FURTHER RESULTS ON L-ORDERED FUZZIFYING CONVERGENCE SPACES

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Abstract. In this paper, it is shown that the category of L-ordered fuzzifying convergence spaces contains the category of pretopological L-ordered fuzzifying convergence spaces as a bireflective subcategory and the latter contains the category of topological L-ordered fuzzifying convergence spaces as a bireflective subcategory. Also, it is proved that the category of L-ordered fuzzifying convergence spaces can be embedded in the category of stratified L-ordered convergence spaces as a coreflective subcategory.

1. **Introduction**

The theory of filters provides a good tool for defining convergence structures. With the development of fuzzy set theory, many researchers extended convergence structures to fuzzy setting using different kinds of fuzzy filters. In [5], Höhle and Šostak introduced the idea of an (resp., stratified) L-filter as a mapping from L^X to L and showed that stratified L-filters provided a fruitful tool in the development of lattice-valued topological spaces. For L a frame, Jäger [8] proved that the category SL-**GCS** of stratified L-fuzzy convergence spaces (which are called L-generalized convergence spaces in [9]) is Cartesian closed and the category of stratified L-topological spaces can be embedded in SL-**GCS** as a reflective subcategory. Following Jäger's suggestion in [9], Yao in [22] replaced stratified L-filters as studied in [8, 9] by L-filters of ordinary subsets to define L-fuzzifying convergence structures. It is shown that the category of L-fuzzifying topological spaces, as a reflective subcategory, can be embedded in the category of L-fuzzifying convergence spaces and the latter is Cartesian closed. Considering lattice-valued convergence structures that are compatible with both the fuzzy inclusion order of L-subsets and that of stratified L-filters, Fang [2] modified the definition of Jäger's stratified L-generalized convergence structures to obtain so called stratified L-ordered convergence structures. Afterwards, Wu and Fang [19] introduced L-ordered fuzzifying convergence structures and showed that the resulting category, as a bireflective full subcategory of the category of L-fuzzifying convergence spaces [22], is a Cartesian closed topological category. There are many works related to kinds of lattice-valued convergence structures (see e.g. [3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]).

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In this paper, our purpose is twofold. Firstly, we study the relations among L-ordered fuzzifying convergence spaces, pretopological and topological L-ordered fuzzifying convergence spaces in the categorical sense. Secondly, we discuss the relations between L-ordered fuzzifying convergence spaces and stratified L-ordered convergence spaces.

2. Preliminaries

In this paper we consider complete lattices L where finite meets is distributive over arbitrary joins, i.e.,

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i) \qquad \text{(ID)}$$

holds for all $a,\ b_i \in L$ $(i \in I)$. These lattices are called *complete Heyting algebras (or frames):* The bottom (resp. top) element of L is denoted by $\bot$ (resp.$\top$). We can then define a *residual implication* by

$$a \to b = \bigvee\{c \in L : a \wedge c \leqslant b\}.$$

We will often use, without explicitly mentioning, the following properties of the residual implication.

Lemma 2.1. [6] *Let L be a complete Heyting algebra. The following holds:*
(H1) $\top \to a = a$.
(H2) $a \leqslant b$ *if and only if* $a \to b = \top$.
(H3) $(a \to b) \to b \geqslant a$.
(H4) $(a \wedge b) \to (a \wedge c) \geqslant b \to c$.
(H5) $a \to \bigwedge_{j\in J} a_j = \bigwedge_{j\in J}(a \to a_j)$, *hence* $a \to b \leqslant a \to c$ *whenever* $b \leqslant c$.
(H6) $\bigvee_{j\in J} a_j \to b = \bigwedge_{j\in J}(a_j \to b)$, *hence* $a \to c \geqslant b \to c$ *whenever* $a \leqslant b$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. The smallest element and the largest element in L^X are denoted by $\underline{\bot}$ and $\underline{\top}$, respectively. Let $f : X \to Y$ be a mapping. Define $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively. In order to distinguish L-subsets and ordinary subsets, we usually denote L-subsets by A, B, C, D and denote ordinary subsets by U, V, W.

Definition 2.2. [2] The mapping $\mathcal{S}(-,-) : L^X \times L^X \to L$ defined by

$$\forall A,\ B \in L^X,\ \ \mathcal{S}(A, B) = \bigwedge_{x\in X}\Big(A(x) \to B(x)\Big)$$

is called the fuzzy inclusion order of L-subsets.

