ESTIMATORS BASED ON FUZZY RANDOM VARIABLES AND THEIR MATHEMATICAL PROPERTIES

M. G. AKBARI AND M. KHANJARI SADEGH

Abstract. In statistical inference, the point estimation problem is very crucial and has a wide range of applications. When, we deal with some concepts such as random variables, the parameters of interest and estimates may be reported/observed as imprecise. Therefore, the theory of fuzzy sets plays an important role in formulating such situations. In this paper, we first recall the crisp uniformly minimum variance unbiased (UMVU) and Bayesian estimators and then develop the concept of fuzzy estimators for fuzzy parameters based on fuzzy random variables.

1. Introduction

Statistical analysis, in traditional form, is based on crispness of data, random variable, point estimation, hypotheses, parameter and so on. As there are many different situations in which the above mentioned concepts are imprecise. Therefore, the fuzzy set theory is naturally an appropriate tool in statistical inference (point estimation) when fuzzy datas use considered. Suppose, for example, a random variable X has density function $f(x|\theta)$ with unknown parameter θ. Suppose further that a sample $X_1, X_2, ..., X_n$ is taken on X. The problem of point estimation (UMVU and Bayes estimation) is to pick a statistic $\delta(X_1, X_2, ..., X_n)$ that best estimates the parameter θ. *In this paper we study the theory of point estimation by using fuzzy random variables* $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$ *based on fuzzy parameter* $\widetilde{\theta}$.

The point estimation approaches are used in statistical inference frequently. On the other hand, the theory of fuzzy sets is a well known and useful tool in formulation and analysis imprecise and subjective concepts. Therefore, the FUMVU and fuzzy Bayesian estimators with fuzzy data play an important role. The problem of point estimation for an unknown parameter, using fuzzy data, is developed in different approaches.

Kruse [11], and Kruse and Meyer [12] explain some methods for point and interval estimation using fuzzy random variables. Viertl [24] [25] [26] studies some nonparametric methods in estimation using fuzzy data. Sadeghpour and Gien [21] study the Rao-Blackwell type theorem for fuzzy random variables. Some methods of statistical inference with fuzzy data, are considered by Viertl [27]. Buckley [5][6] studies the problems of statistical inference in fuzzy environment. Akbari and

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Rezaei [1] proposed a new method for uniformly minimum variance unbiased fuzzy point estimator.

There are some researches regarding the Bayesian point estimation methods combined with ideas from fuzzy set theory.

Uemura [22][23] formulated the fuzzy Bayes decision rule to facilitate determination of the loss function of a Bayes decision rule in a fuzzy environment. Gertner and Zhu [8], based on two extensions of likelihood function, have generalized the Bayesian estimates when sample information and prior distribution are fuzzy. They used their method to forest survey. Hryniewicz [10] proposed the notion of fuzzy Bayes point estimation for fuzzy data. Finally Hong-Zhong et al. [9] proposed a new method to determine the membership function of the estimate parameter of multi-parameter distribution. This method can be used to determine the membership function of the Bayesian estimates of multi-parameter distribution.

In this paper we give a new method for UMVU and Bayesian estimators based on Yao-Wu signed distance and L_2-metric in fuzzy environment which is completely different from those mentioned above. For this purpose we organize the matter in the following way:
in section 2 we describe some basic concepts of canonical fuzzy numbers, fuzzy random variable, fuzzy parameter, Yao-Wu signed distance and L_2-metric. Using Yao-Wu signed distance and L_2-metric, two types of the FUMVU estimators are considered in section 3. We come up in section 4 two kinds of the fuzzy Bayes estimators. Finally, a brief conclusion is provided in section 5.

2. **Preliminaries**

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a random variable (RV) X is a measurable function from $(\Omega, \mathcal{F}, P)$ to $(\mathcal{X}, \mathcal{B}, P_X)$, where P_X is the probability measure induced by X and is called the distribution of the RV X , i.e.,

$$P_X(A) = P(X \in A) = \int_{X \in A} dP \qquad \forall A \in \mathcal{B}.$$

If P_X be dominated by $\sigma-$finite measure υ, i.e. $P_X << \upsilon$ from Radon-Nikodym theorem (see Billingsley, [4]) we have

$$P_X(A) = \int_{X \in A} f(x|\theta) d\upsilon(x) \qquad \forall A \in \mathcal{B},$$

where $f(x|\theta)$ is the Radon-Nikodym derivative of P_X with respect to υ and is called probability density function (PDF) of X with respect to υ. In statistical context, the measure υ is usually a "counting measure" or a "Lebesgue measure", hence $P_X(A)$ is calculated by $\sum_{x \in A} f(x|\theta)$ or $\int_A f(x|\theta)dx$, respectively.

Let $S_X = \{x \in \mathcal{X} | f(x|\theta) > 0\}$ be the "support" or "sample space" of X, then a fuzzy subset $\widetilde{x}$ of S_X is defined by its membership function $\mu_{\widetilde{x}} : S_X \to [0, 1]$. We denote by $\widetilde{x}_\alpha = \{x : \mu_{\widetilde{x}}(x) \geq \alpha\}$ the $\alpha-$cut set of $\widetilde{x}$ and $\widetilde{x}_0$ is the closure of the set $\{x : \mu_{\widetilde{x}}(x) > 0\}$, and

(1) $\widetilde{x}$ is called a normal fuzzy set if there exist $x \in S_X$ such that $\mu_{\widetilde{x}}(x) = 1$;

(2) $\widetilde{x}$ is called a convex fuzzy set if $\mu_{\widetilde{x}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\widetilde{x}}(x)\ ,\ \mu_{\widetilde{x}}(y))$ for all $\lambda \in [0,1]$;

(3) the fuzzy set $\widetilde{x}$ is called a fuzzy number if $\widetilde{x}$ is a normal convex fuzzy set and its $\alpha-$cut sets, is bounded $\forall \alpha \neq 0$;

(4) $\widetilde{x}$ is called a closed fuzzy number if $\widetilde{x}$ is a fuzzy number and its membership function $\mu_{\widetilde{x}}$ is upper semicontinues;

(5) $\widetilde{x}$ is called a bounded fuzzy number if $\widetilde{x}$ is a fuzzy number and its membership function $\mu_{\widetilde{x}}$ has compact support.

If $\widetilde{x}$ is a closed and bounded fuzzy number with $x^L_\alpha = \inf\{x : x \in \widetilde{x}_\alpha\}$ and $x^U_\alpha = \sup\{x : x \in \widetilde{x}_\alpha\}$ and its membership function is strictly increasing on the interval $[x^L_\alpha, x^L_1]$ and strictly decreasing on the interval $[x^U_1, x^U_\alpha]$, then $\widetilde{x}$ is called canonical fuzzy number.

Given a real number $x \in S_X$, we can induce a fuzzy number $\widetilde{x}$ with membership function $\mu_{\widetilde{x}}(r)$ such that $\mu_{\widetilde{x}}(x) = 1$ and $\mu_{\widetilde{x}}(r) < 1$ for $r \neq x$. We call $\widetilde{x}$ a fuzzy real number induced by the real number x.

