A GENERALIZATION OF THE CHEN-WU DUALITY INTO QUANTALE-VALUED SETTING

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Abstract. With the unit interval [0,1] as the truth value table, Chen and Wu presented the concept of possibility computation over dcpos. Indeed, every possibility computation can be considered as a [0,1]-valued Scott open set on a dcpo. The aim of this paper is to study Chen-Wu's duality on quantale-valued setting. For clarity, with a commutative unital quantale L as the truth value table, we introduce a concept of fuzzy possibility computations over fuzzy dcpos and then establish an equivalence between their denotational semantics and their logical semantics.

1. Introduction

As has been documented in [22], the semantics of programming languages has been intensively studied by both mathematicians and computer scientists. By proposing domain theory, in the late 1960s Dana S. Scott invented appropriate semantic domains in [18, 19]. In domain theory, computation in general involves two classes: determinism and non-derterminism. Deterministic computation means that the computed results are deterministic for a given input. However, if a non-deterministic program runs several times with a same input, it may produce different outputs. As we know, an important problem in domain theory is the modelling of non-deterministic features of programming languages and of parallel features treated in a non-deterministic way. In order to describe this non-determinism, the concept of powerdomain [8, 12, 13, 20] is introduced. Another method for description of non-determinism consists in quantifying this non-determinism by means of probability or possibility. Probabilistic non-determinism has also been studied and has led to the probabilistic powerdomain as a model [9, 10, 14, 21]. Different runs of a probabilistic program with the same input may again result in different outputs. In this situation, we need to know how likely these outputs are. Thus, a probability distribution or a continuous valuation on the domain of final states is chosen to describe such a behaviour. Based on this idea, Chen and Wu [3] proposed the concept of possibility valuations.

In computer science, standard or classic predicates are subsets of states, or can equivalently be regarded as $\{0,1\}$-valued functions defined on states. Chen and Jung [4] generalized classic predicates to the fuzzy case in such a way that fuzzy

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predicates on a dcpo D are Scott-continuous functions from D to the unit interval $[0,1]$. Also, Chen and Wu [3] introduced the notion of healthy fuzzy predicate transformers and gave the logical semantics of possibility computations. They successfully established the equivalence between denotational and logical semantics of possibility computations.

In this paper, we will consider a new kind of non-deterministic computations, called L-fuzzy possibility computations. The L-fuzzy possibility computations, in contrast with the possibility computations, consider the L-fuzzy possibility degree (i.e., the results of possibility value replaced by a general ordered structure L) as a state that the fuzzy domain of states lies in a fuzzy Scott open set on that fuzzy domain, which are more extensive than the possibility computation in [3]. In other words, we will define an L-fuzzy possibility distribution of states on the fuzzy Scott open topology of fuzzy dcpos by giving the L-fuzzy possibility that a state belongs to a certain fuzzy Scott open set. The goal of this kind of non-deterministic computations is to give, for a given input, the L-fuzzy possibility distributions of final states on fuzzy Scott topology of the fuzzy domain. Precisely, for a given fuzzy dcpo, we will establish a one-to-one correspondence between an L-fuzzy possibility computation and its L-fuzzy logical semantics.

Let L be a complete lattice. The greatest element of L is denoted by 1 and the least element of L is denoted by 0. For $S \subseteq L$, write $\bigvee S$ for the least upper bound of S and $\bigwedge S$ for the greatest lower bound of S. In particular, $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$.

A commutative quantale is a pair $(L, *)$, where L is a complete lattice and $* : L \times L \to L$ is a commutative, associative, and monotone operation such that $a * (\_)$ has a right adjoint for each $a \in L$. The operation $*$ is called a *tensor*, or a *tensor product*, on L. The right adjoint of $a * (\_)$ is denoted $a \to (\_)$. The resulting binary operation $\to: L \times L \to L$, given by $\to (a, b) = a \to b$, is called the *cotensor* corresponding to $*$. A commutative quantale is called *unital* if the tensor has a unit I , i.e., $a * I = a$ for each $a \in L$. It should be noted that the unit I need not be the greatest element 1 of L.

Some basic properties of the tensor $*$ and the implication operation $\to$ are collected in the following proposition.