Definition 2.3. [6] A mapping $\mathcal{F} : 2^X \to L$ is called an L-filter of ordinary subsets on X if it satisfies
(F1) $\mathcal{F}(\emptyset) = \bot, \mathcal{F}(X) = \top$;
(F2) $U \subseteq V \Rightarrow \mathcal{F}(U) \leqslant \mathcal{F}(V)$;
(F3) $\mathcal{F}(U \cap V) \geqslant \mathcal{F}(U) \wedge \mathcal{F}(V)$.

The family of all L-filters of ordinary subsets on X will be denoted by $\mathcal{F}_L(X)$. For every $x \in X$, $\dot{x} \in \mathcal{F}_L(X)$ is defined by $\forall U \in 2^X$, $\dot{x}(U) = \top$ for all $x \in U$, and $\dot{x}(U) = \bot$ for all $x \notin U$.

Let $f : X \to Y$ be a mapping. For $U \in 2^Y$, the set $\{x \in X \mid f(x) \in U\}$ is denoted by $f^{-1}(U)$. Moreover, for each $\mathcal{F} \in \mathcal{F}_L(X)$, the mapping $f^{\Rightarrow}(\mathcal{F}) : 2^Y \to L$ defined by $\forall U \in 2^Y,\ f^{\Rightarrow}(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ is an L-filter of ordinary subsets on Y and is called the image of $\mathcal{F}$ under f.

Definition 2.4. [22] A mapping

$$\lim : \mathcal{F}_L(X) \to L^X,\ \mathcal{F} \mapsto \lim \mathcal{F}$$

subjects to the conditions

(LYC1) $\forall x \in X,\ \lim \dot{x}(x) = \top$;

(LYC2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X),\ \mathcal{F} \leqslant \mathcal{G} \Rightarrow \forall x \in X, \lim \mathcal{F}(x) \leqslant \lim \mathcal{G}(x)$,

is called an L-fuzzifying convergence structure on X, and the pair $(X, \lim)$ an L-fuzzifying convergence space.

A mapping $f : X \to Y$ between L-fuzzifying convergence spaces $(X, \lim^X)$ and $(Y, \lim^Y)$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L(X),\ x \in X,\ \lim^X \mathcal{F}(x) \leqslant \lim^Y f^{\Rightarrow}(\mathcal{F})(f(x))$. The category of L-fuzzifying convergence spaces and continuous mappings is denoted by L-**FYC**.

Let $\mathcal{S}_{\mathcal{F}}$ denote the fuzzy inclusion order on $\mathcal{F}_L(X)$, i.e., for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$,

$$\mathcal{S}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) = \bigwedge_{U \in 2^X} \Big(\mathcal{F}(U) \to \mathcal{G}(U)\Big).$$

Then we have the following definition.

Definition 2.5. [19] An L-fuzzifying convergence structure $\lim : \mathcal{F}_L(X) \to L^X$ satisfying the following condition

$$\text{(OLYC)}\ \ \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X),\ \mathcal{S}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) \leqslant \mathcal{S}(\lim \mathcal{F}, \lim \mathcal{G}),$$

is called an L-ordered fuzzifying convergence structure, and the pair $(X, \lim)$ an L-ordered fuzzifying convergence space.

A mapping $f : X \to Y$ between L-ordered fuzzifying convergence spaces $(X, \lim^X)$ and $(Y, \lim^Y)$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L(X),\ x \in X,\ \lim^X \mathcal{F}(x) \leqslant \lim^Y f^{\Rightarrow}(\mathcal{F})(f(x))$. Obviously, the category L-**OFYC** consisting of L-ordered fuzzifying convergence spaces is a full subcategory of L-**FYC**.

In [22], $\mathcal{N}^x_{\lim}$ is defined for an L-fuzzifying convergence space $(X, \lim)$ by

$$\forall U \in 2^X,\ \ \mathcal{N}^x_{\lim}(U) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L(X)} \Big(\lim \mathcal{F}(x) \to \mathcal{F}(U)\Big).$$

Then $\mathcal{N}^x_{\lim}$ is an L-filter of ordinary subsets on X satisfying $\mathcal{N}^x_{\lim} \leqslant \dot{x}$. It is not difficult to see that this definition is an L-valued interpretation of :" U is a neighborhood of x iff U belongs to every filter converging to x". Therefore, we call $\mathcal{N}^x_{\lim}$ the neighborhood L-filter of x.

Definition 2.6. [19] An L-ordered fuzzifying convergence space $(X, \lim)$ is called pretopological if lim satiesfies

$$\text{(OLYPC)}\ \ \forall \mathcal{F} \in \mathcal{F}_L(X), \forall x \in X,\ \lim \mathcal{F}(x) = \mathcal{S}_{\mathcal{F}}(\mathcal{N}^x_{\lim}, \mathcal{F}).$$

It will be called topological if lim satisfies moreover,

$$\text{(OLYTC)}\ \ \forall U \in 2^X,\ \forall x \in X,\ \mathcal{N}^x_{\lim}(U) = \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} \mathcal{N}^y_{\lim}(V).$$

The full subcategories of L-**OFYC** consisted of pretopological and topological L-ordered fuzzifying convergence spaces are denoted by L-**OFYPC** and L-**OFYTC**, respectively.

