THE PERCENTILES OF FUZZY NUMBERS AND THEIR APPLICATIONS

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Abstract. The purpose of this study is to find the percentiles of fuzzy numbers and to demonstrate their applications, which include finding weighted means, dispersion indices, and the percentile intervals of fuzzy numbers. The crisp approximations of fuzzy numbers introduced in this paper are new and interesting for the comparison of fuzzy environments, such as a variety of economic, financial, and engineering systems control problems.

1. Introduction

The approximation of fuzzy quantities by scalar or crisp intervals has been studied by many researchers [1, 3, 4, 5, 6]. Finding a crisp approximation of a fuzzy set is often called the process of defuzzification. Various defuzzification methods have been proposed and successfully applied in the area of fuzzy control and modeling [8, 11]. Approximations of fuzzy numbers by scalars derived from descriptive statistics are used in the statistical analysis of crisp numerical data. In statistics, the center of the distribution of a quantitative variable is determined by the mean value, mode, and median. Furthermore, many other types of information are also expressed via percentiles. Concentration of a fuzzy number on the percentile value and interquartile range are useful factors for making decisions in many economic problems. The weighted means and some of the spread or dispersion indices such as the skewness and kurtosis coefficients of fuzzy numbers based on percentiles can be applied to fuzzy data analysis. For both academic and financial communities these are familiar issues; for instance, stock market returns have negative skewness and excess kurtosis, so percentiles play an important role in economic analysis in a fuzzy environment, and they are useful for the comparison of fuzzy numbers. Bodjanova et al. [1, 5] have introduced the median value and median interval of a fuzzy number. The percentiles of a trapezoidal fuzzy number were introduced in [13]. In this paper, percentiles are formalized and extended for a large family of fuzzy numbers. Three types of weighted means for fuzzy numbers are also introduced. These means can be considered as a combination of the mean and median. For symmetric fuzzy numbers, these approximations can be applied as well as the mean, but when a fuzzy number membership function has one high tail on the left

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or right side, these criteria may be better than the mean and median as a measure of central tendency. This paper is organized as follows:

In section 2, we give basic definitions and notations. In section 3, percentiles of fuzzy numbers are introduced and measures of central tendency of fuzzy numbers based on these percentiles are defined in section 4. In section 5, the interquartile range of fuzzy numbers is presented and the coefficient of quartile variation is studied in section 6. The coefficients of skewness and kurtosis of a fuzzy number based on percentiles are introduced in section 7. Finally, in section 8, we obtain the $\alpha - cut$ percentiles from the tails of a fuzzy number, and suggest an interesting interval approximation for fuzzy numbers.

2. **Basic Definition and Notations**

Let $\mathbb{R}$ be the set of all real numbers. We assume that for all $x \in \mathbb{R}$, the fuzzy number A can be expressed in the following form:

$$A(x) = \left\{ \begin{array}{ll} g(x) & x \in [a, b), \\ 1 & x \in [b, c), \\ h(x) & x \in [c, d), \\ 0 & otherwise, \end{array} \right\} \tag{1}$$

where a, b, c, d are real numbers such that $a < b \leq c < d$,g is a real valued increasing and right continuous function , and h is a real valued function that is decreasing and left continuous. Notice that (1) is in fact an **L-R** fuzzy number with a strictly monotonic shape function, as proposed by Dubois and Prade in 1981 and also described in [5]. A fuzzy number A with shape functions g and h given by

$$g(x) = (\frac{x-a}{b-a})^n, \qquad h(x) = (\frac{d-x}{d-c})^n, \tag{2}$$

will be denoted by $A = (a, b, c, d)_n$, $n > 0$, (see Figure 1).

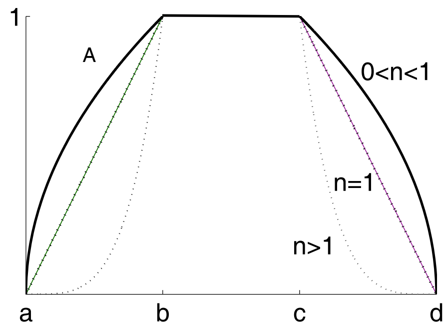

FIGURE 1. Fuzzy number $A = (a, b, c, d)_n$

If $n = 1$, then $A = (a, b, c, d)$, is called a trapezoidal fuzzy number. If $n \neq 1$, the fuzzy number $A^* = (a, b, c, d)_n$ is a modification of a trapzoidal fuzzy number

$A = (a, b, c, d)$. If $n > 1$, then A^* is a concentration of A. If $0 < n < 1$, then A^* is a dilatation of A. The concentration of A when $n = 2$ is often interpreted as the linguistic hedge "*very*"; the dilatation of A, when $n = 0.5$ is often interpreted as the linguistic hedge "*more or less*". More information on linguistic hedges can be found in [15].

A fuzzy number A described by (1) has the following α-level sets($\alpha - cuts$):

(i) $A_\alpha = [g^{-1}(\alpha), h^{-1}(\alpha)]$ for all $\alpha \in (0, 1)$,
(ii) $A_0 = [a, d]$,
(iii) $A_1 = [b, c]$.

If $A = (a, b, c, d)_n$, then for all $\alpha \in (0, 1)$

$$A_\alpha = [a + \alpha^{1/n}(b - a), d - \alpha^{1/n}(d - c)] = [a_\alpha, b_\alpha]. \tag{3}$$

The cardinality of the fuzzy number A described by (1) is defined by

$$card(A) = \int_a^b A(x)dx = \int_0^1 (b_\alpha - a_\alpha)d\alpha. \tag{4}$$

If $A = (a, b, c, d)_n$, then

$$card(A) = \frac{b - a}{n + 1} + (c - b) + \frac{d - c}{n + 1}. \tag{5}$$

In this paper, we shall only consider **L-R** fuzzy numbers.

