we compare proposed model with the corresponding fuzzy number model and intuitionistic fuzzy value model.

6.2.2. *Comparison Analysis with IFS Based MCDM Model.* As mentioned earlier, IFN is a further generalization of IFS and its universe is continuous rather than discrete, so its use is more convenient. In order to make the comparison, we use IFSs to express expert's opinion by assuming expert's satisfaction and dissatisfaction degrees (i.e., (μ_{ij}, ν_{ij})) of his(er) original opinion, which was modeled by TrIFN $\mathcal{A}_i (i = 1, 2, 3, 4)$, as an intuitionistic fuzzy value ratings. Afterwards, we employ the weighted intuitionistic fuzzy arithmetic mean operator [53] to compute the alternatives overall ratings. After all computation, we obtain the final aggregated values of the alternatives $\mathcal{A}_i (i = 1, 2, 3, 4)$ as shown in Table 15.

Alternatives	Final aggregation	Score Value	Ranking order
$\mathcal{A}_1$	$S_1 = (0.6189, 0.3097)$	$Score(S_1) = 0.3092$	
$\mathcal{A}_2$	$S_2 = (0.7027, 0.2065)$	$Score(S_1) = 0.4962$	$\mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_1$
$\mathcal{A}_3$	$S_3 = (0.6707, 0.2117)$	$Score(S_1) = 0.4590$	
$\mathcal{A}_4$	$S_4 = (0.6166, 0.2663)$	$Score(S_1) = 0.3503$	

TABLE 15. IFS Final Aggregated Values and Ranking Values

Finally, the decision results are obtained by using score function [9]. The ranking order of the alternatives is obtained as follows: $\mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_4 > \mathcal{A}_1$, i.e., supplier $\mathcal{A}_2$ will be the best choice.

It is to be noted that the resultant ranking result is quite different from the ranking order obtained by our proposed method with TrIFNs information. This deviation mainly occurs due to the conversion of original TrIFNs information into intuitionistic fuzzy values. In other words, due to this transformation, TrIFN losses its inherent structure as all TrFNs (trapezoidal fuzzy numbers) are discarded from the corresponding TrIFNs. Such a conversion distorts experts' original assessments and also weakens the ability of information representation of TrIFNs. As TrIFNs provide a framework to maintain the integrity in information processing, therefore, TrIFNs may better capture expert's subjective estimation in a decision making problem than IFSs. Therefore, we urge that MAGDM problems with TrIFNs data provides more appropriate results.

6.2.3. *Comparison Analysis with TrFN Based MCDM Problem.* In this section, the proposed method is compared with the corresponding TrFN based method. By adapting the membership function with full satisfaction in original ratings of the alternatives, *i.e.,* by putting $\mu_{ij}^t = 1$ and $\nu_{ij}^t = 0$ ($\forall\ i, j, t = 1, 2, 3, 4$), TrIFN decision matrix is transformed to TrFN decision matrix. Afterwards, by utilizing the weighted trapezoidal fuzzy harmonic mean operator [54] (for calculating individual overall attribute values) and weighted trapezoidal fuzzy arithmetic mean operator [8] (for calculating collective overall attribute values), we aggregate the TrFN ratings of the alternatives. The final aggregated values of the alternatives $\mathcal{A}_i (i = 1, 2, 3, 4)$ are shown in Table 16.

Alternatives	Final aggregation	Centroid point	Ranking order
$\mathcal{A}_1$	$S_1 = [0.1487, 0.2364, 0.3082, 0.3717]$	$X_{S_1} = 0.2614, Y_{S_1} = 0.4320$	
$\mathcal{A}_2$	$S_2 = [0.1594, 0.2317, 0.2992, 0.3616]$	$X_{S_2} = 0.2610, Y_{S_2} = 0.4334$	$\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_3$
$\mathcal{A}_3$	$S_3 = [0.1331, 0.2269, 0.3033, 0.3816]$	$X_{S_3} = 0.2581, Y_{S_1} = 0.4302$	
$\mathcal{A}_4$	$S_4 = [0.1933, 0.3111, 0.3904, 0.4605]$	$X_{S_4} = 0.3290, Y_{S_4} = 0.4289$	

TABLE 16. TrFN Final Aggregated Values and Ranking Values

Finally, the decision results are obtained by using the ranking method defined in Section 2.3 by putting $\mu = 1$ and $\nu = 0$ in the equations (3) and (4). The ranking results of the alternatives are obtained as follows: $\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_3$.

The desirable alternative is $\mathcal{A}_4$ and it is consistent with the ranking result of our proposed TrIFN based approach. However, the ranking order for other three alternatives, obtained by using the TrFNs based method is somewhat different from our proposed TrIFNs based method. This result shows that experts' satisfaction, dissatisfaction, hesitation related to human thinking, reasoning, etc. play an important role in decision results of MAGDM problem. This uncertainty related to experts' subjective assessments are captured by TrIFNs in a logical and meaningful way and, therefore, MAGDM with TrIFNs data provides more appropriate ranking result.

7. Conclusions

In this paper, we have proposed an aggregation operator which we refer to as TrIFWPHM operator to aggregate TrIFNs. We have discussed variety of special cases of TrIFWPHM operator. Moreover, TrIFWPHM operator satisfies the properties of mean-type aggregation operator, such as, monotonicity, idempotency, boundedness. Applying TrIFWPHM operator, a MAGDM approach has been presented, where experts' opinions on alternatives are modeled by using TrIFNs. Finally, by the help of supplier selection problem, the performance of TrIFWPHM operator has been compared with other existing aggregation operators and applicability of modeling expert's opinion by TrIFNs has also been demonstrated.

The main contribution of this work can be pointed out as follows: (i) the proposed operator is based on TrIFNs as TrIFNs suitably reflect the uncertainty and hesitation of human thinking and, thus, give more flexibility to the expert while expressing their opinion regarding each alternative over the attribute; (ii) the TrIFN representation of linguistic variables enable the expert to express their judgements with the corresponding degrees of satisfaction and dissatisfaction; (iii) the operator TrIFWPHM reduces to TrIFWHM, TrIFWGM, TrIFWAM and TrIFWQM operators depending on the values of the parameter 'r'; (iv) when some extreme outliers exist in the decision matrix, the operator TrIFWHM (r=1) operator is used as it reduces the influence of unfair data on the decision result.

Although, the proposed operators are used in a numerical supplier selection problem, it can also be applied to any other areas of decision problems where uncertainty and hesitation are involved in the evaluation process. This will be topic of our future research work.

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