Definition 2.7. [5] A mapping $\mathscr{F} : L^X \to L$ is called a stratified L-filter on X if it satisfies
(LF1) $\mathscr{F}(\underline{\bot}) = \bot, \mathscr{F}(\underline{\top}) = \top$;
(LF2) $A \leqslant B \Rightarrow \mathscr{F}(A) \leqslant \mathscr{F}(B)$;
(LF3) $\mathscr{F}(A \wedge B) \geqslant \mathscr{F}(A) \wedge \mathscr{F}(B)$;
(LFs) $a \wedge \mathscr{F}(A) \leqslant \mathscr{F}(a \wedge A)$.

The family of all stratified L-filters on X will be denoted by $\mathscr{F}^s_L(X)$. For every $x \in X$, $[x] \in \mathscr{F}^s_L(X)$ is defined by $[x](A) = A(x)$ for all $A \in L^X$.

Definition 2.8. [8] A mapping

$$Lim : \mathscr{F}^s_L(X) \to L^X,\ \mathscr{F} \mapsto Lim\mathscr{F}$$

subjects to the conditions
(LGC1) $\forall x \in X,\ Lim[x](x) = \top$;
(LGC2) $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X),\ \mathscr{F} \leqslant \mathscr{G} \Rightarrow \forall x \in X, Lim\mathscr{F}(x) \leqslant Lim\mathscr{G}(x)$,
is called a stratified L-generalized convergence structure on X, and the pair (X, Lim) a stratified L-generalized convergence space.

A mapping $f : X \to Y$ between stratified L-generalized convergence spaces (X, Lim^X) and (Y, Lim^Y) is called continuous if for all $\mathscr{F} \in \mathscr{F}^s_L(X)$ and $x \in X$, $Lim^X \mathscr{F}(x) \leqslant Lim^Y f^{\Rightarrow}(\mathscr{F})(f(x))$. The category of stratified L-generalized convergence spaces and their continuous mappings is denoted by SL-**GCS**.

We can also define the fuzzy inclusion order $\mathcal{S}_{\mathscr{F}}$ on $\mathscr{F}^s_L(X)$, i.e., for any $\mathscr{F}, \mathscr{G} \in \mathscr{F}^s_L(X)$,

$$\mathcal{S}_{\mathscr{F}}(\mathscr{F}, \mathscr{G}) = \bigwedge_{A \in L^X} \Big(\mathscr{F}(A) \to \mathscr{G}(A)\Big).$$

Definition 2.9. [2] A stratified L-generalized convergence structure $Lim : \mathscr{F}^s_L(X) \to L^X$ satisfying the following condition

$$\text{(OLGC)}\ \ \forall \mathscr{F}, \mathcal{G} \in \mathscr{F}_L(X),\ \mathcal{S}_{\mathscr{F}}(\mathscr{F}, \mathscr{G}) \leqslant \mathcal{S}(Lim\mathscr{F}, Lim\mathscr{G}),$$

is called an L-ordered convergence structure, and the pair (X, Lim) a stratified L-ordered convergence space.

A mapping $f : X \to Y$ between stratified L-ordered convergence spaces (X, Lim^X) and (Y, Lim^Y) is called continuous if for all $\mathscr{F} \in \mathscr{F}^s_L(X),\ x \in X,\ \lim^X \mathscr{F}(x) \leqslant \lim^Y f^{\Rightarrow}(\mathscr{F})(f(x))$. The category of stratified L-ordered convergence spaces and continuous mappings is denoted by SL-**OGCS**. It is easy to check that the category SL-**OGCS** is a full subcategory of SL-**GCS**.

In order to avoid confusion, we repeat some notations for emphasis. The family of all L-filters of ordinary subsets on X is denoted by $\mathcal{F}_L(X)$ and elements in $\mathcal{F}_L(X)$ are denoted by $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. The fuzzy inclusion order on $\mathcal{F}_L(X)$ is denoted by $\mathcal{S}_{\mathcal{F}}$. Similarly, the family of all stratified L-filters on X is denoted by $\mathscr{F}_L^s(X)$ and elements in $\mathscr{F}_L^s(X)$ are denoted by $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$. The fuzzy inclusion order on $\mathscr{F}_L^s(X)$ is denoted by $\mathcal{S}_{\mathscr{F}}$.