Proposition 1.1. [15, 23] *Suppose that $(L, *, I)$ is a commutative unital quantale and $\to$ is the cotensor corresponding to $*$. Then for all $a, b, c, r \in L$ and $\{a_j\}_{j\in J} \subseteq L$, the following conditions hold:*

(I1) $a * b \leq c \Longleftrightarrow a \leq b \to c$,
(I2) $a \to b = \bigvee\{r \mid a * r \leq b\}$,
(I3) $I \to a = a$,
(I4) $(a \to b) * (b \to c) \leq (a \to c)$,
(I5) $a * \bigvee_{j\in J} a_j = \bigvee_{j\in J} a * a_j$,
(I6) $(\bigvee_{j\in J} a_j) \to b = \bigwedge_{j\in J}(a_j \to b)$,
(I7) $a \to (\bigwedge_{j\in J} a_j) = \bigwedge_{j\in J}(a \to a_j)$,
(I8) $(a \to b) \to (a \to c) \geq b \to c$,
(I9) $(a \to c) \to (b \to c) \geq b \to a$,
(I10) $a \to (b \to c) = (a * b) \to c$.

Throughout this paper, the triple $(L, *, I)$, simply L when there is no confusion, always denotes a commutative unital quantale.

Let X be a set. Each $A \in L^X$ is called an L-subset of X. For an element a in L, we use the symbol a_X to stand for the constant map from X to L with the value a. For $A, B \in L^X$, $A * B \in L^X$ denotes the L-subset of X sending each $x \in X$ to $A(x) * B(x)$. For $a \in L$, we write $a_X * B$ as $a * B$ for short.

Definition 1.2. [11] Let X be a nonempty set. An *L-order* (or, a *fuzzy order*) on X is a map $e : X \times X \longrightarrow L$ such that

(E1) Self-reflexivity: $e(x, x) \geq I$ for each $x \in X$,

(E2) Transitivity: $e(x, y) * e(y, z) \leq e(x, z)$ for all $x, y, z \in X$,

(E3) Antisymmetric: if $e(x, y) \geq I$ and $e(y, x) \geq I$, then $x = y$.

The pair (X, e) is called an *L-ordered set* (or, a *fuzzy poset*).

Example 1.3. (1) Define $e_L : L \times L \to L$ by $e_L(x, y) = x \to y$ for all $x, y \in L$. Then e_L is an L-order on L [1].

(2) For a given set X, the fuzzy inclusion order of fuzzy subsets or subsethood degree [7] of A in B is defined by $sub_X(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$ $(\forall A, B \in L^X)$. Then $sub_X : L^X \times L^X \to L$ is an L-order on L^X [1].

(3) For a given L-ordered set (X, e), define $\leq_e \subseteq X \times X$ as $x \leq_e y \Leftrightarrow e(x, y) \geq I$ for all $x, y \in X$. It is trivial to check that $\leq_e$ is an ordinary partial order on X. Conversely, for each ordinary partially ordered set $(X, \leq)$, define $e_{\leq} : X \times X \to L$ as $e_{\leq}(x, y) = I$ if $x \leq y$, and $e_{\leq}(x, y) = 0$ otherwise. It is easy to see that $e_{\leq}$ is an L-order on X.

Definition 1.4. [30, 31] For $x \in X$, $\downarrow x \in L^X$ (resp., $\uparrow x \in L^X$) is defined as

$$\forall y \in X,\ \downarrow x(y) = e(y, x)\ (\text{resp.}, \forall y \in X,\ \uparrow x(y) = e(x, y)).$$

Definition 1.5. [6] Let (X, e) be an L-ordered set. $A \in L^X$ is called a *fuzzy upper set* (resp., *fuzzy lower set*) if $A = \uparrow A$ (resp., $A = \downarrow A$), where $\uparrow A(x) = \bigvee_{y \in X} e(y, x) * A(y)$ (resp., $\downarrow A(x) = \bigvee_{y \in X} e(x, y) * A(y)$).

It is easy to verify that A is a fuzzy upper set (resp., a fuzzy lower set) if and only if $A(x) * e(x, y) \leq A(y)$ (resp., $A(x) * e(y, x) \leq A(y)$) for all $x, y \in X$.

Definition 1.6. [30, 31] Let (X, e) be an L-ordered set with $x_0 \in X$ and $A \in L^X$. The element x_0 is called a *join* (resp., *meet*) of A (w.r.t. the L-order e), in symbols $x_0 = \bigsqcup A$ (resp., $x_0 = \bigsqcap A$), if

(1) $\forall x \in X, A(x) \leq e(x, x_0)$ (resp., $A(x) \leq e(x_0, x)$);

(2) $\forall y \in X$, $\bigwedge\limits_{x \in X} A(x) \to e(x, y) \leq e(x_0, y)$ (resp., $\bigwedge\limits_{x \in X} A(x) \to e(y, x) \leq e(y, x_0)$).