3. Percentiles of Fuzzy Numbers

In this section, we introduce the percentiles of fuzzy numbers. A definition for the median value of a fuzzy number was introduced by Bodjanova [1] as:

Definition 3.1. The median value of a fuzzy number A characterized by (1) is the real number m_A from the support of A such that:

$$\int_a^{m_A} A(x)dx = \int_{m_A}^d A(x)dx. \tag{6}$$

For practical purposes (6) can be rewritten as:

$$\int_a^{m_A} A(x)dx = \frac{card(A)}{2}. \tag{7}$$

Bodjanova classified fuzzy numbers with respect to the "distribution" of their cardinality into four cases, and studied the location of the median value m_A in the support of A [1].

We define a percentile of a fuzzy number as the numerical value of the support function that divides the area under the membership function into 100 equal parts; e.g. quartiles divide the area under the membership function into four equal parts. The first quartile is the 25^{th} percentile, the second is the 50^{th} percentile (also known as the median), and the third is the 75^{th} percentile.

Definition 3.2. [13] The i^{th} percentile of fuzzy number A characterized by (1) is the real number $P_i, (i = 1, 2, ..., 99)$ from the support of A such that:

$$\int_a^{P_i} A(x)dx = \frac{i}{100}\int_a^d A(x)dx = \frac{i}{100}card(A). \tag{8}$$

Let $S_1 = \int_a^b A(x)dx, S_2 = \int_b^c A(x)dx, S_3 = \int_c^d A(x)dx$, then

$$S = S_1 + S_2 + S_3 = card(A). \tag{9}$$

We write:

$$\int_a^y g(x) = G(y),\ y \in [a, b],\ \int_c^y h(x) = H(y), y \in [c, d], \tag{10}$$

so

$$\int_y^d h(x) = S_3 - \int_c^y h(x)dx = S_3 - H(y), y \in [c, d]. \tag{11}$$

We shall assume that the functions $G^{-1}(.)$ and $H^{-1}(.)$ exist. Now, in order to find the i^{th} percentile $(P_i, i = 1, 2, ..., 99)$ of a fuzzy number A, we consider the following three cases:

Case 1. If $S_1 \geq \frac{i}{100}S$, then $G(P_i) = \frac{i}{100}S$, so

$$P_i = G^{-1}(\frac{i}{100}S)\ ,\ for\ every\ i \in \{1, 2, 3..., 99\}. \tag{12}$$

Case 2. If $S_1 + S_2 \geq \frac{i}{100}S > S_1$, then $G(b) + (P_i - b) = \frac{i}{100}S$, so

$P_i = b + \frac{i}{100}S - G(b)$, or

$$P_i = b + \frac{i}{100}S - S_1\ ,\ for\ every\ i \in \{1, 2, 3..., 99\}. \tag{13}$$

Case 3. If $S_3 > (1 - \frac{i}{100})S$, then $S_3 - H(P_i) = (1 - \frac{i}{100})S$, so

$$P_i = H^{-1}(S_3 - (1 - \frac{i}{100})S)\ ,\ for\ every\ i \in \{1, 2, 3..., 99\}. \tag{14}$$

For the fuzzy number $A = (a, b, c, d)_n$ with $g(x) = (\frac{x-a}{b-a})^n$, $h(x) = (\frac{d-x}{d-c})^n$ we get:

$$\int_a^y g(x) = \int_a^y (\frac{x-a}{b-a})^n dx = \frac{b-a}{n+1}(\frac{y-a}{b-a})^{n+1} = G(y),\ y \in [a, b]. \tag{15}$$

$$\int_c^y h(x) = \int_c^y (\frac{d-x}{d-c})^n dx = \frac{d-c}{n+1} - \frac{d-c}{n+1}(\frac{d-y}{d-c})^{n+1} = H(y),\ y \in [c, d]. \tag{16}$$

So

$S_1 = G(b) = \frac{b-a}{n+1}\ ,\ S_2 = (c - b)\ ,\ S_3 = H(d) = \frac{d-c}{n+1}$, then

$$S = card(A) = \frac{b-a}{n+1} + (c-b) + \frac{d-c}{n+1}. \quad (17)$$

$$G^{-1}(y) = a + (b-a)[\frac{(n+1)y}{b-a}]^{\frac{1}{n+1}} \ , \ y \in [a,b], \quad (18)$$

$$H^{-1}(y) = d - (d-c)[1 - \frac{(n+1)y}{d-c}]^{\frac{1}{n+1}} \ , \ y \in [c,d]. \quad (19)$$

Therefore, for the fuzzy number $A = (a,b,c,d)_n$, the percentiles are obtained as follows:

Case 1. If: $S_1 = \frac{b-a}{n+1} \geq \frac{i}{100}(\frac{b-a}{n+1} + c - b + \frac{d-c}{n+1}) = \frac{i}{100}S$, then

$$P_i = G^{-1}(\tfrac{i}{100}S), \quad \text{so}$$

$$P_i = a + (b-a)[\frac{i(n+1)}{100(b-a)}(\frac{b-a}{n+1} + c - b + \frac{d-c}{n+1})]^{\frac{1}{n+1}}. \quad (20)$$