Definition 2.10. [Adámek et al. [1]] (1) Let **B** be a category and E be a class of **B**-bimorphisms.

(1) A full subcategory **A** of **B** is called bireflective in **B** provided that each **B**-object has an **A**-reflection arrow in E as a bimorphism. This means that, for any **B**-object B, there exists an **A**-reflection bimorphism $r : B \to A$ from B to an **A**-object A with the following universal property: for any morphism $f : B \to A'$ from B into some **A**-object A', there exists a unique **A**-morphism $f' : A \to A'$ such that $f' \circ r = f$.

(2) A full subcategory **A** of **B** is called bicoreflective in **B** provided that each **B**-object has an **A**-coreflection arrow in E as a bimorphism. This means that, for any **B**-object B, there exists an **A**-reflection bimorphism $c : A \to B$ from A to an **B**-object B with the following universal property: for any morphism $f : A' \to B$ from A' into some **B**-object B, there exists a unique **A**-morphism $f' : A' \to A$ such that $c \circ f' = f$.

The class of objects of a category **A** is denoted by $|\mathbf{A}|$. For more notions related to category theory we refer to [1].

3. Relations Between Stratified L-filters and L-filters of Ordinary Subsets

In [4], the authors discussed the relations between stratified L-filters and L-filters of ordinary subsets (under the name "generalised filters") in the case that L is a GL-Monoid. In this section, we will further study these relations based on a frame L. We will give two ways of transformations between them and show that they are equivalent.

Lemma 3.1. *Let $\mathcal{F} \in \mathcal{F}_L(X)$ and define $\mathscr{F}^{\mathcal{F}} : L^X \to L$ as follows:*

$$\forall A \in L^X, \quad \mathscr{F}^{\mathcal{F}}(A) = \bigvee_{a \in L} a \wedge \mathcal{F}(A_{[a]}),$$

where $A_{[a]} = \{x \mid A(x) \geqslant a\}$. Then

(1) $\mathscr{F}^{\mathcal{F}} \in \mathscr{F}_L^s(X)$.
(2) $\mathscr{F}^{\dot{x}} = [x]$.
(3) $\mathscr{F}^{\mathcal{F} \wedge \mathcal{G}} = \mathscr{F}^{\mathcal{F}} \wedge \mathscr{F}^{\mathcal{G}}$.
(4) $f^{\Rightarrow}(\mathscr{F}^{\mathcal{F}}) = \mathscr{F}^{f^{\Rightarrow}(\mathcal{F})}$.

Proof. (1) Refer to Theorem 4.3 in [4].

(2) Take any $A \in L^X$. Then

$$\mathscr{F}^{\dot{x}}(A) = \bigvee_{a \in L} a \wedge \dot{x}(A_{[a]}) = \bigvee \{a \mid A(x) \geqslant a\} = A(x) = [x](A).$$

(3) From the definition of $\mathscr{F}^{\mathcal{F}}$, we have $\mathscr{F}^{\mathcal{F}} \leqslant \mathscr{F}^{\mathcal{G}}$ whenever $\mathcal{F} \leqslant \mathcal{G}$. Hence, $\mathscr{F}^{\mathcal{F} \wedge \mathcal{G}} \leqslant \mathscr{F}^{\mathcal{F}} \wedge \mathscr{F}^{\mathcal{G}}$. Further, we have

$$\begin{aligned}
\mathscr{F}^{\mathcal{F}} \wedge \mathscr{F}^{\mathcal{G}}(A) &= \left(\bigvee_{a \in L} a \wedge \mathcal{F}(A_{[a]})\right) \wedge \left(\bigvee_{b \in L} b \wedge \mathcal{G}(A_{[b]})\right) \\
&= \bigvee_{a \in L} \bigvee_{b \in L} a \wedge \mathcal{F}(A_{[a]}) \wedge b \wedge \mathcal{G}(A_{[b]}) \quad \text{(by (ID))} \\
&= \bigvee_{a \in L} \bigvee_{b \in L} a \wedge b \wedge \mathcal{F}(A_{[a]}) \wedge \mathcal{G}(A_{[b]}) \\
&\leqslant \bigvee_{a \in L} \bigvee_{b \in L} a \wedge b \wedge \mathcal{F}(A_{[a \wedge b]}) \wedge \mathcal{G}(A_{[a \wedge b]}) \\
&\leqslant \bigvee_{c \in L} c \wedge (\mathcal{F} \wedge \mathcal{G})(A_{[c]}) \\
&= \mathscr{F}^{\mathcal{F} \wedge \mathcal{G}}(A).
\end{aligned}$$

Therefore, $\mathscr{F}^{\mathcal{F} \wedge \mathcal{G}} = \mathscr{F}^{\mathcal{F}} \wedge \mathscr{F}^{\mathcal{G}}$.