It is easy to verify by (E3) that if x_1 and x_2 are two joins (or meets) of A, then $x_1 = x_2$. That is, each $A \in L^X$ has at most one join (or meet).

Proposition 1.7. [24, 30, 31] *Let (X, e) be an L-ordered set and $S \in L^X$. Then*

(1) $x_0 = \bigsqcup S$ *if and only if* $e(x_0, x) = \bigwedge_{y \in X}(S(y) \to e(y, x))$.

(2) $x_0 = \bigsqcap S$ *if and only if* $e(x, x_0) = \bigwedge_{y \in X}(S(y) \to e(x, y))$.

(3) $\forall x \in X,\ x = \bigsqcup \downarrow x$.
(4) $\forall x \in X,\ x = \bigsqcap \uparrow x$.

Definition 1.8. [1, 2] An L-ordered set (X, e) is called a *fuzzy complete lattice* if for each $A \in L^X$, $\bigsqcap A$ and $\bigsqcup A$ exist.

Example 1.9. For the L-ordered set (L^X, sub_X), where sub_X is the subsethood degree function on L^X, it follows from [1] that both $\bigsqcup \mathscr{R}$ and $\bigsqcap \mathscr{R}$ exist for each $\mathscr{R} : L^X \to L$, and they can be determined by the following formulas:

$$\bigsqcap \mathscr{R} = \bigwedge_{A \in L^X} \mathscr{R}(A) \to A \quad and \quad \bigsqcup \mathscr{R} = \bigvee_{A \in L^X} \mathscr{R}(A) * A.$$

Hence, (L^X, sub_X) is a fuzzy complete lattice.

Theorem 1.10. [24] *Let (X, e) be an L-ordered set. The following are equivalent:*
(1) *(X, e) is fuzzy complete;*
(2) *For each $A \in L^X$, $\bigsqcap A$ exists;*
(3) *For each $A \in L^X$, $\bigsqcup A$ exists.*

Definition 1.11. [30, 31] Let (X, e_X) and (Y, e_Y) be two L-ordered sets. A map $f : (X, e_X) \longrightarrow (Y, e_Y)$ is called *order-preserving* or *isotone* (resp., *antitone*) if for all $x, y \in X$, $e_X(x, y) \leq e_Y(f(x), f(y))$ (resp., $e_X(x, y) \leq e_Y(f(y), f(x))$.

Two L-ordered sets (X, e_X) and (Y, e_Y) are *order-isomorphic* if and only if there exist order-preserving maps $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$. Such maps are called *order-isomorphisms*.

Definition 1.12. [11, 25] Let (X, e) be an L-ordered set. $D \in L^X$ is called a *fuzzy directed subset* if
(1) $\bigvee_{x \in X} D(x) \geq I$, and
(2) for all $x, y \in X$, $D(x) * D(y) \leq \bigvee_{z \in X} D(z) * e(x, z) * e(y, z)$.

A fuzzy directed subset is a *fuzzy ideal* if it is a fuzzy lower set additionally. The sets of all fuzzy directed subsets and all fuzzy ideals on X are denoted by $\mathcal{D}_L(X)$ and $\mathcal{I}_L(X)$, respectively. An L-ordered set is called a *fuzzy dcpo* if each fuzzy directed subset has a join.

For each map $f : X \longrightarrow Y$, we have a map $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called the *L-forward powerset operator*, cf. [16, 17]) defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x) \ (\forall y \in Y, \forall A \in L^X).$$

The right adjoint of $f_L^{\rightarrow}$ is denoted by $f_L^{\leftarrow}$ (called the *L-backward powerset operator*, cf. [16, 17]) and given by

$$\forall B \in L^Y,\ f_L^{\leftarrow}(B) = B \circ f.$$

Definition 1.13. [26] Given two fuzzy dcpos (X, e_X) and (Y, e_Y), a fuzzy order-preserving map $f : (X, e_X) \longrightarrow (Y, e_Y)$ is said to be *fuzzy Scott continuous* if for each $D \in \mathcal{D}_L(X)$, $f(\bigsqcup D) = \bigsqcup f_L^{\rightarrow}(D)$.