Case 2. If: $S_1 + S_2 \geq \frac{i}{100}S > S_1$, then

$$P_i = b + \frac{i}{100}(\frac{b-a}{n+1} + c - b + \frac{d-c}{n+1}) - \frac{b-a}{n+1}. \quad (21)$$

Case 3. If: $S_3 = \frac{d-c}{n+1} > (1 - \frac{i}{100})(\frac{b-a}{n+1} + c - b + \frac{d-c}{n+1}) = (1 - \frac{i}{100})S$, then

$$P_i = H^{-1}(S_3 - (1 - \frac{i}{100})S),$$

so

$$P_i = d - (d-c)[\frac{(100-i)(n+1)}{100(d-c)}(\frac{b-a}{n+1} + c - b + \frac{d-c}{n+1})]^{\frac{1}{n+1}}. \quad (22)$$

To simplify formulas (20),(21) and (22), let $\theta = b - a$ and, $\beta = d - c$,

$$k_1 = \frac{100\theta}{\theta + \beta + (n+1)(c-b)} \quad , \ k_2 = \frac{100(\theta + (n+1)(c-b))}{\theta + \beta + (n+1)(c-b)}.$$

Hence the i^{th} percentile ($P_i, i = 1, 2, ..., 99$) of the fuzzy number $A = (a,b,c,d)_n$ is obtained as follows:

$$if\ k_1 \geq i\ \ then\ \ P_i = a + \theta[\frac{i}{100\theta}(\theta + \beta + (n+1)(c-b))]^{\frac{1}{n+1}}, \quad (23)$$

$$if\ k_1 < i \leq k_2\ \ then\ \ P_i = b + \frac{i}{100}(\frac{\theta + \beta}{n+1} + c - b) - \frac{\theta}{n+1}, \quad (24)$$

$$if\ i > k_2\ then\ P_i = d - \beta[\frac{(100-i)}{100\beta}(\theta + \beta + (n+1)(c-b))]^{\frac{1}{n+1}}. \quad (25)$$

Remark 3.3. Without loss of generality, we can suppose $i \in (0, 100)$.

Corollary 3.4. *Let $A = (a, b, d)$ be a triangular fuzzy number, $\theta = b - a$, $\beta = d - b$ and $k = \frac{100\theta}{\theta+\beta}$. Then,*

$$if: \; k \geq i \; then \; P_i = a + \theta\sqrt{\frac{i(\theta+\beta)}{100\theta}}, \tag{26}$$

$$else: \; P_i = d - \beta\sqrt{\frac{(100-i)(\theta+\beta)}{100\beta}}. \tag{27}$$

Theorem 3.5. *Let $A = (a, b, c, d)_n$. Then,*

$$A(P_i) \in [min((\frac{i}{100})^{\frac{n}{n+1}}, (\frac{100-i}{100})^{\frac{n}{n+1}}), 1].$$

Proof. For every $i \in \{1, 2, 3, ..., 99\}$ if $P_i \in (a, b]$ then:

$$P_i = a + (b-a)[\frac{i}{100(b-a)}(b - a + d - c + (n+1)(c-b)]^{\frac{1}{n+1}}, \tag{28}$$

$$A(P_i) = (\frac{P_i - a}{b-a})^n = [\frac{i}{100(b-a)}(b - a + d - c + (n+1)(c-b)]^{\frac{n}{n+1}}. \tag{29}$$

since $\frac{b-a+d-c+(n+1)(c-b)}{b-a} \geq 1$, hence:

$$[\frac{i}{100}(\frac{b-a+d-c+(n+1)(c-b}{b-a})]^{\frac{n}{n+1}} \geq (\frac{i}{100})^{\frac{n}{n+1}},$$

Or, equivalently, $A(P_i) \geq (\frac{i}{100})^{\frac{n}{n+1}}$, if $P_i \in (b, c]$ obviously $A(P_i) = 1$.

Similarly, for $P_i \in (c, d)$, since $\frac{b-a+d-c+(n+1)(c-b)}{d-c} \geq 1$, hence:

$$[\frac{100-i}{100(d-c)}(b - a + d - c + (n+1)(c-b))]^{\frac{n}{n+1}} \geq (\frac{100-i}{100})^{\frac{n}{n+1}},$$

or equivalently $A(P_i) \geq (\frac{100-i}{100})^{\frac{n}{n+1}}$,
Consequently, $A(P_i) \in [min((\frac{i}{100})^{\frac{n}{n+1}}, (\frac{100-i}{100})^{\frac{n}{n+1}}), 1]$. □

Corollary 3.6. *Let $A = (a, b, c, d)$ be a trapezoidal fuzzy number. Then the membership grade of the 25^{th} and 75^{th} percentiles are always at least $(\frac{25}{100})^{\frac{1}{2}} = 0.5$ and the membership grade of the 50^{th} percentile (median) is always at least $(\frac{50}{100})^{\frac{1}{2}} =$ 0.707.*

Example 3.7. Consider the fuzzy number A with the following membership function:

$$A(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \; x, \mu \in \mathbb{R}, \; \sigma > 0. \tag{30}$$

We obtain the percentiles of the fuzzy number A as:

$\int_{-\infty}^{P_i} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{i}{100} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{i}{100} \sqrt{2 positive implicative}\sigma$, so

$$\int_{-\infty}^{P_i} \frac{1}{\sqrt{2 positive implicative}\sigma} e^{(-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{i}{100}.$$

Consequently

$$\phi(\frac{P_i - \mu}{\sigma}) = \frac{i}{100},$$

therefore

$$P_i = \mu + \sigma\phi^{-1}(\frac{i}{100}), \quad i = 1, 2, ...99,$$

where $\phi(.)$ is the standard normal distribution function. For example:

$$P_{25} = \mu + \sigma\phi^{-1}(\frac{25}{100}) = \mu - 0.68\sigma.$$

$$P_{50} = \mu + \sigma\phi^{-1}(\frac{50}{100}) = \mu.$$

$$P_{75} = \mu + \sigma\phi^{-1}(\frac{75}{100}) = \mu + 0.68\sigma.$$

These percentiles are given in Figure 2.