(4) For each $A \in L^Y$, we have

$$\begin{aligned}
f^{\Rightarrow}(\mathscr{F}^{\mathcal{F}})(A) &= \mathscr{F}^{\mathcal{F}}(f^{\leftarrow}(A)) \\
&= \bigvee_{a \in L} a \wedge \mathcal{F}((f^{\leftarrow}(A))_{[a]}) \\
&= \bigvee_{a \in L} a \wedge \mathcal{F}(f^{-1}(A_{[a]})) \\
&= \bigvee_{a \in L} a \wedge f^{\Rightarrow}(\mathcal{F})(A_{[a]}) \\
&= \mathscr{F}^{f^{\Rightarrow}(\mathcal{F})}(A),
\end{aligned}$$

where the third equality holds by the following fact

$$\begin{aligned}
x \in (f^{\leftarrow}(A))_{[a]} &\Leftrightarrow f^{\leftarrow}(A)(x) \geqslant a \\
&\Leftrightarrow A(f(x)) \geqslant a \\
&\Leftrightarrow f(x) \in A_{[a]} \Leftrightarrow x \in f^{-1}(A_{[a]}).
\end{aligned}$$

□

Corollary 3.2. *Let $\mathcal{F},\ \mathcal{G} \in \mathcal{F}_L(X)$ and $\mathcal{F} \leqslant \mathcal{G}$. Then $\mathscr{F}^{\mathcal{F}} \leqslant \mathscr{F}^{\mathcal{G}}$.*

Lemma 3.3. *Let $\mathscr{F} \in \mathscr{F}_L^s(X)$ and define $\mathcal{F}^{\mathscr{F}} : 2^X \to L$ as follows:*

$$\forall U \in 2^X, \quad \mathcal{F}^{\mathscr{F}}(U) = \mathscr{F}(\chi_U),$$

where χ_U denotes the characteristic function of U. Then

(1) $\mathcal{F}^{\mathscr{F}} \in \mathcal{F}_L(X)$.
(2) $\mathcal{F}^{[x]} = \dot{x}$.
(3) $\mathcal{F}^{\mathscr{F}\wedge\mathscr{G}} = \mathcal{F}^{\mathscr{F}} \wedge \mathcal{F}^{\mathscr{G}}$.
(4) $f^{\Rightarrow}(\mathcal{F}^{\mathscr{F}}) = \mathcal{F}^{f^{\Rightarrow}(\mathscr{F})}$.

Proof. (1)Refer to Theorem 4.3 in [4].

(2) $\forall U \in 2^X,\ \ \mathcal{F}^{[x]}(U) = [x](\chi_U) = \chi_U(x) = \dot{x}(U)$.

(3) $\mathcal{F}^{\mathscr{F}\wedge\mathscr{G}}(U) = (\mathscr{F}\wedge\mathscr{G})(\chi_U) = \mathscr{F}(\chi_U)\wedge\mathscr{G}(\chi_U) = \mathcal{F}^{\mathscr{F}}(U)\wedge\mathcal{F}^{\mathscr{G}}(U)$.

(4) Take any $U \in 2^X$. Then

$$\begin{aligned} f^{\Rightarrow}(\mathcal{F}^{\mathscr{F}})(U) &= \mathcal{F}^{\mathscr{F}}(f^{-1}(U)) = \mathscr{F}(\chi_{f^{-1}(U)}) \\ &= \mathscr{F}(f^{\leftarrow}(\chi_U)) = f^{\Rightarrow}(\mathscr{F})(\chi_U) = \mathcal{F}^{f^{\Rightarrow}(\mathscr{F})}(U). \end{aligned}$$

□

Corollary 3.4. *Let $\mathscr{F}$, $\mathscr{G} \in \mathscr{F}_L^s(X)$ and $\mathscr{F} \leqslant \mathscr{G}$. Then $\mathcal{F}^{\mathscr{F}} \leqslant \mathcal{F}^{\mathscr{G}}$.*

Theorem 3.5. *If $\mathcal{F} \in \mathcal{F}_L(X)$ and $\mathscr{F} \in \mathscr{F}_L^s(X)$, then $\mathcal{F}^{\mathscr{F}^{\mathcal{F}}} = \mathcal{F}$ and $\mathscr{F}^{\mathcal{F}^{\mathscr{F}}} \leqslant \mathscr{F}$.*