Definition 1.14. [26] Let (X, e) be a fuzzy dcpo. $A \in L^X$ is said to be a *fuzzy Scott open set* if A is a fuzzy upper set and $A(\bigsqcup J) \leq \bigvee_{x\in X} A(x) * J(x)$ for each $J \in \mathcal{I}_L(X)$. The set of all fuzzy Scott open sets on X is denoted by $\sigma_L(X)$, called the *fuzzy Scot topology* on (X, e).

2. **Fuzzy Scott Topology on Fuzzy Dcpos Revisited**

With a complete Heyting algebra as the truth value table, fuzzy Scott topology on fuzzy dcpos are defined and studied in [26]. Fuzzy Scott topology on fuzzy dcpos seems a reasonable fuzzy version of the classic one since the Scott convergence of stratified L-filters is closely related to continuity of the underlying fuzzy dcpos [26] and the fuzzy Scott topology on a continuous fuzzy dcpo is modified L-sober [27].

There is always a question of great concern that, can we generalize the fuzzy Scott topology for some more general lattices? In this section, we would like to answer this question and give some preliminary results.

Definition 2.1. [5] An L-topology on a set X is a subset $\tau \subseteq L^X$ such that
(O1) $0_X \in \tau$, $1_X \in \tau$,
(O2) $U \wedge V \in \tau$ for all $U, V \in \tau$,
(O3) $\bigvee_{i\in I} U_i \in \tau$ for each subfamily $\{U_i|\ i \in I\}$ of τ.
An L-topology τ is stratified if
(O4) $a * U \in \tau$ for all $a \in L$ and $U \in \tau$.
An L-topology τ is co-stratified if
(O5) $a \to U \in \tau$ for all $a \in L$ and $U \in \tau$.
An L-topology τ is strong if it is both stratified and co-stratified.

Remark 2.2. Let L be a commutative unital quantale. For a fuzzy dcpo (X, e), although we call $\sigma_L(X)$ the fuzzy Scott topology on X, it indeed is not an L-topology since (O2) is not satisfied necessarily [29]. But it is straightforward to show that $\sigma_L(X)$ is stratified. Specifically, let $U \in \sigma_L(X)$ and $J \in \mathcal{I}_L(X)$. On one hand, it is easy to verify that $a * U$ is an upper set; on another hand, we have that

$$a * U(\bigsqcup J) \leq a * (\bigvee_{x\in X} U(x) * J(x)) = \bigvee_{x\in X} (a * U)(x) * J(x).$$

Hence, $a * U \in \sigma_L(X)$. The reason for the name of fuzzy Scott topology is that it really acts as a fuzzy counterpart of the classic Scott topology in domain theory.

Example 2.3. [5] Let $(L, *) = ([0, 1], \wedge)$, $X = [0, 1]$, and $\mathrm{id} : [0, 1] \longrightarrow [0, 1]$ be the identity function.

(1) The family $\tau = \{a \wedge (b \vee \mathrm{id})|\ a, b \in [0, 1]\}$ is a stratified but not co-stratified L-topology on X since $a \to \mathrm{id} \notin \tau$ for all $0 < a < 1$.

(2) For each $a \in [0, 1]$, let

$$\mu_a(x) = \begin{cases} x, & 0 \leq x \leq a; \\ 1, & a < x < 1. \end{cases}$$

Then $\delta = \{a \to \mathrm{id}|\ a \in [0, 1]\} \cup \{\mu_a|\ a \in [0, 1]\} \cup \{0_X\}$ is a co-stratified but not stratified L-topology on X.

A complete lattice L is called a frame if L satisfies the infinite distributive law of finite meets over arbitrary joins, i.e.,

$$a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b$$

for any $a \in L$, $B \subseteq L$. Clearly, a frame is a commutative unital quantale by regarding $\wedge$ as $*$.