Let $\mathcal{F}(S_X)$ be the set of all fuzzy real numbers induced by the real number system S_X. We define the relation $\sim$ on $\mathcal{F}(S_X)$ as $\widetilde{x}_1 \sim \widetilde{x}_2$ iff $\widetilde{x}_1$ and $\widetilde{x}_2$ are induced by the same real number x. Hence $\sim$ is an equivalence relation, which induces the equivalence classes $[\widetilde{x}] = \{\widetilde{a} : \widetilde{a} \sim \widetilde{x}\}$. The quotient set $\mathcal{F}(S_X)/\sim$ is the set of all equivalence classes. We call $\mathcal{F}(S_X)/\sim$ the fuzzy real number system. In practice, we take only one element $\widetilde{x}$ from each equivalence class $[\widetilde{x}]$ to the fuzzy real number system $(\mathcal{F}(S_X)/\sim)$ that is,

$$(\mathcal{F}(S_X)/\sim) = \{\widetilde{x} : \widetilde{x} \in [\widetilde{x}], \widetilde{x}\ \textit{is the only element from}\ [\widetilde{x}]\}.$$

If the fuzzy real number system $(\mathcal{F}(S_X)/\sim)$ consists of all canonical fuzzy real numbers then we call $(\mathcal{F}(S_X)/\sim)$ the canonical fuzzy real number system.

Let X be a RV with support S_X and $\mathcal{F}(S_X)$ be the set of all fuzzy real numbers induced by the real numbers S_X.

Definition 2.1. A fuzzy random variable (FRV) is a Borel measurable function $\widetilde{X} : \Omega \to \mathcal{F}(S_X)$ where

$$\{(\omega, x) : \omega \in \Omega\ ,\ x \in \widetilde{X}_\alpha(\omega)\} \in\ \mathcal{F} \times \mathcal{B} \qquad \forall \alpha \in [0,1].$$

All $\alpha-$cuts of $\widetilde{X}$ are a compact convex random set and furthermore the above definition is equivalent to that given by Puri and Ralescu [18], and for $n = 1$ to the definition given by Kwakernaak [13].

In the above definition, we note that $\mathcal{F}(S_X)$ is the support of fuzzy random variable $\widetilde{X}$, and each $\alpha-$cut set of $\widetilde{X}$ is depend on random variable X, thus we can claim that the fuzzy random variable $\widetilde{X}$ is induced by X.

Proposition 2.2. *Let $\mathcal{F}(\mathcal{R})$ be a canonical fuzzy real number system, then $\widetilde{X}$ is a FRV iff X^L_α and X^U_α are random variables for all $\alpha \in [0,1]$.*

The fuzzy parameter $\widetilde{\theta}$ is a fuzzy subset of Θ with membership function $\mu_{\widetilde{\theta}}$. Here the fuzzy parameter $\widetilde{\theta}$ is assumed to be a canonical fuzzy real number. If $\widetilde{\theta}$ is a fuzzy parameter, then for all $\alpha \in [0,1]$ θ^L_α and θ^U_α are in the parameter space Θ.

Suppose RV X has a distribution $F(x)$ with parameter θ and $\widetilde{X}$ be a FRV induced by X.

Definition 2.3. We say that $\widetilde{X}$ has a distribution with fuzzy parameter $\widetilde{\theta}$ induced by X, if X^L_α and X^U_α have the same distribution as X with parameters θ^L_α and θ^U_α, respectively.

For example, we say that $\widetilde{X}$ has a fuzzy normal distribution with fuzzy $\widetilde{\theta}$ induced by random variable X with normal distribution $N(\theta, 1)$. Then X^L_α and X^U_α have normal distributions $N(\theta^L_\alpha, 1)$ and $N(\theta^U_\alpha, 1)$, respectively, for all $\alpha \in [0,1]$.

Let $\widetilde{\theta}$ be any fuzzy parameter of fuzzy random variable induced by X. *We confine our attention to estimate $\widetilde{\theta}$ under fuzzy environments.*

Now we define two distance between fuzzy numbers which is used in the sequel. Several ranking methods have been proposed so far, see for example Cheng[7], Yao and Wu [28] Modarres and Sadi-Nezhad [14], Nojavan and Ghazanfari [16], Puri and Ralescu [17], and Akbari and Rezaei [2].

2.1. **Yao-Wu Signed Distance.** In this subsection we use another ranking system for canonical fuzzy numbers which is very realistic and is defined by Yao and Wu as the follows.

Definition 2.4. For each $a, b \in \mathcal{R}$, the signed distance d^* of a and b is defined by $d^*(a,b) = a - b$. Thus, we define the rank of any two real numbers a, b,

$$\begin{aligned} d^*(a,b) > 0 \;&\Leftrightarrow\; d^*(a,0) > d^*(b,0) \;\Leftrightarrow\; a > b,\\ d^*(a,b) < 0 \;&\Leftrightarrow\; d^*(a,0) < d^*(b,0) \;\Leftrightarrow\; a < b,\\ d^*(a,b) = 0 \;&\Leftrightarrow\; d^*(a,0) = d^*(b,0) \;\Leftrightarrow\; a = b.\end{aligned}$$

Definition 2.5. For each $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathcal{R})$, the signed distance of $\widetilde{a}$ and $\widetilde{b}$ is defined as follows:

$$\begin{aligned} d(\widetilde{a}, \widetilde{b}) &= \int_0^1 (M_\alpha(\widetilde{a}) - M_\alpha(\widetilde{b}))d\alpha \\ &= \int_0^1 d^*(M_\alpha(\widetilde{a}), M_\alpha(\widetilde{b}))d\alpha,\end{aligned}$$

where $M_\alpha(\widetilde{a}) = \frac{a^L_\alpha + a^U_\alpha}{2}$ and $M_\alpha(\widetilde{b}) = \frac{b^L_\alpha + b^U_\alpha}{2}$; furthermore $d(\widetilde{a}, \widetilde{b})$ means the distance of $\widetilde{a}$ to $\widetilde{b}$.

Definition 2.6. [28] For each $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathcal{R})$, the ranking of $\widetilde{a}$ and $\widetilde{b}$ is given by

$$\begin{aligned} d(\widetilde{a},\widetilde{b}) > 0 \;&\Leftrightarrow\; d(\widetilde{a},0) > d(\widetilde{b},0) \;\Leftrightarrow\; \widetilde{a} \succ \widetilde{b},\\ d(\widetilde{a},\widetilde{b}) < 0 \;&\Leftrightarrow\; d(\widetilde{a},0) < d(\widetilde{b},0) \;\Leftrightarrow\; \widetilde{a} \prec \widetilde{b},\\ d(\widetilde{a},\widetilde{b}) = 0 \;&\Leftrightarrow\; d(\widetilde{a},0) = d(\widetilde{b},0) \;\Leftrightarrow\; \widetilde{a} \approx \widetilde{b}.\end{aligned}$$

2.2. L_2-**metric.** For each $\alpha-$cuts of $\widetilde{a} \in \mathcal{F}(\mathcal{R}^n)$ the support function $S_{\widetilde{a}_\alpha}$ is defined as $S_{\widetilde{a}_\alpha}(t) = \sup_{x \in \widetilde{a}_\alpha} \ll x, t \gg$, $t \in S^{n-1}$ where S^{n-1} is the $(n-1)-$dimensional unit sphere in $\mathcal{R}^n$. Using support function we define L_2-metric as follow

$$\delta_2(\widetilde{a}, \widetilde{b}) = (n \int_0^1 (\rho_2(\widetilde{a}_\alpha, \widetilde{b}_\alpha))^2 d\alpha)^{\frac{1}{2}} \qquad \widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathcal{R}^n),$$

where

$$\rho_2(\widetilde{a}_\alpha, \widetilde{b}_\alpha) = (\int_{S^{n-1}} |S_{\widetilde{a}_\alpha}(t) - S_{\widetilde{b}_\alpha}(t)|^2 \mu(dt))^{\frac{1}{2}}.$$

Note that μ is the normalized Lebesgue measure on S^{n-1}.

This metric is very realistic because

- which implies very good statistical properties in connection with variance;
- it involves distances between extreme points;
- it is distance with convenient statistical features.

3. The FUMVU Estimators of Fuzzy Parameter

In this section, we first propose the UMVU estimator and then use it to define FUMVU estimators. Their properties are also considered.