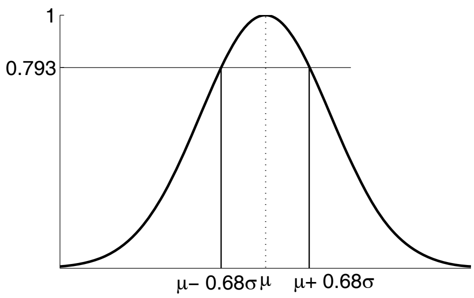

FIGURE 2. Fuzzy number A from Example 1

Example 3.8. Let $A = (0, 10, 11, 12)_1$, $B = (0, 10, 11, 12)_2$ and $C = (0, 10, 11, 12)_{1/2}$ be three fuzzy numbers, then

$$\theta = 10, \beta = 1, b = 10, c = 11,$$

we obtain the percentiles $P_{25}^A, P_{50}^A, P_{75}^A$ as follows:

$$k_1 = \frac{100\theta}{\theta+\beta+(n+1)(c-b)} = \frac{100\times 10}{10+1+(1+1)(11-10)} = 76.9$$

$$k_2 = \frac{100(\theta+(n+1)(c-b))}{\theta+\beta+(n+1)(c-b)} = \frac{100(10+(1+1)(11-10))}{10+1+(1+1)(11-10)} = 92.31$$

1. Since $k_1 = 76.9 \geq 25 = i$, thus:

$$P_{25}^A = 0 + 10[\frac{25}{100\times 10}(10+1+(1+1)(11-10)]^{\frac{1}{2}} = 5.701\,, A(P_{25}^A) = 0.5701,$$

2. Since $k_1 = 76.9 \geq 50 = i$ thus:

$$P_{50}^A = 0 + 10[\frac{50}{100\times 10}(10+1+(1+1)(11-10)]^{\frac{1}{2}} = 8.062\,, A(P_{50}^A) = 0.8062,$$

3. Since $k_1 = 76.9 \geq 75 = i$ thus:

$$P_{75}^A = 0 + 10[\frac{75}{100\times 10}(10+1+(1+1)(11-10)]^{\frac{1}{2}} = 9.874\,, A(P_{75}^A) = 0.9874.$$

Similarly, for fuzzy numbers B and C we have
$P_{25}^B = 7.0473$, $P_{50}^B = 8.879$, $P_{75}^B = 10.1667$
$B(P_{25}^B) = 0.4966$, $B(P_{50}^B) = 0.7884$, $B(P_{75}^B) = 1$.
$P_{25}^C = 4.6050$, $P_{50}^C = 7.3100$, $P_{75}^C = 9.5789$
$C(P_{25}^C) = 0.6786$, $C(P_{50}^C) = 0.8550$, $C(P_{75}^C) = 0.9787$.

4. Measures of Central Tendency Based on Percentiles

Sometimes the low grade membership functions of fuzzy numbers are not very important. In this case, we can use the weighted values obtained from percentiles (particularly quartiles) as measures of central tendency. Furthermore, we can use these quantities as the salient points that are composed of the percentiles [2] and these indices can be used for ranking fuzzy numbers and the parametric and non-parametric inferential statistics of fuzzy numbers.
For quartiles of a fuzzy number A we write:
$Q_1^A = P_{25}^A$, $Q_2^A = P_{50}^A$, $Q_3^A = P_{75}^A$.

Definition 4.1. Let Q_1^A, Q_2^A and Q_3^A be the first, second and third quartiles of a fuzzy number A, respectively; the trimean of A is defined as

$$\overline{x}_Q^A = \frac{Q_1^A A(Q_1^A) + 2Q_2^A A(Q_2^A) + Q_3^A A(Q_3^A)}{A(Q_1^A) + 2A(Q_2^A) + A(Q_3^A)}. \tag{31}$$

The trimean is a measure of central tendency , being a weighted mean of the quartiles. It may be considered a crisp point approximation of the corresponding fuzzy number.

Definition 4.2. [13] Let P_i be i^{th} percentile of a fuzzy number A, the trimmed mean of A is defined as

$$\overline{x}_{Tr}^{A} = \frac{\int_{P_{25}^{A}}^{P_{75}^{A}} xA(x)dx}{\int_{P_{25}^{A}}^{P_{75}^{A}} A(x)dx}. \tag{32}$$

The above relation is a 25% trimmed mean. Note that we can compute other trimmed means; for example, to find a 10% trimmed mean, use P_{10} and P_{90} in equation (32). To find a 20% trimmed mean, use P_{20} and P_{80} in equation (32).

The trimmed mean is another measure of the center of a fuzzy number and is more resistant than the mean, but it is still sensitive with respect to the upper and lower values of the support function of a fuzzy number. However, it has some advantages over both the mean and median. For instance, the trimmed mean inflation is one of the measures underlying (core) inflation. The underlying or core inflation measure is mostly used by central banks in formulating monetary policy, rather than headline inflation. This is because headline inflation is usually influenced by temporary price shocks. There is a general consensus amongst policymakers that monetary policy should not respond to these temporary price shocks, since attempting to offset the resulting short-term rise in inflation could lead to large fluctuations in real gross domestic product (GDP) [9, 10, 12].
Figure 3 shows that fuzzy number A is trimmed and transformed to the fuzzy number B.