Lemma 2.4. [28] *Considering a frame L as a fuzzy dcpo on itself, we have*

$$\sigma_L(L) = \{a_L \vee (b_L \wedge \mathrm{id}_L) |\ a \leq b\}.$$

Remark 2.5. The fuzzy Scott topology $\sigma_L(X)$ on a fuzzy dcpo X need not be co-stratified even if L is a frame. Here we have an example. Let $L = \{0, a, 1\}$ be a lattice with the order $0 \leq a \leq 1$. Clearly, $(L, \wedge)$ is a frame. By Lemma 2.3, we get the fuzzy Scott topology $\sigma_L(L) = \{0_L, A_1, \mathrm{id}_L, A_2, a_L, 1_L\}$ as follows:

	0_L	A_1	id_L	A_2	a_L	1_L
0	0	0	0	a	a	1
a	0	a	a	a	a	1
1	0	a	1	1	a	1

Then we have that

$$(a \to \mathrm{id}_L)(0) = a \to 0 = 0;\ \ (a \to \mathrm{id}_L)(a) = a \to a = 1;\ \ (a \to \mathrm{id}_L)(1) = a \to 1 = 1.$$

It follows that $a \to \mathrm{id}_L \notin \sigma_L(L)$. Hence, $\sigma_L(L)$ is not co-stratified.

Theorem 2.6. *The fuzzy poset $(\sigma_L(E), sub_E)$ is a fuzzy complete lattice, where for each $\mathcal{A} \in L^{\sigma_L(E)}$, $\bigsqcup \mathcal{A} = \bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U$.*

Proof. We first show that $\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U$ is a fuzzy upper set. For all $x, y \in E$, we have that

$$\begin{aligned} (\textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U)(x) * e_E(x,y) &= (\textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U(x)) * e_E(x,y) \\ &= \textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * (U(x) * e_E(x,y)) \\ &\leq \textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U(y) \\ &= (\textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U)(y). \end{aligned}$$

Secondly for each $J \in \mathcal{I}_L(E)$, we have

$$\begin{aligned} (\textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U)(\bigsqcup J) &= \textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U(\bigsqcup J) \\ &\leq \textstyle\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * \bigvee_{x \in E} U(x) * J(x) \\ &= \textstyle\bigvee_{x \in E} (\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U(x)) * J(x) \\ &= \textstyle\bigvee_{x \in E} (\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U)(x) * J(x). \end{aligned}$$

Then $\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U \in \sigma_L(E)$. Furthermore,

$$\forall A \in \sigma_L(E), sub_E(\bigwedge_{U \in \sigma_L(E)} \mathcal{A}(U) * U, A) = \bigwedge_{U \in \sigma_L(E)} \mathcal{A}(U) \to sub_E(U, A).$$

Hence, $\bigsqcup \mathcal{A} = \bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * U$. □

3. A Duality Between L-fuzzy Possibility Computation and Its L-fuzzy Logical Semantics

In domain theory, non-determinism is one class of computations which means that the computed results are more than one possibility for a given input. An important problem is how to describe non-determinism. In general, there are two methods: the powerdomains [8, 12, 13, 20] and a possibilistic (or probabilistic) model [10, 14]. Chen and Wu [3] considered the model of possibility computation, in which the possibility measures/valuations are defined as maps from the family of Scott open sets to the unit interval $[0, 1]$, satisfying the axiom of possibility measures. In a pure mathematical aspect, we wish to extend the results by replacing the unit interval with certain general lattices. As we assumed, L is a commutative unital quantale. Clearly, the unit interval is such a lattice. In this section, we will generalize the contents in [3] for L a commutative unital quantale.

Definition 3.1. Let (E, e_E) be a fuzzy dcpo. A map $\Pi : \sigma_L(E) \longrightarrow L$ is called an *L-fuzzy possibility valuation* of E if it preserves arbitrary fuzzy unions, i.e., $\Pi(\bigsqcup \mathcal{A}) = \bigsqcup \Pi_L^{\rightarrow}(\mathcal{A})$ for each $\mathcal{A} \in L^{\sigma_L(E)}$. Denote by $\pi_L(E)$ the set of all L-fuzzy possibility valuations of E.

Remark 3.2. In Definition 3.1, we regard L as a $\{0, 1\}$-ordered set (i.e., a fuzzy poset with the valuation $\{0, 1\}$ referring to a special case of Example 1.3(3)). For L a complete lattice, L can be seen as a $\{0, 1\}$-fuzzy complete lattice, where $\bigsqcup B = \bigvee_{a \in L} B(a) * a = \bigvee \{a \in L | \ B(a) = 1\}$ for each $B : L \longrightarrow \{0, 1\}$. Let $\Pi \in \pi_L(E)$. Since for each $\mathcal{A} \in L^{\sigma_L(E)}$, we have that

$$\bigsqcup \Pi_L^{\rightarrow}(\mathcal{A}) = \bigvee_{a \in L} \Pi_L^{\rightarrow}(\mathcal{A})(a) * a = \bigvee_{a \in L} \bigvee_{U \in \sigma_L(E), \Pi(U) = a} \mathcal{A}(U) * a = \bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * \Pi(U).$$

Theorem 3.3. *For each fuzzy dcpo (E, e), $(\pi_L(E), sub)$ is a fuzzy complete lattice, where*

$$\bigsqcup \mathscr{B} = \bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * \Pi$$

for each $\mathscr{B} \in L^{\pi_L(E)}$.