Let $X_1, X_2, ..., X_n$ be an *iid* random sample of size n, where $X_i^{'}$s have the PDF $f(x|\theta)$ with unknown parameter θ, $\theta \in \Theta$, and $x_1, x_2, ..., x_n$ are observations of $X_1, X_2, ..., X_n$, respectively.

Recall that an estimator $T(X_1, X_2, ..., X_n)$ of θ is unbiased iff $E[T(X_1, X_2, ..., X_n)] = \theta$ for any $\theta \in \Theta$. If there exists an unbiased estimator of θ, then θ is called an estimable parameter.

Definition 3.1. An unbiased estimator $T(X_1, X_2, ..., X_n)$ of θ is called the uniformly minimum variance unbiased estimator $(UMVUE)$ iff

$$Var_\theta[(T(X_1, X_2, ..., X_n))] \leq Var_\theta[(U(X_1, X_2, ..., X_n)],$$

for any $\theta \in \Theta$ and any other unbiased estimator $U(X_1, X_2, ..., X_n)$ of θ.

The derivation of UMVUE is relatively simple if there exists a complete sufficient statistic for $\theta \in \Theta$.

Theorem 3.2. *(Lehmann-Scheffé Theorem) Suppose there exists a complete sufficient statistic $T(X_1, X_2, ..., X_n)$ for $\theta \in \Theta$. If θ is estimable, then there is a unique unbiased estimator of $g(\theta)$ which is of the form $h(T)$ with a Borel function h. Furthermore, $h(T)$ is the unique $UMVUE$ of θ.*

There are two typically ways to derive the UMVUE when a complete sufficient statistic $T(X_1, X_2, ..., X_n)$ is available.

The first one is solving for h when the distribution of T is available. The second method (we use it) is conditioning on T, i.e., if $U(X_1, X_2, ..., X_n)$ is any unbiased estimators of θ, then $E[(U(X_1, X_2, ..., X_n)|\ t(x_1, x_2, ..., x_n))]$ is the UMVUE of θ. To apply this method, we do not need the distribution of T, but need to work out the conditional expectation $E[U(X_1, X_2, ..., X_n|t)]$. By uniqueness of the UMVUE, it does not matter which $U(X_1, X_2, ..., X_n)$ is used. Hence we choose $U(X_1, X_2, ..., X_n)$ so as to make the calculation of $E[U(X_1, X_2, ..., X_n)|t]$ as easy as possible. For a review in more details, see Shao [20]).

Definition 3.3. Let $\widetilde{X}$ and $\widetilde{Y}$ be two FRV's induced by X and Y. We say that $\widetilde{X}$ and $\widetilde{Y}$ are independent iff each random variable in the set $\{X_\alpha^L, X_\alpha^U : 0 \le \alpha \le 1\}$ is independent of each random variable in the set $\{Y_\alpha^L, Y_\alpha^U : 0 \le \alpha \le 1\}$.

Definition 3.4. We say that $\widetilde{X}$ and $\widetilde{Y}$are identically distributed iff X_α^L, Y_α^L are identically distributed, and X_α^U, Y_α^U are identically distributed for all $\alpha \in [0, 1]$.

Definition 3.5. We say that $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ is a fuzzy random sample iff $\widetilde{X}_i$'s is independent and identically distributed.

Let $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ be a fuzzy random sample of size n induced by random sample $\mathbf{X} = (X_1, X_2, ..., X_n)$ with fuzzy parameter $\widetilde{\theta}$.

We have the $\alpha-$cut set $\widetilde{\theta}_\alpha = [\theta_\alpha^L, \theta_\alpha^U]$. Since θ_α^L and θ_α^U are continuous with respect to α, $[\theta_\alpha^L, \theta_\alpha^U]$ is continuously shrinking with respect to α, then

$$\forall\ \theta \in [\theta_\alpha^L, \theta_\alpha^U] \qquad \exists\ \theta_\beta^L\ or\ \theta_\beta^U \quad s.t. \quad \theta = \theta_\beta^L\ or\ \theta = \theta_\beta^U,\ for\ some\ \beta \ge \alpha,$$

thus for any parameter $\theta \in \widetilde{\theta}_\alpha$, we can associate an estimate $\delta(\mathbf{x})$ to θ.

- ***If*** $\theta = \theta_\beta^L$***,*** $\beta \ge \alpha$

we define δ_α^L and δ_α^U as follows:

$$\delta_\alpha^L = \delta_\alpha^L(\widetilde{\mathbf{x}}) = \inf_{\alpha \le \beta \le 1} \{\delta(\mathbf{x}) : x_i = x_{i\beta}^L; \qquad i = 1, 2, ..., n\},$$

$$\delta_\alpha^U = \delta_\alpha^U(\widetilde{\mathbf{x}}) = \sup_{\alpha \le \beta \le 1} \{\delta(\mathbf{x}) : x_i = x_{i\beta}^L; \qquad i = 1, 2, ..., n\},$$

and the interval $I(\delta_\alpha) = [\delta_\alpha^L, \delta_\alpha^U]$ contains all of the crisp estimators $\theta \in [\theta_\alpha^L, \theta_1^L]$ of each RV $X_{i\beta}^L$ and $X_{i\beta}^U$ for $\beta \ge \alpha$.

- ***If*** $\theta = \theta_\beta^U$***,*** $\beta \ge \alpha$

we define δ_α^L and δ_α^U as follows:

$$\delta_\alpha^L = \delta_\alpha^L(\widetilde{\mathbf{x}}) = \inf_{\alpha \le \beta \le 1} \{\delta(\mathbf{x}) : x_i = x_{i\beta}^U; \qquad i = 1, 2, ..., n\},$$

$$\delta_\alpha^U = \delta_\alpha^U(\widetilde{\mathbf{x}}) = \sup_{\alpha \le \beta \le 1} \{\delta(\mathbf{x}) : x_i = x_{i\beta}^U; \qquad i = 1, 2, ..., n\},$$

and the interval $I(\delta_\alpha)$ contains all of the crisp estimators $\theta \in [\theta_1^U, \theta_\alpha^U]$ of each RV $X_{i\beta}^L$ and $X_{i\beta}^U$ for $\beta \ge \alpha$.

- ***Generally if*** $\theta = \theta_\beta^L$ ***or*** $\theta = \theta_\beta^L$***,*** $\beta \ge \alpha$

we define δ_α^L and δ_α^U as follows:

$$\delta_\alpha^L = \min[\inf_{\alpha\leq\beta\leq 1}\{\delta(\mathbf{x}) : x_i = x_{i\beta}^L\}, \inf_{\alpha\leq\beta\leq 1}\{\delta(\mathbf{x}) : x_i = x_{i\beta}^U\}]$$

$$\delta_\alpha^U = \max[\sup_{\alpha\leq\beta\leq 1}\{\delta(\mathbf{x}) : x_i = x_{i\beta}^L\}, \sup_{\alpha\leq\beta\leq 1}\{\delta(\mathbf{x}) : x_i = x_{i\beta}^U\}],$$

and the interval $I(\delta_\alpha)$ contains all of the crisp estimators $\theta \in \widetilde{\theta}_\alpha$ of each RV $X_{i\beta}^L$ and $X_{i\beta}^U$ for $\beta \geq \alpha$.