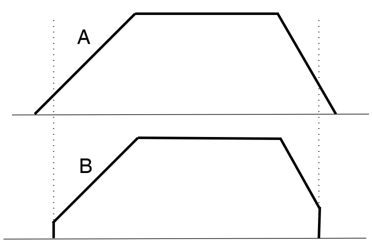

FIGURE 3. Fuzzy number A and its trimmed (B)

Definition 4.3. [13] The winsorized mean of a fuzzy number A is defined as:

$$\overline{x}_{W}^{A} = \frac{P_{25}^{A}A(P_{25}^{A}) + \int_{P_{25}^{A}}^{P_{75}^{A}} xA(x)dx + P_{75}^{A}A(P_{75}^{A})}{A(P_{25}^{A}) + \int_{P_{25}^{A}}^{P_{75}^{A}} A(x)dx + A(P_{75}^{A})}. \tag{33}$$

In other words, the fuzzy number A is winsorized in each of the two tails. We can also compute various winsorized means of a fuzzy number A.

5. **Interquartile Range(IQR) of Fuzzy Numbers**

Variability provides a second dimension for statistical evaluations where the central tendency alone cannot be used for decision making. A useful measure of variability, or dispersion, is the interqurtile range [7], and this quantity represents scatter in the middle 50% of a fuzzy number, so that its value for a fuzzy number A is the difference between the third (Q_3^A) and first (Q_1^A) quartiles.

$$IQR^A = Q_3^A - Q_1^A. \tag{34}$$

We can also obtain other percentile range of fuzzy numbers. For example, the 80% central range of a fuzzy number A is: $P_{90}^A - P_{10}^A$.

Definition 5.1. [13] Let Q_1^A, Q_3^A be the first and third quartiles of a fuzzy number A, respectively. Then the midquartile of A is defined as:

$$MQ^A = \frac{Q_1^A + Q_3^A}{2}. \tag{35}$$

The MQ^A is the midpoint between the first and third quartiles. We now state a new and interesting definition using percentiles as follows:

Definition 5.2. Let P_i^A be the i^{th} percentile of a fuzzy number A. Then, for any $i \in \{1, 2, ...50\}$, we define the interval $PI_{(100-2i)}^A = [P_i^A, P_{100-i}^A]$ as a (100-2i)% interval of A.

This interval approximation of fuzzy numbers is a new and useful tool for obtaining a measure of the variability or dispersion of a fuzzy number. For instance, the length of the interval $PI_{(50)}^A = [P_{25}^A, P_{75}^A]$ represents the scatter of a fuzzy number A in the middle 50% of the area under membership function of A.

Example 5.3. Consider the fuzzy numbers $A = (0, 10, 11, 12)_1$,
$B = (0, 10, 11, 12)_2$ and $C = (0, 10, 11, 12)_{1/2}$. Then the 25% trimmed, winsorized means, quartile range, and midquartile are computed as follows:

$$\overline{x}_{Tr}^A = \frac{\int_{5.7009}^{9.874} x(\frac{x}{10})dx}{\int_{5.7009}^{9.874} \frac{x}{10}dx} = 7.9738,$$

$$\overline{x}_W^A = \frac{5.7009 \times 0.5701 + \int_{5.7009}^{9.874} x(\frac{x}{10})dx + 9.874 \times 0.9874}{0.5701 + \int_{5.7009}^{9.874} \frac{x}{10}dx + 0.9874} = 8.0945,$$

$$\overline{x}_Q^A = \frac{5.7009 \times 0.5701 + 2 \times 8.0623 \times 0.8062 + 9.8742 \times 0.9874}{0.5701 + 2 \times 0.8062 + 0.9874} = 8.202,$$

$$MQ^A = \frac{5.7009 + 9.8742}{2} = 7.7875.$$

Similarly, for fuzzy numbers B and C we obtain:

$$\overline{x}_{Tr}^B = 8.7918 \ ,\overline{x}_W^B = 8.9245 \ \overline{x}_Q^B = 9.002.$$

$$\overline{x}^C_{Tr} = 7.2395 \ \ , , \overline{x}^C_W = 7.3256 \ \ , \overline{x}^C_Q = 7.424.$$

Also, $IQR^A = 9.874 - 5.7009 = 4.1731, IQR^B = 10.1667 - 7.0473 = 3.1194$, $IQR^C = 9.5789 - 4.605 = 4.9739$,

$$MQ^B = \frac{7.0473 + 10.1667}{2} = 8.6070, MQ^C = \frac{4.605 + 9.5789}{2} = 7.0919.$$

Clearly, $IQR^B \leq IQR^A \leq IQR^C$. Therefore, the dispersion of the fuzzy number B is less than that of fuzzy numbers A and C (Figure 4). The main result of this section is new and interesting because we can order fuzzy numbers with equal support by this method.

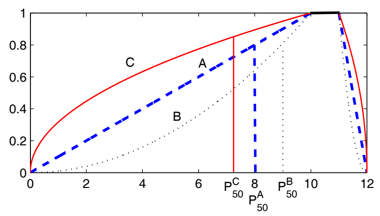

Figure 4. Fuzzy numbers A, B, C from Example 3

6. Coefficient of Quartile Variation

The coefficient of quartile is an alternative to the standard deviation as a measure of variation. This index, which is found from relative indices is used for measuring the dispersion in data.