Proof. It suffices to prove that $\bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * \Pi \in \pi_L(E)$. For each $\mathcal{A} \in L^{\sigma_L(E)}$, we have

$$\begin{aligned}
(\bigsqcup \mathscr{B})(\bigsqcup \mathcal{A}) &= \textstyle\bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * \Pi(\bigsqcup \mathcal{A}) \\
&= \textstyle\bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * (\bigsqcup \Pi_L^{\rightarrow}(\mathcal{A})) \\
&= \textstyle\bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * (\bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * \Pi(U)) \\
&= \textstyle\bigvee_{U \in \sigma_L(E)} (\bigvee_{\Pi \in \pi_L(E)} \mathscr{B}(\Pi) * \Pi(U)) * \mathcal{A}(U) \\
&= \textstyle\bigvee_{U \in \sigma_L(E)} (\bigsqcup \mathscr{B})(U) * \mathcal{A}(U) \\
&= \textstyle\bigsqcup (\bigsqcup \mathscr{B})_L^{\rightarrow}(\mathcal{A}).
\end{aligned}$$

It follows that $\bigsqcup \mathscr{B} \in \pi_L(E)$. □

Definition 3.4. Let (D, e_D) and (E, e_E) be two fuzzy dcpos. The *denotational semantics* assigns to an L-fuzzy possibility computation F from D to E a fuzzy

Scott continuous function $[\![F]\!] : D \longrightarrow \pi_L(E)$. Denote by $[D \longrightarrow \pi_L(E)]_S$ the set of all L-fuzzy possibility computation of F from D to E.

From this definition, we can see the result of a possibility computation at one input state. For example, $d \in D$ is an L-fuzzy possibility valuation on the fuzzy Scott topology $\sigma_L(E)$, which gives the possibility that one element, for example $e \in E$, belongs to a fuzzy Scott open set $U \in \sigma_L(E)$.

Definition 3.5. An L-*fuzzy predicate transformer* from D to E is a map $t : \sigma_L(E) \longrightarrow \sigma_L(D)$. An L-fuzzy predicate transformer is called *healthy*, if it satisfies the following healthy conditions:

(H) Fuzzy sups-preserving: for $\mathcal{A} \in L^{\sigma_L(E)}$, $t(\bigsqcup \mathcal{A}) = \bigsqcup t_L^{\rightarrow}(\mathcal{A})$.

Denote by $[\sigma_L(E) \longrightarrow \sigma_L(D)]$ the set of all healthy L-fuzzy predicate transformers from D to E with the subsethood order.

Remark 3.6. If $t \in [\sigma_L(E) \longrightarrow \sigma_L(D)]$, by the condition (H) and Theorem 2.6, we have that

$$\begin{aligned} t(\bigsqcup \mathcal{A}) &= \bigsqcup t_L^{\rightarrow}(\mathcal{A}) \\ &= \bigvee_{V \in \sigma_L(D)} t_L^{\rightarrow}(\mathcal{A})(V) * V \\ &= \bigvee_{V \in \sigma_L(D)} \bigvee_{U \in \sigma_L(E),\ t(U)=V} \mathcal{A}(U) * V \\ &= \bigvee_{U \in \sigma_L(E)} \mathcal{A}(U) * t(U). \end{aligned}$$

Proposition 3.7. *Let $t : \sigma_L(E) \longrightarrow \sigma_L(D)$ be a healthy L-fuzzy predicate transformer. Then it satisfies*

(H′) Level-preserving: for $r \in L$ and $U \in \sigma_L(E)$, $t(r * U) = r * t(U)$.