Proof. We first show $\mathcal{F}^{\mathscr{F}^{\mathcal{F}}} = \mathcal{F}$ as follows:

$$\mathcal{F}^{\mathscr{F}^{\mathcal{F}}}(U) = \mathscr{F}^{\mathcal{F}}(\chi_U) = \bigvee_{a\in L} a \wedge \mathcal{F}((\chi_U)_{[a]}) = \bigvee_{a\in L\setminus\{\perp\}} a \wedge \mathcal{F}(U) = \mathcal{F}(U).$$

The inequality $\mathscr{F}^{\mathcal{F}^{\mathscr{F}}} \leqslant \mathscr{F}$ is shown by

$$\begin{aligned} \mathscr{F}^{\mathcal{F}^{\mathscr{F}}}(A) &= \bigvee_{a\in L} a \wedge \mathcal{F}^{\mathscr{F}}(A_{[a]}) \\ &= \bigvee_{a\in L} a \wedge \mathscr{F}(\chi_{A_{[a]}}) \\ &\leqslant \bigvee_{a\in L} \mathscr{F}(a \wedge \chi_{A_{[a]}}) \qquad \text{(by (LFs))} \\ &\leqslant \mathscr{F}(A), \end{aligned}$$

where the last inequality follows from $a \wedge \chi_{A_{[a]}} \leqslant A$ for all $a \in L$. □

About the transformation between stratified L-filters and L-filters of ordinary subsets, there are some other ways, which will be shown in the sequel.[1]

In general topology, there is a conclusion with respect to classical filters that for a classical filter $\mathbf{F}$ on X, $U \subseteq X$ is an element of $\mathbf{F}$ iff there exists $B \in \mathbf{F}$ such that $B \subseteq A$. We now apply this conclusion to the transformation between stratified L-filters and L-filters of ordinary subsets.

Theorem 3.6. *Let $\mathcal{F} \in \mathcal{F}_L(X)$ and define $\mathscr{F}_{\mathcal{F}} : L^X \to L$ as follows:*

$$\forall A \in L^X,\ \mathscr{F}_{\mathcal{F}}(A) = \bigvee_{U\subseteq X} \left(\mathcal{F}(U) \wedge \left(\bigwedge_{x\in U} A(x) \right) \right) = \bigvee_{U\subseteq X} \bigwedge_{x\in U} \mathcal{F}(U) \wedge A(x).$$

Then $\mathscr{F}_{\mathcal{F}} \in \mathscr{F}_L^s(X)$ and $\mathscr{F}_{\mathcal{F}} = \mathscr{F}^{\mathcal{F}}$.

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[1]The following conclusions are suggested by one of the anonymous reviewers.

Proof. We need only show that $\mathscr{F}_{\mathcal{F}}(A) = \mathscr{F}^{\mathcal{F}}(A)$ for all $A \in L^X$. On one hand, for each $a \in L$, take $U = A_{[a]}$. Then

$$\begin{aligned}\mathscr{F}_{\mathcal{F}}(A) &= \bigvee_{U\subseteq X}\left(\mathcal{F}(U) \wedge \left(\bigwedge_{x\in U} A(x)\right)\right)\\ &\geqslant \mathcal{F}(A_{[a]}) \wedge \bigwedge_{x\in A_{[a]}} A(x)\\ &= \mathcal{F}(A_{[a]}) \wedge \bigwedge_{A(x)\geqslant a} A(x)\\ &\geqslant a \wedge \mathcal{F}(A_{[a]}).\end{aligned}$$

From the arbitrariness of a, we obtain

$$\mathscr{F}_{\mathcal{F}}(A) \geqslant \bigvee_{a\in L} a \wedge \mathcal{F}(A_{[a]}) = \mathscr{F}^{\mathcal{F}}(A).$$

On the other hand, for each $U \subseteq X$, let $b = \bigwedge_{x\in U} A(x)$. Then $A(x) \geqslant b$ for all $x \in U$, i.e., $U \subseteq A_{[b]}$. This implies that

$$b \wedge \mathcal{F}(U) \leqslant b \wedge \mathcal{F}(A_{[b]}) \leqslant \bigvee_{a\in L} a \wedge \mathcal{F}(A_{[a]}).$$

From the arbitrariness of U, we have

$$\mathscr{F}_{\mathcal{F}}(A) = \bigvee_{U\subseteq X} b \wedge \mathcal{F}(U) \leqslant \bigvee_{a\in L} a \wedge \mathcal{F}(A_{[a]}) = \mathscr{F}^{\mathcal{F}}(A).$$

Therefore, $\mathscr{F}_{\mathcal{F}}(A) = \mathscr{F}^{\mathcal{F}}(A)$, as desired. □