Definition 6.1. Let Q^A_1, Q^A_3 be the first and third quartiles of a fuzzy number A, respectively. Then the coefficient of quartile variation of A is defined as:

$$V^A_Q = \frac{\frac{Q^A_3 - Q^A_1}{2}}{\frac{Q^A_1 + Q^A_3}{2}} \times 100\,\% = \frac{Q^A_3 - Q^A_1}{Q^A_1 + Q^A_3} \times 100\,\%. \tag{36}$$

Example 6.2. Consider the fuzzy number A of Example 1. We obtain:

$$V^A_Q = \frac{(\mu + 0.68\sigma) - (\mu - 0.68\sigma)}{(\mu - 0.68\sigma) + (\mu + 0.68\sigma)} \times 100\,\% = \frac{68\sigma}{\mu}\%.$$

One can see that when the value of σ is small, the coefficient of quartile variation of a fuzzy number A is also small. Therefore, the spread of A is only dependent on the value of σ, μ being a local parameter of A.

7. Coefficients of Skewness and Kurtosis

The skewness and kurtosis coefficients allow a more precise inference about the shape of the membership functions of fuzzy numbers. These coefficients play an important role for problems with a fuzzy environment such as in the academic and financial communities and times series models.

The coefficient of skewness is a measure of the degree of symmetry of the fuzzy number membership function and indicates whether fuzzy number has greater weight on towards the left or right. A fuzzy number is said to be symmetric if its membership function can be folded along an axis so that the two halves coincide with each other. A membership function that lacks symmetry with respect to the vertical axis is said to be skewed. The fuzzy number illustrated in Figure 5(a) is said to be skewed to the right; in this case the coefficient of skewness is positive, it has a long right tail and a very short left tail. In Figure 5(b) the fuzzy number is symmetric and the coefficient of skewness is zero, while in Figure 5(c) it is skewed to the left and the coefficient of skewness is negative.

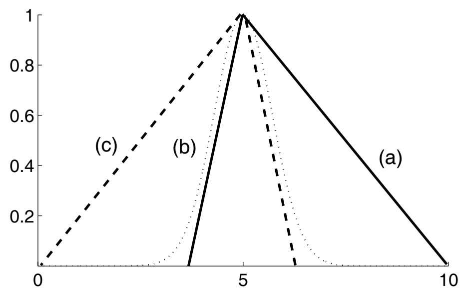

FIGURE 5. Skewness of fuzzy numbers

Definition 7.1. Let Q_1^A, Q_2^A and Q_3^A be the first, second and third quartiles of a fuzzy number A, respectively. Then the coefficient of skewness of A is defined as:

$$S_Q^A = \frac{(Q_3^A - Q_2^A) - (Q_2^A - Q_1^A)}{(Q_3^A - Q_2^A) + (Q_2^A - Q_1^A)} = \frac{Q_3^A - 2Q_2^A + Q_1^A}{Q_3^A - Q_1^A}. \tag{37}$$

Definition 7.2. Let P_{10}^A, P_{50}^A and P_{90}^A be the 10^{th}, 50^{th} and 90^{th} percentiles of a fuzzy number A, respectively. Then the coefficient of percentiles skewness of A is defined as:

$$S_P^A = \frac{(P_{90}^A - P_{50}^A) - (P_{50}^A - P_{10}^A)}{(P_{90}^A - P_{50}^A) + (P_{50}^A - P_{10}^A)} = \frac{P_{90}^A - 2P_{50}^A + P_{10}^A}{P_{90}^A - P_{10}^A}. \tag{38}$$

Example 7.3. Again consider the fuzzy number A of Example 1. We have

$$P_i = \mu + \sigma\phi^{-1}(\frac{i}{100}),\ i = 1, 2, ...99,$$

so

$$P_{10}^{A} = \mu - 1.28\sigma,\ P_{50}^{A} = \mu,\ P_{90}^{A} = \mu + 1.28\sigma.$$

Consequently,

$$S_Q^A = \frac{(\mu + 0.68\sigma) - 2\mu + (\mu - 0.68\sigma)}{(\mu + 0.68\sigma) - (\mu - 0.68\sigma)} = 0.$$

$$S_P^A = \frac{(\mu + 1.28\sigma) - 2\mu + (\mu - 1.28\sigma)}{(\mu + 1.28\sigma) - (\mu - 1.28\sigma)} = 0.$$

These coefficients show that the fuzzy number A is symmetric.

Example 7.4. Let $D = (0, 4, 5)$ be a triangular fuzzy number. Then

$$Q_1^D = P_{25}^D = 2.236,\ Q_2^D = P_{50}^D = 3.162,\ Q_3^D = P_{75}^D = 3.950,$$

$$P_{10}^D = 1.414,\ P_{90}^D = 4.293,$$

$$S_Q^D = \frac{3.950 - 2 \times 3.162 + 2.236}{3.950 - 2.236} = -0.08.$$

$$S_P^D = \frac{4.293 - 2 \times 3.162 + 1.414}{4.293 - 1.414} = -0.214.$$

Since the coefficients of skewness are negative, the fuzzy number D is skewed to the left (see Figure 6).