Definition 3.6. The fuzzy estimator of $\widetilde{\theta}$ is denoted by $\widetilde{\delta}(\widetilde{\mathbf{x}})$ and the membership function of $\widetilde{\delta}(\widetilde{\mathbf{x}})$ is defined by

$$\mu_{\widetilde{\delta}(\widetilde{\mathbf{x}})}(y) = \sup_{0\leq\alpha\leq 1} \alpha\ I_{I(\delta_\alpha)}(y).$$

Now, we are in the situation to discuss the FUMVU estimators for any fuzzy parameter $\widetilde{\theta}$, where θ_α^L and θ_α^U are the parameters of the random variables X_α^L and X_α^U respectively. Thus for any parameter $\theta \in \widetilde{\theta}_\alpha$, we can find the UMVU estimate $\widehat{\theta}$ of each $\theta \in \widetilde{\theta}_\alpha$.

We define $\widehat{\theta}_\alpha^L$ and $\widehat{\theta}_\alpha^U$ as follows.

$$\widehat{\theta}_\alpha^L = \min[\inf_{\alpha\leq\beta\leq 1}\{\widehat{\theta}(\mathbf{x}) : x_i = x_{i\beta}^L\}, \inf_{\alpha\leq\beta\leq 1}\{\widehat{\theta}(\mathbf{x}) : x_i = x_{i\beta}^U\}],$$

$$\widehat{\theta}_\alpha^U = \max[\sup_{\alpha\leq\beta\leq 1}\{\widehat{\theta}(\mathbf{x}) : x_i = x_{i\beta}^L\}, \sup_{\alpha\leq\beta\leq 1}\{\widehat{\theta}(\mathbf{x}) : x_i = x_{i\beta}^U\}],$$

where the interval $[\widehat{\theta}_\alpha^L, \widehat{\theta}_\alpha^U]$ will contain all of the UMVU estimators of each $\theta \in \widetilde{\theta}_\alpha$. The FUMVU estimator of $\widetilde{\theta}$ is denoted by $\widehat{\widetilde{\theta}}$ and its membership function is

$$\mu_{\widehat{\widetilde{\theta}}}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{[\widehat{\theta}_\alpha^L, \widehat{\theta}_\alpha^U]}(y).$$

Example 3.7. Let X be a RV from $N(\theta, \sigma^2)$ i.e.,

$$f(x|\theta) = \frac{1}{\sqrt{6.28}\sigma} \exp(-\frac{(x-\theta)^2}{2\sigma^2})\ x, \theta\ \in\ R,$$

and $\widetilde{X}_1$, $\widetilde{X}_2$, ..., $\widetilde{X}_n$ be a fuzzy random sample of size n with fuzzy parameter $\widetilde{\theta}$ induced by the normal distribution $N(\theta, 1)$. Then $X_{i\alpha}^L$ and $X_{i\alpha}^U$ have normal distribution $N(\theta_\alpha^L, 1)$ and $N(\theta_\alpha^U, 1)$, respectively. The UMVU estimate is

$$\widehat{\theta}_\alpha^L = \frac{\sum_{i=1}^n X_{i\alpha}^L}{n}$$

$$\widehat{\theta}_\alpha^U = \frac{\sum_{i=1}^n X_{i\alpha}^U}{n}.$$

Proposition 3.8. *If $\widehat{\theta}(\mathbf{x})$ is a continuous estimator of θ in order of x_i's, then $\widehat{\widetilde{\theta}}$ is a fuzzy random variable.*

Proof. Based on Proposition 2.1 it is enough to show that $\widehat{\theta}^L_\alpha$ and $\widehat{\theta}^U_\alpha$ are crisp random variables.

Since $\widetilde{X}_i$'s are FRVs induced by X_i's, by Proposition 2.1 $X^L_{i\beta}$ and $X^U_{i\beta}$ are crisp random variables for $\beta \geq 1$.

Note that $\widehat{\theta}(\mathbf{x})$ is a continuous function of x_i, so $\widehat{\theta}(X^L_{1\beta}, X^L_{2\beta}, ..., X^L_{n\beta})$ and $\inf_{\alpha \leq \beta \leq 1}\{\widehat{\theta}(\mathbf{X}) : X_i \in X^L_{i\beta}\}$ are crisp random variables for $\beta \geq 1$.

In a similar way, $\inf_{\alpha \leq \beta \leq 1}\{\widehat{\theta}(\mathbf{X}) : X_i \in X^U_{i\beta}\}$ is a crisp random variable for $\beta \geq 1$. It implies that $\widehat{\theta}^L_\alpha$ is a crisp random variable. The results can be used for $\widehat{\theta}^U_\alpha$. □

3.1. The FUMVU Estimators Based on Yao-Wu Signed Distance.

Definition 3.9. The fuzzy estimator $\widetilde{\delta}(\widetilde{\mathbf{X}})$ of $\widetilde{\theta}$ is a fuzzy unbiased estimator iff

$$\begin{cases} E[\delta(X^L_{1\alpha}, X^L_{2\alpha}, ..., X^L_{n\alpha})] = \theta^L_\alpha \\ E[\delta(X^U_{1\alpha}, X^U_{2\alpha}, ..., X^U_{n\alpha})] = \theta^U_\alpha \end{cases} \quad i = 1, 2, ..., n, \ \alpha \in [0, 1].$$

Example 3.10. Let X be a RV from $N(\theta, 1)$, population, i.e.,

$$f(x|\theta) = \frac{1}{\sqrt{6.28}} \exp(-\frac{(x-\theta)^2}{2}) \ x, \theta \ \in \ R,$$

and $\widetilde{X}_1, \ \widetilde{X}_2, \ ..., \ \widetilde{X}_n$ be a fuzzy random sample with triangular membership functions $(X_i - a, X_i, X_i + b)$ given by

$$\mu_{\widetilde{x}_i}(y) = \begin{cases} \frac{y - X_i + a}{a} & X_i - a \leq y \leq X_i \\ \frac{X_i + b - y}{b} & X_i \leq y \leq X_i + b, \end{cases}$$

for each $1 \leq i \leq n$ and $a, b \geq 0$. We can interpret the fuzzy FRV's $\widetilde{X}_i$ as the values of "near to X_i".

We have

$$\mu_{\widetilde{\delta}(\widetilde{\mathbf{X}})}(y) = \sup_{0 \leq \alpha \leq 1} \alpha I_{[\delta^L_\alpha, \delta^U_\alpha]}(y),$$

such that

$$\delta^L_\alpha = \overline{X} - (1-\alpha)a, \ \delta^U_\alpha = \overline{X} + (1-\alpha)b,$$

and

$$E[\delta^L_\alpha] = \theta - (1-\alpha)a = \theta^L_\alpha, \ E[\delta^U_\alpha] = \theta + (1-\alpha)b = \theta^U_\alpha.$$

We note that in crisp form for this example the unbiased estimator is $\delta(\mathbf{X}) = \overline{\mathbf{X}}$ and also $\mu_{\widetilde{\delta}(\widetilde{\mathbf{x}})}(\overline{X}) = 1$.