Remark 3.7. In fact, the definition $\mathscr{F}_{\mathcal{F}}$ is a lattice-valued extension of the transformation of a stratified L-filter from a classical filter $\mathbf{F}$ in [7] that

$$\mathscr{F}_{\mathbf{F}}(A) = \bigvee_{U\in\mathbf{F}} \bigwedge_{x\in U} A(x).$$

Theorem 3.8. *Let $\mathscr{F} \in \mathscr{F}_L^s(X)$ and define $\mathcal{F}_{\mathscr{F}} : 2^X \to L$ as follows:*

$$\forall U \in 2^X,\ \mathcal{F}_{\mathscr{F}}(U) = \bigvee_{A\in L^X} \mathscr{F}(A) \wedge \mathcal{S}(A, \chi_U) = \bigvee_{A\in L^X} \bigwedge_{x\notin U} \mathscr{F}(A) \wedge (A(x) \to \bot),$$

where χ_U denotes the characterization function of U. Then $\mathcal{F}_{\mathscr{F}} \in \mathcal{F}_L(X)$ and $\mathcal{F}_{\mathscr{F}} = \mathcal{F}^{\mathscr{F}}$.

Proof. It suffices to show that $\mathcal{F}_{\mathscr{F}}(U) = \mathcal{F}^{\mathscr{F}}(U)$ for all $U \in 2^X$. This can be obtained by the following fact.

$$\begin{aligned}\mathcal{F}_{\mathscr{F}}(U) &= \bigvee_{A\in L^X} \mathscr{F}(A) \wedge \mathcal{S}(A, \chi_U)\\ &\leqslant \bigvee_{A\in L^X} \mathscr{F}(\mathcal{S}(A, \chi_U) \wedge A) \quad \text{(by (LFs))}\\ &\leqslant \bigvee_{A\in L^X} \mathscr{F}(\chi_U)\\ &= \mathscr{F}(\chi_U)\\ &= \mathscr{F}(\chi_U) \wedge \mathcal{S}(\chi_U, \chi_U)\\ &\leqslant \bigvee_{A\in L^X} \mathscr{F}(A) \wedge \mathcal{S}(A, \chi_U)\\ &= \mathcal{F}_{\mathscr{F}}(U),\end{aligned}$$

where the second inequality holds since $\mathcal{S}(A,\chi_U)\wedge A\leqslant \chi_U$. Hence, $\mathcal{F}_{\mathscr{F}}(U)=\mathscr{F}(\chi_U)=\mathcal{F}^{\mathscr{F}}(U)$, as desired. □

4. Relations Among L-OFYC, L-OFYPC and L-OFYTC

In this section, the relations among L-ordered fuzzifying convergence spaces, pretopological L-ordered fuzzifying convergence spaces and topological L-ordered fuzzifying convergence spaces are discussed in the categorical sense.

Lemma 4.1. *Let* $(X,\lim_X)$, $(Y,\lim_Y)\in|L$-**OFYPC**$|$. *Then* $f:(X,\lim_X)\to(Y,\lim_Y)$ *is continuous iff* $\mathcal{N}^x_{\lim_X}(f^{-1}(U))\geqslant\mathcal{N}^{f(x)}_{\lim_Y}(U)$ *for each* $x\in X$ *and* $U\in 2^Y$.

Proof. Necessity. Since $f:(X,\lim_X)\to(Y,\lim_Y)$ is continuous, we have

$$\forall x\in X,\ \forall\mathcal{F}\in\mathcal{F}_L(X),\ \lim_X\mathcal{F}(x)\leqslant\lim_Y f^{\Rightarrow}(\mathcal{F})(f(x)).$$

Then for each $U\in 2^Y$,

$$\begin{aligned}\mathcal{N}^x_{\lim_X}(f^{-1}(U)) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)}\Big(\lim_X\mathcal{F}(x)\to\mathcal{F}(f^{-1}(U))\Big)\\ &\geqslant \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)}\Big(\lim_Y f^{\Rightarrow}(\mathcal{F})(f(x))\to f^{\Rightarrow}(\mathcal{F})(U)\Big)\\ &\geqslant \bigwedge_{\mathcal{G}\in\mathcal{F}_L(Y)}\Big(\lim_Y\mathcal{G}(f(x))\to\mathcal{G}(U)\Big)\\ &= \mathcal{N}^{f(x)}_{\lim_Y}(U).\end{aligned}$$

Sufficiency. Let $\mathcal{F}\in\mathcal{F}_L(X)$ and $x\in X$. By (OLYPC) we find

$$\begin{aligned}\lim_Y f^{\Rightarrow}(\mathcal{F})(f(x)) &= \bigwedge_{U\in 2^Y}\Big(\mathcal{N}^{f(x)}_{\lim_Y}(U)\to f^{\Rightarrow}(\mathcal{F})(V)\Big)\\ &\geqslant \bigwedge_{U\in 2^Y}\Big(\mathcal{N}^x_{\lim_X}(f^{-1}(U))\to\mathcal{F}(f^{-1}(U))\Big)\\ &\geqslant \bigwedge_{V\in 2^X}\Big(\mathcal{N}^x_{\lim_X}(V)\to\mathcal{F}(V)\Big)\\ &= \lim_X\mathcal{F}(x).\end{aligned}$$

This completes the proof. □

Theorem 4.2. L-**OFYPC** *is a bireflective subcategory of* L-**OFYC**.