Proof. For each $r \in L$ and each $U \in \sigma_L(E)$, we have that

$$\begin{aligned} sub(\bigsqcup r * \chi_U, V) &= \bigwedge_{W \in \sigma_L(E)} (r * \chi_U(W)) \to sub(W, V) \\ &= r \to sub(U, V) \\ &= r \to (\bigwedge_{x \in E} U(x) \to V(x)) \\ &= \bigwedge_{x \in E} r \to (U(x) \to V(x)) \\ &= \bigwedge_{x \in E} (r * U(x)) \to V(x) \\ &= sub(r * U, V) \end{aligned}$$

for each $V \in \sigma_L(E)$. Hence, $\bigsqcup r * \chi_U = r * U$. By the condition (H), we have that

$$t(r * U) = t(\bigsqcup r * \chi_U) = \bigsqcup t_L^{\rightarrow}(r * \chi_U) = \bigvee_{W \in \sigma_L(E)} r * \chi_U(W) * t(W) = r * t(U).$$

□

Definition 3.8. Let (D, e_D) and (E, e_E) be two fuzzy dcpos. The L-*fuzzy logical semantics* assigns to an L-fuzzy possibility computation F from D to E a healthy L-fuzzy predicate transformer $[\![F]\!]$from D to E.

Proposition 3.9. *Let (X, e) be a fuzzy dcpo. Then $\sigma_L(X) = [X \longrightarrow L]_S$ (see in* [26] *for L a frame).*

Proof. (1) Let $U \in \sigma_L(X)$ and $D \in \mathcal{D}_L(X)$. In fact, on one hand, we have

$$\begin{aligned} U(\bigsqcup D) &= U(\bigsqcup \downarrow D) \\ &= \bigvee_{x \in X} U(x) * (\downarrow D)(x) \\ &= \bigvee_{x \in X} U(x) * (\bigvee_{y \in X} D(y) * e(x, y)) \\ &= \bigvee_{y \in X} (\bigvee_{x \in X} U(x) * e(x, y)) * D(y) \\ &= \bigvee_{y \in X} U(y) * D(y). \end{aligned}$$

On the other hand, we have

$$\bigsqcup U_L^{\rightarrow}(D) = \bigvee_{a\in L} U_L^{\rightarrow}(D)(a) * a = \bigvee_{a\in L}\ \bigvee_{y\in X,\ U(y)=a} D(y) * a = \bigvee_{y\in X} D(y) * U(y).$$

Then we have $U(\bigsqcup D) = \bigsqcup U_L^{\rightarrow}(D)$, which follows that $U \in [X \longrightarrow L]_S$.

(2) Let $f \in [X \longrightarrow L]_S$. For each $x \in X$, we have

$$f(x) = f(\bigsqcup \downarrow x) = \bigsqcup f_L^{\rightarrow}(\downarrow x) = \bigvee_{y\in X} f(y) * \downarrow x(y) = \bigvee_{y\in X} f(y) * e(y,x) = \uparrow f(x).$$

It follows that f is a fuzzy upper set. Further, for each $J \in \mathcal{I}_L(X)$, we have

$$f(\bigsqcup J) = \bigsqcup f_L^{\rightarrow}(J) = \bigvee_{x\in X} f(x) * J(x).$$

Hence, $f \in \sigma_L(X)$. □

Next, we will show how to establish a relationship between an L-fuzzy possibility computation and its L-fuzzy logical semantics.

Proposition 3.10. *Let $t \in [\sigma_L(E) \longrightarrow \sigma_L(D)]$, $x \in D$ and $U \in \sigma_L(E)$. The transformation*

$$\alpha : [\sigma_L(E) \longrightarrow \sigma_L(D)] \longrightarrow [D \longrightarrow \pi_L(E)]_S$$

by $\alpha(t)(x)(U) = t(U)(x)$ defines a map.

Proof. (1) For each $x \in D$, we have $\alpha(t)(x) \in \pi_L(E)$. In fact, we have

$$\begin{aligned}
\alpha(t)(x)(\bigsqcup \mathcal{A}) &= t(\bigsqcup \mathcal{A})(x) \\
&= (\bigsqcup t_L^{\rightarrow}(\mathcal{A}))(x) \\
&= (\bigvee_{U\in\sigma_L(E)} t(U) * \mathcal{A}(U))(x) \\
&= \bigvee_{U\in\sigma_L(E)} t(U)(x) * \mathcal{A}(U) \\
&= \bigvee_{U\in\sigma_L(E)} \alpha(t)(x)(U) * \mathcal{A}(U) \\
&= \bigsqcup(\alpha(t)(x))_L^{\rightarrow}(\mathcal{A})
\end{aligned}$$

for each $\mathcal{A} \in L^{\sigma_L(E)}$.