Example 3.11. Let X be a RV from $E(\theta, 1)$, population, i.e.,

$$f(x|\theta) = \exp(-(x-\theta)) \ x > \theta, \ \theta \ > 0,$$

and $\widetilde{X}_i's$, be our fuzzy random samples with membership functions as follows

$$\mu_{\widetilde{x}_i}(y) = \begin{cases} \exp(-(y-X_i)^2) & X_i - 0.5 \leq y \leq X_i + 0.5 \\ 0 & otherwise, \end{cases}$$

for each $1 \leq i \leq n$.
We have

$$\mu_{\widetilde{\delta^*}(\widetilde{\mathbf{x}})}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{[\delta_\alpha^L, \delta_\alpha^U]}(y),$$

such that

$$\delta_\alpha^L = \begin{cases} X_{(1)} - \frac{1}{n} - 0.5 & 0 \leq \alpha \leq \exp(-0.25) \\ X_{(1)} - \frac{1}{n} - \sqrt{-\ln\alpha} & \exp(-0.25) \leq \alpha \leq 1, \end{cases}$$

$$\delta_\alpha^U = \begin{cases} X_{(1)} - \frac{1}{n} + 0.5 & 0 \leq \alpha \leq \exp(-0.25) \\ X_{(1)} - \frac{1}{n} + \sqrt{-\ln\alpha} & \exp(-0.25) \leq \alpha \leq 1 \end{cases}$$

and

$$\theta_\alpha^L = \begin{cases} \theta - 0.5 & 0 \leq \alpha \leq \exp(-0.25) \\ \theta - \sqrt{-\ln\alpha} & \exp(-0.25) \leq \alpha \leq 1, \end{cases}$$

$$\theta_\alpha^U = \begin{cases} \theta + 0.5 & 0 \leq \alpha \leq \exp(-0.25) \\ \theta + \sqrt{-\ln\alpha} & \exp(-0.25) \leq \alpha \leq 1. \end{cases}$$

Let $\widetilde{\mathcal{D}}$ denote the set of all fuzzy unbiased estimators of $\widetilde{\theta}$.

Definition 3.12. The fuzzy estimator $\widetilde{\delta}(\widetilde{\mathbf{X}})$ of $\widetilde{\theta}$ is a fuzzy sufficient estimator iff

(i) the crisp estimator $\delta(X_{1\alpha}^L, X_{2\alpha}^L, ..., X_{n\alpha}^L)$ be a sufficient estimator for θ_α^L, $\alpha \in [0,1]$,

(ii) the crisp estimator $\delta(X_{1\alpha}^U, X_{2\alpha}^U, ..., X_{n\alpha}^U)$ be a sufficient estimator for θ_α^U $\alpha \in [0,1]$.

Definition 3.13. We define the membership function of the fuzzy variance function of $\widetilde{\delta}(\widetilde{\mathbf{X}})$, denoted by $\widetilde{\nu}(\widetilde{\delta}(\widetilde{\mathbf{X}}))$, as the following

$$\mu_{\widetilde{\nu}}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{[\nu_\alpha^L, \nu_\alpha^U]}(y)$$

where ν_α^L and ν_α^U are defined by

$$\nu_\alpha^L = \min[\inf_{\alpha\leq\beta\leq 1}\{Var_{\theta_\beta^L}(\delta(\mathbf{X})) : X_i = X_{i\beta}^L, \inf_{\alpha\leq\beta\leq 1}\{Var_{\theta_\beta^U}(\delta(\mathbf{X})) : X_i = X_{i\beta}^U\}],$$

$$\nu_\alpha^U = \max[\sup_{\alpha\leq\beta\leq 1}\{Var_{\theta_\beta^L}(\delta(\mathbf{X})) : X_i = X_{i\beta}^L, \sup_{\alpha\leq\beta\leq 1}\{Var_{\theta_\beta^U}(\delta(\mathbf{X})) : X_i = X_{i\beta}^U\}].$$

Proposition 3.14. *Let $\widetilde{U}(\widetilde{\mathbf{X}}) \in \widetilde{\mathcal{D}}$ and $\widetilde{T}(\widetilde{\mathbf{X}})$ be a sufficient estimator. If*

$$\widehat{\theta}_*(\mathbf{X}^L_\alpha) = \widehat{\theta}_*(X^L_{1\alpha}, X^L_{2\alpha}, ..., X^L_{n\alpha}) = E[U(\mathbf{X}^L_\alpha)|T(\mathbf{X}^L_\alpha)],$$

$$\widehat{\theta}_*(\mathbf{X}^U_\alpha) = \widehat{\theta}_*(X^U_{1\alpha}, X^U_{2\alpha}, ..., X^U_{n\alpha}) = E[U(\mathbf{X}^U_\alpha)|T(\mathbf{X}^U_\alpha)],$$

then the fuzzy estimator $\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})$ with membership function

$$\mu_{\widetilde{\widehat{\theta}}_*}(y) = \sup_{0\leq\alpha\leq1} \alpha I_{[\widehat{\theta}^L_{*\alpha}, \widehat{\theta}^U_{*\alpha}]}(y),$$

is a FUMVU estimator, i.e.,

$$\widetilde{\nu}(\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})) \precsim \widetilde{\nu}(\widetilde{U}(\widetilde{\mathbf{X}})),$$

where

$$\widehat{\theta}^L_{*\alpha} = \min[\inf_{\alpha\leq\beta\leq1}\{\widehat{\theta}_*(\mathbf{X}^L_\alpha)\}, \inf_{\alpha\leq\beta\leq1}\{\widehat{\theta}_*(\mathbf{X}^U_\alpha)\}],$$

$$\widehat{\theta}^U_{*\alpha} = \max[\sup_{\alpha\leq\beta\leq1}\{\widehat{\theta}_*(\mathbf{X}^L_\alpha)\}, \sup_{\alpha\leq\beta\leq1}\{\widehat{\theta}_*(\mathbf{X}^U_\alpha)\}].$$

Proof. Since $\widetilde{U}(\widetilde{\mathbf{X}}) \in \widetilde{\mathcal{D}}$, based on Definition 3.6 we have

$$\forall\ \alpha \in [0,1],\ \theta \in \widetilde{\theta}_\alpha \quad \exists\ X_i \in \widetilde{X}_{i\alpha} \quad s.t \quad E[U_\alpha(\mathbf{X})] = \theta.$$

and based on definition $\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})$ we have

$$\widehat{\theta}_{*\alpha}(\mathbf{X}) = E[U_\alpha(\mathbf{X})|T(\mathbf{X})] \qquad X_i \in \widetilde{X}_{i\alpha}\ i = 1, 2, ..., n$$

and $E[\widehat{\theta}_{*\alpha}(\mathbf{X})] = \theta$, so the fuzzy estimator $\widetilde{\widehat{\theta}}_*$ is a fuzzy unbiased estimator.

On the other hand, $\widetilde{T}(\widetilde{\mathbf{X}})$ is a fuzzy sufficient estimator, thus from Theorem 3.1 we have

$$Var_\theta[\widehat{\theta}_{*\alpha}(\mathbf{X})] \leq Var_\theta[U_\alpha(\mathbf{X})] \qquad X_i \in \widetilde{X}_{i\alpha},\ \theta \in \widetilde{\theta}_\alpha.$$

By Definition 3.8 we can write

$$\nu^L_\alpha[\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})] \leq \nu^L_\alpha[\widetilde{U}(\widetilde{\mathbf{X}})]$$

and similarly we have

$$\nu^U_\alpha[\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})] \leq \nu^U_\alpha[\widetilde{U}(\widetilde{\mathbf{X}})].$$

Hence according to the Yao-Wu singed distance we have

$$\widetilde{\nu}(\widetilde{\widehat{\theta}}_*(\widetilde{\mathbf{X}})) \precsim \widetilde{\nu}(\widetilde{U}(\widetilde{\mathbf{X}})).$$

□

3.2. **The FUMVU Estimators Based on L_2−metric.** Without loss of generality, we assume that $S_X = \mathcal{R}$, $\mathcal{F}(S_X) = \mathcal{F}(\mathcal{R})$ and $\mathcal{F}(S_{\mathbf{X}}) = \mathcal{F}(\mathcal{R}^n)$. Therefore, a FRV $\widetilde{\mathbf{X}}$ can be defined as a Borel measurable function

$$\widetilde{\mathbf{X}} : \Omega \to \mathcal{F}(\mathcal{R}^n).$$

Generally, the expected value $\widetilde{E}(\widetilde{\mathbf{X}})$ of the FRV $\widetilde{\mathbf{X}}$ is defined by

$$\widetilde{E}_\alpha(\widetilde{\mathbf{X}}) = \{E(\mathbf{X}) | \mathbf{X} : \Omega \to \mathcal{R}^n,\ \mathbf{X}(\omega) = \widetilde{\mathbf{X}}_\alpha(\omega)\},$$

where $\widetilde{E}_\alpha(\widetilde{\mathbf{X}})$ is Aumann's integral .