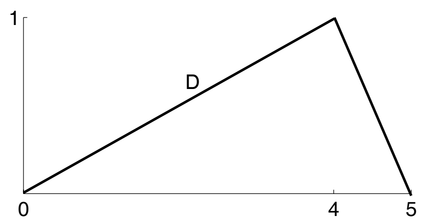

FIGURE 6. Fuzzy number D

In order to define two more coefficients for the skewness of fuzzy numbers, three indices based on the comparison of areas and probabilistic concepts are employed: the expected value, variance, and mode value. These indices are defined as follows:

1. The expected value is:

$$E(A) = \frac{\int_{-\infty}^{+\infty} xA(x)dx}{\int_{-\infty}^{+\infty} A(x)dx}, \tag{39}$$

provided that the integral converges absolutely, that is:

$$\int_{-\infty}^{+\infty} |xA(x)|dx < \infty. \tag{40}$$

Otherwise, A has no finite expected value.

2. The variance:

$$\sigma_A^2 = \frac{\int_{-\infty}^{+\infty}[x - E(A)]^2 A(x)dx}{\int_{-\infty}^{+\infty} A(x)dx}. \tag{41}$$

3. The mode value:

$$Mode(A) = \{x|A(x) = supA(x), x \in supp(A)\}, \tag{42}$$

where $A(x)$ is the membership function of fuzzy number A.
Now we define two types of the coefficient of the skewness of fuzzy numbers that are called the first and second Pearson's skewness coefficients.

Definition 7.5. Let $E(A)$, $Mode(A)$ and σ_A^2 be the expected value, mode, and variance of a fuzzy number A, respectively. Then the first and second Pearson's skewness coefficients of A, denoted by Sk_1^A, Sk_2^A, are defined as follows:

$$Sk_1^A = \frac{E(A) - Mode(A)}{\sigma_A}. \tag{43}$$

$$Sk_2^A = \frac{3(E(A) - Q_2^A)}{\sigma_A}. \tag{44}$$

For the fuzzy number A of Example 1 we have

$$E(A) = \mu,\ Mode(A) = \mu,\ Q_2^A = \mu,\ \sigma_A^2 = \sigma^2,$$

so

$$Sk_1^A = 0,\ Sk_2^A = 0.$$

The skewness coefficients are zero, showing that the fuzzy number A in Example 1 is symmetric. The coefficient of kurtosis is a measure of the degree of peakedness or flatness in the variable distribution [14]. The kurtosis of a fuzzy number is based on the size of the tails of the membership function. Fuzzy numbers with relatively small tails are called "leptokurtic" and those with large tails are called "platykurtic" (see Figure 7).

Definition 7.6. Let P_i^A, (i=1,2,...99) be i^{th} percentile of a fuzzy number A. Then the percentile kurtosis coefficient of A is defined by

$$KR(A) = \frac{P_{75}^A - P_{25}^A}{2(P_{90}^A - P_{10}^A)} = \frac{Q_3^A - Q_1^A}{2(P_{90}^A - P_{10}^A)}. \tag{45}$$

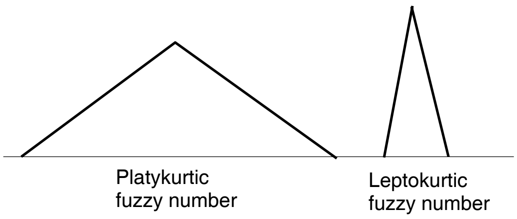

Figure 7. Kurtosis of fuzzy numbers

The conventional measure of the kurtosis (KR) of a fuzzy number can be interpreted as a measure of the dispersion of the fuzzy number around the two values $\mu_A \pm \sigma_A$. Hence, $KR(A)$ is large when the mass is concentrated about the mean value (μ_A). For the fuzzy number A of Example 1 we obtain:

$$KR(A) = \frac{(\mu + \sigma\phi^{-1}(\frac{75}{100})) - (\mu + \sigma\phi^{-1}(\frac{25}{100}))}{2[(\mu + \theta\phi^{-1}(\frac{90}{100})) - (\mu + \sigma\phi^{-1}(\frac{10}{100}))]} = \frac{0.68 - (-0.68)}{2[1.28 - (-1.28)]} = 0.266.$$

The value 0.266 is the standard for comparing the kurtosis coefficients of fuzzy numbers; a fuzzy number A such that $KR(A) > 0.266$ is considered to be leptokurtic and if $KR(A) < 0.266$ the fuzzynumber A is platykurtic.

8. *Alpha − cut* percentiles Derived from the Tails of Fuzzy Numbers

In this section, we will introduce percentiles for the fuzziness segments of a fuzzy number. Let A be a fuzzy number described by (1) and $A_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$, then for each $\alpha - cut$ and $i \in \{1, 2, ..., 99\}$, the i^{th} percentile from the left tail of a fuzzy number A ($PLT^A_{(i,\alpha)}$) and the i^{th} percentile from the right tail of a fuzzy number A ($PRT^A_{(i,\alpha)}$) are defined by

$$\int_a^{PLT^A_{(i,\alpha)}} A(x)dx = \frac{i}{100} \int_a^{\underline{a}(\alpha)} A(x)dx, \tag{46}$$

$$\int_{\overline{a}(\alpha)}^{PRT^A_{(i,\alpha)}} A(x)dx = \frac{i}{100} \int_{\overline{a}(\alpha)}^{d} A(x)dx. \tag{47}$$

Replacing the functions $g(x)$ and $h(x)$ in the equations (46) and(47), respectively, we get:

$$PLT^A_{(i,\alpha)} = G^{-1}[\frac{i}{100} G(\underline{a}(\alpha))], \tag{48}$$

$$PRT^A_{(i,\alpha)} = H^{-1}[(1-\frac{i}{100})H(\overline{a}(\alpha)) + \frac{i}{100}H(d)]. \quad (49)$$