(2) $\alpha(t) \in [D \longrightarrow \pi_L(E)]_S$, that is, $\alpha(t) : D \longrightarrow \pi_L(E)$ is a fuzzy Scott continuous function. In fact, for each $\mathcal{B} \in \mathcal{D}_L(D)$ and each $U \in \sigma_L(E)$, we have

$$\begin{aligned}
\alpha(t)(\bigsqcup \mathcal{B})(U) &= t(U)(\bigsqcup \mathcal{B}) \\
&= \bigsqcup(t(U))_L^{\rightarrow}(\mathcal{B}) \\
&= \bigvee_{x\in D} t(U)(x) * \mathcal{B}(x) \\
&= \bigvee_{x\in D} \alpha(t)(x)(U) * \mathcal{B}(x) \\
&= (\bigvee_{x\in D} \alpha(t)(x) * \mathcal{B}(x))(U) \\
&= (\bigsqcup(\alpha(t))_L^{\rightarrow}(\mathcal{B}))(U).
\end{aligned}$$

□

Proposition 3.11. *Let $h \in [D \longrightarrow \pi_L(E)]_S$, $x \in D$ and $U \in \sigma_L(E)$. The transformation*

$$\beta : [D \longrightarrow \pi_L(E)]_S \longrightarrow [\sigma_L(E) \longrightarrow \sigma_L(D)]$$

by $\beta(h)(U)(x) = h(x)(U)$ defines a map.

Proof. (1) On one hand, for each $U \in \sigma_L(E)$ and each $\beta(h)(U) \in \sigma_L(D)$, we have

$$\beta(h)(U)(x) * e_D(x,y) = h(x)(U) * e_D(x,y) = (h(x) * e_D(x,y))(U) \leq \beta(h)(U)(y).$$

It follows that $\beta(h)(U)$ is a fuzzy upper set.

On the other hand, for each $B \in \mathcal{I}_L(D)$, we have

$$\begin{array}{rcl} \beta(h)(U)(\bigsqcup B) & = & h(\bigsqcup B)(U) \\ & = & (\bigsqcup h_L^{\rightarrow}(B))(U) \\ & = & (\bigvee_{x\in D} B(x) * h(x))(U) \\ & = & \bigvee_{x\in D} B(x) * \beta(h)(U)(x). \end{array}$$

(2) $\beta(h) \in [\sigma_L(E) \longrightarrow \sigma_L(D)]$, that is, $\beta(h) : \sigma_L(E) \longrightarrow \sigma_L(D)$ preserves arbitrary fuzzy unions. In fact, for each $\mathcal{A} \in L^{\sigma_L(E)}$, we have

$$\begin{array}{rcl} \beta(h)(\bigsqcup \mathcal{A})(x) & = & h(x)(\bigsqcup \mathcal{A})(x) \\ & = & \bigsqcup h(x)_L^{\rightarrow}(\mathcal{A}) \\ & = & \bigvee_{U\in\sigma_L(E)} \mathcal{A}(U) * h(x)(U) \\ & = & \bigvee_{U\in\sigma_L(E)} \mathcal{A}(U) * \beta(h)(U)(x) \\ & = & (\bigsqcup \beta(h)_L^{\rightarrow}(\mathcal{A}))(x) \end{array}$$

□

With the help of Propositions 3.10 and 3.11, we now arrive at the main result of this paper.

Theorem 3.12. *Let (D, e_D) and (E, e_E) be two fuzzy dcpos. Then $[\sigma_L(E) \longrightarrow \sigma_L(D)] \cong [D \longrightarrow \pi_L(E)]_S$ via the pair of functions (α, β).*

Proof. (1) For each $h \in [D \longrightarrow \pi_L(E)]_S$, $x \in D$ and $U \in \sigma_L(E)$, we have

$$(\alpha \circ \beta)(h)(x)(U) = \beta(h)(U)(x) = h(x)(U).$$

Hence, $\alpha \circ \beta = \mathrm{id}_{[D\longrightarrow\pi_L(E)]_S}$.

(2) For each $t \in [\sigma_L(E) \longrightarrow \sigma_L(D)]$, $U \in \sigma_L(E)$ and $x \in D$, we have

$$(\beta \circ \alpha)(t)(U)(x) = \alpha(t)(x)(U) = t(U)(x).$$

Hence, $\beta \circ \alpha = \mathrm{id}_{[\sigma_L(E)\longrightarrow\sigma_L(D)]}$. □

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