Definition 3.15. The variance of a FRV $\widetilde{\mathbf{X}}$ is defined as $\upsilon(\widetilde{\mathbf{X}}) = E[\Delta^2(\widetilde{\mathbf{X}}, E(\widetilde{\mathbf{X}}))]$.

Using $E_\alpha(\widetilde{\mathbf{X}}) = E(\widetilde{\mathbf{X}}_\alpha)$ and $S_{E(\widetilde{\mathbf{X}}_\alpha)}(t) = E(S_{\widetilde{\mathbf{X}}_\alpha}(t))$ it can be written as

$$\upsilon(\widetilde{\mathbf{X}}) = n \int_0^1 \int_{S^{n-1}} Var(S_{\widetilde{\mathbf{X}}_\alpha}(t))\mu(dt)d\alpha.$$

Näther [15] defined an scalar multiplication between $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ as

$$< \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} >= n \int_0^1 \int_{S^{n-1}} S_{\widetilde{\mathbf{X}}_\alpha}(t) S_{\widetilde{\mathbf{Y}}_\alpha}(t)\mu(dt)d\alpha,$$

which implies that

$$\upsilon(\widetilde{\mathbf{X}}) = E < \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}} > - < E(\widetilde{\mathbf{X}}), E(\widetilde{\mathbf{X}}) > .$$

Definition 3.16. [19] The conditional expectation of $\widetilde{\mathbf{X}}$ with respect to the σ−algebra $\mathcal{G}$ is the fuzzy random variable $E(\widetilde{\mathbf{X}}|\mathcal{G})$ with the following properties:

(a) $E(\widetilde{\mathbf{X}}|\mathcal{G})$ is $\mathcal{G}$ measurable.
(b) $\int_G E(\widetilde{\mathbf{X}}|\mathcal{G})dp = \int_G \widetilde{\mathbf{X}}dp \qquad \forall\ G \in \mathcal{G}$.

Proposition 3.17. [19] *For $\alpha \in [0, 1]$, we have*

$$E(\widetilde{X}_\alpha|\mathcal{G}) = E_\alpha(\widetilde{X}|\mathcal{G}).$$

Proposition 3.18. *If $\mathcal{G} \subseteq \mathcal{F}$, then*

$$E(S_{\widetilde{Y}_\alpha}(t)|\mathcal{G}) = S_{E(\widetilde{Y}_\alpha|\mathcal{G})}(t)) \qquad t \in S^{n-1}.$$

Proof. We know that $E(S_{\widetilde{Y}_\alpha}(t)|\mathcal{G})$ and $S_{E(\widetilde{Y}_\alpha|\mathcal{G})}(t))$ are $\mathcal{G}$−measurable. Hence it is enough to show that

$$\int_G E(S_{\widetilde{Y}_\alpha}(t)|\mathcal{G}))dp = \int_G S_{\widetilde{Y}_\alpha}(t)dp = \int_G S_{E(\widetilde{Y}_\alpha|\mathcal{G})}(t))dp \qquad \forall\ G \in \mathcal{G}.$$

Let $G \in \mathcal{G}$. We have

$$\begin{aligned}\int_G S_{\widetilde{Y}_\alpha}(t)dp &= \int_\Omega I_G S_{\widetilde{Y}_\alpha}(t)dp \\ &= E(I_G S_{\widetilde{Y}_\alpha}(t))dp\end{aligned}$$

$$=^{I_G\geq 0} S_{E(I_G\widetilde{Y}_\alpha)}(t)dp$$
$$= S_{\int_G \widetilde{Y}_\alpha}(t)dp$$
$$= S_{\int_G E(\widetilde{Y}_\alpha|\mathcal{G})}(t)dp$$
$$= \int_G S_{E(\widetilde{Y}_\alpha|\mathcal{G})}(t)dp.$$

□

If $\mathcal{G} = \sigma(\widetilde{\mathbf{Y}})$ is induced by a FRV $\widetilde{\mathbf{Y}}$ we can write $E(\widetilde{\mathbf{X}}|\mathcal{G}) = E(\widetilde{\mathbf{X}}|\widetilde{\mathbf{Y}})$, where $\sigma(\widetilde{\mathbf{Y}}) \subseteq \mathcal{F}$ is the smallest $\sigma-$algebra such that $\widetilde{\mathbf{Y}}$ is measurable.

Proposition 3.19. *Based on Proposition 3.3, we have the following inequality*

$$\upsilon(\widehat{\widetilde{\theta}}_*(\widetilde{\mathbf{X}})) \leq \upsilon(\widetilde{U}(\widetilde{\mathbf{X}})).$$

Proof. We have

$$\begin{aligned}
\upsilon(\widehat{\widetilde{\theta}}_*) &= E[S_{\widehat{\widetilde{\theta}}_{*\alpha}}(t) - E(S_{\widehat{\widetilde{\theta}}_{*\alpha}}(t))]^2 \\
&= E[S_{E(\widetilde{U}_\alpha|\widetilde{T})}(t) - E(S_{\widehat{\widetilde{\theta}}_{*\alpha}}(t))]^2 \\
&=^{Prop\ 3.4} E[S_{E_\alpha(\widetilde{U}|\widetilde{T})}(t) - E(S_{\widehat{\widetilde{\theta}}_{*\alpha}}(t))]^2 \\
&= E[S_{E_\alpha(\widetilde{U}|\widetilde{T})}(t) - S_{E(\widetilde{U}_\alpha)}(t))]^2 \\
&=^{Prop\ 3.5} E[E_\alpha(S_{\widetilde{U}}(t)|\widetilde{T}) - S_{E(\widetilde{U}_\alpha)}(t))]^2 \\
&\leq E[S_{\widetilde{U}}(t) - S_{E(\widetilde{U}_\alpha)}(t))]^2 = \upsilon(\widetilde{U}(\widetilde{\mathbf{X}})).
\end{aligned}$$

□

4. The Fuzzy Bayes Estimators of Fuzzy Parameter

In this section, we first propose the Bayesian estimator and then we use it to define the fuzzy Bayesian estimators. Also we will state their properties.

Let $X_1, X_2, ..., X_n$ be an *iid* random sample of size n, where X_i's have the PDF $f(x|\theta)$ with unknown parameter θ, $\theta \in \Theta$, and $x_1, x_2, ..., x_n$ are observations of $X_1, X_2, ..., X_n$, respectively, also let $\mathcal{A}$ denote the space of actions and $L : \Theta \times \mathcal{A} \to R^{\geq 0}$ be a loss function.

Definition 4.1. The function $\delta : S_{X_1} \times S_{X_2} \times ... \times S_{X_n} \to \mathcal{A}$ is a decision function, and we denote by $\mathcal{D}$ the class of decision functions from $S_{X_1} \times S_{X_2} \times ... \times S_{X_n}$ into $\mathcal{A}$.

Definition 4.2. The function $R : \Theta \times \mathcal{D} \to R^{\geq 0}$ which is defined by

$$R(\theta, \delta) = E[L(\theta, \delta(\mathbf{X}))]$$

is called the risk function.