For the fuzzy number $A = (a,b,c,d)_n$, $g(x) = (\frac{x-a}{b-a})^n, h(x) = (\frac{d-x}{d-c})^n$, we obtain:

$$PLT^A_{(i,\alpha)} = a + (b-a)\alpha^{\frac{1}{n}}(\frac{i}{100})^{\frac{1}{n+1}}, \quad (50)$$

$$PRT^A_{(i,\alpha)} = d - (d-c)\alpha^{\frac{1}{n}}(1-\frac{i}{100})^{\frac{1}{n+1}}. \quad (51)$$

Proposition 8.1. *Let $A = (a,b,c,d)_n$. Then for every $i \in \{1,2,...,99\}$ and each $\alpha - cut$,*

$$A(PLT^A_{(i,\alpha)}) = \alpha(\frac{i}{100})^{\frac{n}{n+1}}, \;\; A(PRT^A_{(i,\alpha)}) = \alpha(1-\frac{i}{100})^{\frac{n}{n+1}}, \quad (52)$$

$$\frac{A(PRT^A_{(i,\alpha)})}{A(PLT^A_{(100-i,\alpha)})} = 1. \quad (53)$$

Proof. By (50) and (51), we get:

$$A(PLT^A_{(i,\alpha)}) = (\frac{PLT^A_{(i,\alpha)} - a}{b-a})^n = \alpha(\frac{i}{100})^{\frac{n}{n+1}},$$

and

$$A(PRT^A_{(i,\alpha)}) = (\frac{d - PRT^A_{(i,\alpha)}}{d-c})^n = \alpha(1-\frac{i}{100})^{\frac{n}{n+1}},$$

consequently, $\frac{A(PRT^A_{(i,\alpha)})}{A(PLT^A_{(100-i,\alpha)})} = 1$.

This proposition shows that for any $0 \leq \alpha \leq 1$ and $i \in \{1,2,...,99\}$

$$A(PLT^A_{(i,\alpha)}) = A(PRT^A_{(100-i,\alpha)}). \quad (54)$$

Furthermore, we have found a new and interesting interval approximation through the $\alpha - cut$ percentiles from tails of a fuzzy number A:

$$PLRTI^A_{(i,\alpha)} = [PLT^A_{(i,\alpha)}, PRT^A_{(100-i,\alpha)}]. \quad (55)$$

□

The interval approximation for fuzzy numbers given above can be applied to describe various fuzzy problems. Also, for any $0 \leq \alpha \leq 1$ and $i < j$ $(i,j \in \{1,2,...,99\})$ we have

$$PLRTI^A_{(j,\alpha)} \subset PLRTI^A_{(i,\alpha)}. \quad (56)$$

Example 8.2. Consider the fuzzy number $D = (0, 1, 2, 4)$ (Figure 8). We have:

$$PLT^D_{(10,\frac{1}{2})} = 0 + (1-0)(\frac{1}{2})(\frac{10}{100})^{\frac{1}{1+1}} = 0.1581,$$

$$PRT^D_{(90,\frac{1}{2})} = 4 - (4-2)(\frac{1}{2})(1 - \frac{90}{100})^{\frac{1}{1+1}} = 3.6838,$$

$$D(PLT^D_{(10,\frac{1}{2})}) = 0.1581, \; D(PRT^D_{(90,\frac{1}{2})}) = 0.1581.$$

$$PLRTI^D_{(10,\frac{1}{2})} = [PLT^D_{(10,\frac{1}{2})}, PRT^D_{(90,\frac{1}{2})}] = [0.1581, 3.6838].$$

$$PLRTI^D_{(20,\frac{1}{2})} = [PLT^D_{(20,\frac{1}{2})}, PRT^D_{(80,\frac{1}{2})}] = [0.2236, 3.5528],$$

$$D(PLT^D_{(20,\frac{1}{2})}) = 0.2236, \; D(PRT^D_{(80,\frac{1}{2})}) = 0.2236.$$

Observe that

$$PLRTI^D_{(20,\frac{1}{2})} = [0.2236, 3.5528] \subset [0.1581, 3.6838] = PLRTI^D_{(10,\frac{1}{2})}.$$

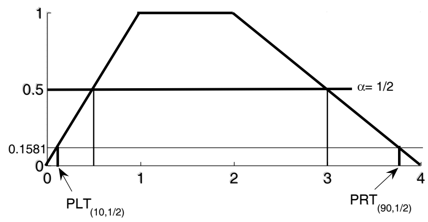

FIGURE 8. Fuzzy number D, $PLT^D_{(10,\frac{1}{2})}, PRT^D_{(90,\frac{1}{2})}$

9. **Conclusion**

In this paper we have introduced percentiles of fuzzy numbers and suggested several weighted means of fuzzy numbers as crisp point approximations for the center of fuzzy numbers. The interquartile range and the coefficients of skewness and kurtosis as dispersion indices of the fuzzy numbers based on the percentiles were also introduced. These criteria are robust measures of spread or dispersion and play an important role in the description of fuzzy numbers, especially in economic problems. Also, some of these indices can be used for ranking fuzzy numbers when the support functions are equal. Furthermore, the $\alpha - cut$ percentiles from the left and right tails of fuzzy numbers were defined, and a new and interesting interval approximation for fuzzy numbers was derived. For future research, we suggest defining other percentile intervals of fuzzy numbers which can be employed in transforming fuzzy comparison matrices to interval and point comparison matrices of fuzzy numbers and hence applied in the analytic hierarchy process by fuzzy numbers (fuzzy AHP). Additionally, future research could perform nonparametric tests on the percentiles of fuzzy numbers using simulation methods.

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