Consider a prior density $\vartheta(\theta)$ for θ and assume that $f(x|\theta)$ is the conditional PDF of X for fixed $\theta \in \Theta$. The conditional density θ given by $X_i = x_i$ $i = 1, 2, ..., n$, is called the posterior density of θ and is denoted by $positiveimplicative(\theta|\mathbf{x})$. The Bayes risk of a decision δ, corresponding to the prior distribution *positiveimplicative*, is defined by

$$r(\vartheta, \delta) = E[R(\Theta, \delta)].$$

From the above definitions, a decision δ^* is a Bayes rule if it minimizes the Bayes risk, i.e., if

$$r(\vartheta, \delta^*) = \min\{r(\vartheta, \delta) : \delta \in \mathcal{D}\}.$$

Theorem 4.3. [20] *Let $X_1, X_2, ..., X_n$ be a random sample of size n, where X_i has the PDF $f(x|\theta)$ with unknown $\theta \in \Theta$ and observed values $x_1, x_2, ..., x_n$. Under the squared error (SE) loss function the Bayes point estimation of θ, is the mean of posterior distribution.*

Now, it is time to discuss the fuzzy Bayes estimators for any fuzzy parameter $\widetilde{\theta}$. For any parameter $\theta \in \widetilde{\theta}_\alpha$, we can find the Bayes estimate $\widehat{\theta^*}$ of θ

$$\widehat{\theta}^{*L}_\alpha = \min[\inf_{\alpha\le\beta\le1}\{\widehat{\theta}^*(\mathbf{x}) : x_i = x^L_{i\beta}\}, \inf_{\alpha\le\beta\le1}\{\widehat{\theta}^*(\mathbf{x}) : x_i = x^U_{i\beta}\}],$$

$$\widehat{\theta}^{*U}_\alpha = \max[\sup_{\alpha\le\beta\le1}\{\widehat{\theta}^*(\mathbf{x}) : x_i = x^L_{i\beta}\}, \sup_{\alpha\le\beta\le1}\{\widehat{\theta}^*(\mathbf{x}) : x_i = x^U_{i\beta}\}],$$

where the interval $[\widehat{\theta}^{*L}_\alpha, \widehat{\theta}^{*U}_\alpha]$ will contain all of the Bayes estimators of each $\theta \in \widetilde{\theta}_\alpha$. The fuzzy Bayes estimator of $\widetilde{\theta}$ is denoted by $\widetilde{\widehat{\theta}}^*$ and its membership function is

$$\mu_{\widetilde{\widehat{\theta}}^*}(y) = \sup_{0\le\alpha\le1} \alpha I_{[\widehat{\theta}^{*L}_\alpha, \widehat{\theta}^{*U}_\alpha]}(y).$$

Example 4.4. Let X be a RV from $N(\theta, \sigma^2)$, population, i.e.,

$$f(x|\theta) = \frac{1}{\sqrt{6.28\sigma}} \exp(-\frac{(x-\theta)^2}{2\sigma^2})\ x, \theta\ \in\ R,$$

and $\widetilde{x}_1,\ \widetilde{x}_2,\ ...,\ \widetilde{x}_n$, be a fuzzy random sample with triangular membership functions given by

$$\mu_{\widetilde{x}_i}(x) = \begin{cases} \frac{x-x_i+a}{a} & x_i - a \le x \le x_i \\ \frac{x_i+b-x}{b} & x_i \le x \le x_i + b, \end{cases}$$

for each $1 \le i \le n$ and $a, b \ge 0$.
We can interpret the canonical fuzzy numbers $\widetilde{x}_i$ as the values of "near to x_i".
Suppose $N(s, b^2)$ is the prior distribution, then we have

$$\mu_{\widetilde{\widehat{\theta}}^*}(y) = \sup_{0\le\alpha\le1} \alpha I_{[\widehat{\theta}^{*L}_\alpha, \widehat{\theta}^{*U}_\alpha]}(y),$$

where

$$\theta^L_\alpha = \theta - (1-\alpha)a \qquad \theta^U_\alpha = \theta + (1-\alpha)b,$$

$$\widehat{\theta}^{*L}_{\alpha} = \frac{\frac{n}{\sigma^2}(\overline{x} - (1-\alpha)a) + \frac{s}{b^2}}{\frac{n}{\sigma^2} + \frac{1}{b^2}}, \ \widehat{\theta}^{*U}_{\alpha} = \frac{\frac{n}{\sigma^2}(\overline{x} + (1-\alpha)b) + \frac{s}{b^2}}{\frac{n}{\sigma^2} + \frac{1}{b^2}},$$

Furthermore, $\mu_{\widetilde{\widehat{\theta}}^*}(\frac{\frac{n}{\sigma^2}\overline{x}+\frac{s}{b^2}}{\frac{n}{\sigma^2}+\frac{1}{b^2}}) = 1$.

4.1. **The Fuzzy Bayes Estimators Based on Yao-Wu Signed Distance.**

Definition 4.5. We define the membership function of the fuzzy risk function of $\widetilde{\delta}(\widetilde{\mathbf{X}})$, denoted by $\widetilde{R}(\widetilde{\delta}(\widetilde{\mathbf{X}}))$, as follow

$$\mu_{\widetilde{R}}(y) = \sup_{0\leq\alpha\leq1} \alpha I_{[R^L_\alpha, R^U_\alpha]}(y)$$

where

$$R^L_\alpha = \min[\inf_{\alpha\leq\beta\leq1}\{R(\theta^L_\beta, \delta(\mathbf{X})) : X_i = X^L_{i\beta}, \inf_{\alpha\leq\beta\leq1}\{R(\theta^U_\beta, \delta(\mathbf{X})) : X_i = X^U_{i\beta}\}],$$

$$R^U_\alpha = \max[\sup_{\alpha\leq\beta\leq1}\{R(\theta^L_\beta, \delta(\mathbf{X})) : X_i = X^L_{i\beta}, \sup_{\alpha\leq\beta\leq1}\{R(\theta^U_\beta, \delta(\mathbf{X})) : X_i = X^U_{i\beta}\}].$$

Definition 4.6. We define the membership function of the fuzzy Bayes risk function of $\widetilde{\delta}(\widetilde{\mathbf{X}})$, denoted by $\widetilde{r}(\widetilde{\delta}(\widetilde{\mathbf{X}}))$, as follow

$$\mu_{\widetilde{r}}(y) = \sup_{0\leq\alpha\leq1} \alpha I_{[r^L_\alpha, r^U_\alpha]}(y)$$

where

$$r^L_\alpha = \min[\inf_{\alpha\leq\beta\leq1}\{r(\vartheta^L_\beta, \delta(\mathbf{X})) : X_i = X^L_{i\beta}, \inf_{\alpha\leq\beta\leq1}\{r(\vartheta^U_\beta, \delta(\mathbf{X})) : X_i = X^U_{i\beta}\}],$$

$$r^U_\alpha = \max[\sup_{\alpha\leq\beta\leq1}\{r(\vartheta^L_\beta, \delta(\mathbf{X})) : X_i = X^L_{i\beta}, \sup_{\alpha\leq\beta\leq1}\{r(\vartheta^U_\beta, \delta(\mathbf{X})) : X_i = X^U_{i\beta}\}],$$

and ϑ^L_β and ϑ^U_β are the posterior distribution functions of θ^L_α and θ^U_α, respectively.

Proposition 4.7. *Let $\widetilde{\mathbf{X}}$ be a fuzzy random sample of size n induced by $\mathbf{X}$ with parameter $\widetilde{\theta}$ and $\widetilde{\mathcal{M}}$ be a set of all fuzzy estimators of $\widetilde{\theta}$. If*

$$\widehat{\theta}_{**}(\mathbf{X}^L_\alpha) = E[\Theta^L_\alpha|\mathbf{X}^L_\alpha],$$
$$\widehat{\theta}_{**}(\mathbf{X}^U_\alpha) = E[\Theta^U_\alpha|\mathbf{X}^U_\alpha],$$

then the fuzzy estimator $\widetilde{\widehat{\theta}}_{**}(\widetilde{\mathbf{X}})$ with membership function

$$\mu_{\widetilde{\widehat{\theta}}_{**}}(y) = \sup_{0\leq\alpha\leq1} \alpha I_{[\widehat{\theta}^L_{**\alpha}, \widehat{\theta}^U_{**\alpha}]}(y),$$

is a fuzzy Bayes estimator estimator, i.e.,

$$\widetilde{r}(\widetilde{\widehat{\theta}}_{**}(\widetilde{\mathbf{X}})) \precsim \widetilde{r}(\widetilde{\delta}(\widetilde{\mathbf{X}})) \qquad \forall\ \widetilde{\delta} \in \widetilde{\mathcal{M}},$$

where

$$\widehat{\theta}^L_{**\alpha} = \min[\inf_{\alpha\leq\beta\leq1}\{\widehat{\theta}_{**}(\mathbf{X}^L_\alpha)\}, \inf_{\alpha\leq\beta\leq1}\{\widehat{\theta}_{**}(\mathbf{X}^U_\alpha)\}],$$

$$\widehat{\theta}^{U}_{**\alpha} = \max[\sup_{\alpha\leq\beta\leq1}\{\widehat{\theta}_{**}(\mathbf{X}^{L}_{\alpha})\}, \sup_{\alpha\leq\beta\leq1}\{\widehat{\theta}_{**}(\mathbf{X}^{U}_{\alpha})\}].$$

4.2. **The Fuzzy Bayes Estimators Based on L_2-metric.**

Definition 4.8. Under the SE loss function, we define the risk value of $\widetilde{\delta}(\widetilde{\mathbf{X}})$ as follow

$$\begin{aligned} R[\widetilde{\theta}, \widetilde{\delta}(\widetilde{\mathbf{X}})] &= E[\Delta^2(\widetilde{\theta}, \widetilde{\delta}(\widetilde{\mathbf{X}}))|\widetilde{\theta}] \\ &= n\int_0^1\int_{S^{n-1}} E_{\widetilde{\mathbf{X}}|\widetilde{\theta}}(S_{\widetilde{\theta}_\alpha}(t) - S_{\widetilde{\delta}_\alpha(\widetilde{\mathbf{X}})}(t))^2\mu(dt)d\alpha. \end{aligned}$$

Definition 4.9. Under the SE loss function, we define the Bayes risk function of $\widetilde{\delta}(\widetilde{\mathbf{X}})$ as the following

$$r(\widetilde{\delta}(\widetilde{\mathbf{X}})) = E_\theta\{R[\widetilde{\Theta}, \widetilde{\delta}(\widetilde{\mathbf{X}})]\}.$$

Definition 4.10. The fuzzy estimator $\widetilde{\delta}^*(\widetilde{\mathbf{X}})$ is a fuzzy Bayes estimator iff for any fuzzy estimator $\widetilde{\delta}(\widetilde{\mathbf{X}})$ of $\widetilde{\theta}$ we have

$$r(\widetilde{\delta}^*(\widetilde{\mathbf{X}})) \leq r(\widetilde{\delta}(\widetilde{\mathbf{X}})).$$

Proposition 4.11. *Let $\widetilde{\mathbf{X}}$ be a random sample of size n induced by $\mathbf{X}$. For any fuzzy estimator $\widetilde{\delta}(\widetilde{\mathbf{X}})$ of $\widetilde{\theta}$ we have*

$$\begin{aligned} r(\widetilde{\delta}^*(\widetilde{\mathbf{X}})) &= r(\widehat{\widetilde{\theta}}_{**}(\widetilde{\mathbf{X}})) \\ &= E(\widetilde{\Theta}|\widetilde{\mathbf{X}}) \\ &\leq r(\widetilde{\delta}(\widetilde{\mathbf{X}})). \end{aligned}$$

Proof. It is obvious that $\widehat{\widetilde{\theta}}_{**}(\widetilde{\mathbf{X}}) = E(\widetilde{\Theta}|\widetilde{\mathbf{X}})$. Furthermore we have

$$\begin{aligned} r(\widetilde{\delta}(\widetilde{\mathbf{X}})) &= E_{\widetilde{\theta}}\{R[\widetilde{\Theta}, \widetilde{\delta}(\widetilde{\mathbf{X}})]\} \\ &= nE_{\widetilde{\theta}}\int_0^1\int_{S^{n-1}} E_{\widetilde{\mathbf{X}}|\widetilde{\theta}}(S_{\widetilde{\Theta}_\alpha}(t) - S_{\widetilde{\delta}_\alpha(\widetilde{\mathbf{X}})}(t))^2\mu(dt)d\alpha \\ &= n\int_0^1\int_{S^{n-1}} E_{\widetilde{\theta}}E_{\widetilde{\mathbf{X}}|\widetilde{\theta}}(S_{\widetilde{\Theta}_\alpha}(t) - S_{\widetilde{\delta}_\alpha(\widetilde{\mathbf{X}})}(t))^2\mu(dt)d\alpha \\ &= n\int_0^1\int_{S^{n-1}} E_{\widetilde{\mathbf{X}}}E_{\widetilde{\theta}|\widetilde{\mathbf{X}}}(S_{\widetilde{\Theta}_\alpha}(t) - S_{\widetilde{\delta}_\alpha(\widetilde{\mathbf{X}})}(t))^2\mu(dt)d\alpha \\ &\geq \int_0^1\int_{S^{n-1}} E_{\widetilde{\mathbf{X}}}E_{\widetilde{\theta}|\widetilde{\mathbf{X}}}(S_{\widetilde{\Theta}_\alpha}(t) - E(S_{\widetilde{\Theta}_\alpha}(t)|\widetilde{\mathbf{X}}))^2\mu(dt)d\alpha. \end{aligned}$$

Thus

$$\begin{aligned} & S_{\widetilde{\delta}^*_\alpha(\widetilde{\mathbf{X}})}(t) = E(S_{\widetilde{\Theta}_\alpha}(t)|\widetilde{\mathbf{X}}) \\ & \Rightarrow^{Prop\ 3.5} S_{\widetilde{\delta}^*_\alpha(\widetilde{\mathbf{X}})}(t) = S_{E(\widetilde{\Theta}_\alpha|\widetilde{\mathbf{X}})}(t) \\ & \Rightarrow \widetilde{\delta}^*_\alpha(\widetilde{\mathbf{X}}) = E(\widetilde{\Theta}_\alpha|\widetilde{\mathbf{X}}) \qquad \forall\ \alpha \in [0,1] \\ & \Rightarrow \widetilde{\delta}^*(\widetilde{\mathbf{X}}) = \widetilde{E}(\widetilde{\Theta}|\widetilde{\mathbf{X}}). \end{aligned}$$

□

5. **Conclusions**

In this paper, a new approach for estimating the fuzzy parameters of interest using fuzzy random variables is proposed. In this approach, we introduce the FUMVU and the fuzzy Bayes estimators based on Yao-Wu singed distance and L_2-metric.

It is noticeable that when the fuzzy random variables are reduced to crisp random variables, the proposed methods for estimating the parameters of interest are induced to the classical cases of UMVU and the Bayes estimators.

Extension of the proposed methods to estimate the parameters of linear models (regression models and design of experiment) can be considered for future research.

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M. G. Akbari*, Department of Statistics, Faculty of Sciences, University of Birjand, Southern Khorasan, Birjand

E-mail address: mga13512@yahoo.com

M. Khanjari Sadegh, Department of Statistics, Faculty of Sciences, University of Birjand, Southern Khorasan, Birjand

E-mail address: g_z_akbari@yahoo.com

*Corresponding author