Proof. For an L-ordered fuzzifying convergence space $(X,\lim)$, define $\lim^*:\mathcal{F}_L(X)\to L^X$ as follows:

$$\lim{}^*\mathcal{F}(x)=\bigwedge_{U\in 2^X}\Big(\mathcal{N}^x_{\lim}(U)\to\mathcal{F}(U)\Big).$$

Then we claim that $id_X:(X,\lim)\to(X,\lim^*)$ is the L-**OFYPC**-bireflector.

For this it suffices to prove:

(1) $(X,\lim^*)$ is a pretopological L-ordered fuzzifying convergence space.
(2) $id_X:(X,\lim)\to(X,\lim^*)$ is continuous.

(3) For each pretopological L-ordered fuzzifying convergence space $(Y, \lim_Y)$ and each mapping $f : X \to Y$, the continuity of $f : (X, \lim) \to (Y, \lim_Y)$ implies the continuity of $f : (X, \lim^*) \to (Y, \lim_Y)$.

(1) (LYC1) follows immediately from $\mathcal{N}^x_{\lim} \leqslant \dot{x}$ and (OLYC) is obvious. For the proof of (OLYPC) we first show that $\mathcal{N}^x_{\lim} = \mathcal{N}^x_{\lim^*}$. Let $U \in 2^X$. Then

$$\begin{aligned}\mathcal{N}^x_{\lim^*}(U) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \Big(\lim{}^*\mathcal{F}(x) \to \mathcal{F}(U)\Big)\\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \left(\left(\bigwedge_{V\in 2^X} (\mathcal{N}^x_{\lim}(V) \to \mathcal{F}(V))\right) \to \mathcal{F}(U)\right)\\ &\geqslant \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \Big((\mathcal{N}^x_{\lim}(U) \to \mathcal{F}(U)) \to \mathcal{F}(U)\Big)\\ &\geqslant \mathcal{N}^x_{\lim}(U).\end{aligned}$$

On the other hand, we conclude from $\lim^*\mathcal{N}^x_{\lim}(x) = \top$ that

$$\mathcal{N}^x_{\lim^*}(U) \leqslant \lim{}^*\mathcal{N}^x_{\lim}(x) \to \mathcal{N}^x_{\lim}(U) = \top \to \mathcal{N}^x_{\lim}(U) = \mathcal{N}^x_{\lim}(U).$$

Hence $\mathcal{N}^x_{\lim^*} = \mathcal{N}^x_{\lim}$. Therefore

$$\lim{}^*\mathcal{F}(x) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}^x_{\lim}(U) \to \mathcal{F}(U)\Big) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}^x_{\lim^*}(U) \to \mathcal{F}(U)\Big).$$

This proves (OLYPC).

(2) From (1) we know $\mathcal{N}^x_{\lim} = \mathcal{N}^x_{\lim^*}$. By Lemma 4.1, we obtain $id_X : (X, \lim) \to (X, \lim^*)$ is continuous.

(3) Since $f : (X, \lim) \to (Y, \lim_Y)$ is continuous, by Lemma 4.1 and (1), we have

$$\mathcal{N}^x_{\lim^*}(f^{-1}(U)) = \mathcal{N}^x_{\lim}(f^{-1}(U)) \geqslant \mathcal{N}^{f(x)}_{\lim_Y}(U).$$

The continuity of $f : (X, \lim^*) \to (Y, \lim_Y)$ is proved. □

Corollary 4.3. *The category* L-**OFYPC** *is topological over* **Set**.

Lemma 4.4. *Let* $(X, \lim)$ *be a pretopological* L*-ordered fuzzifying convergence space. Define* $\mathcal{N}^\star_x : 2^X \to L$ *by*

$$\forall U \in 2^X,\ \mathcal{N}^\star_x(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}^y_{\lim}(V).$$

Then $\mathcal{N}^\star_x(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}^\star_y(V)$.

Proof. $\mathcal{N}^\star_x(U) \geqslant \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}^\star_y(V)$ is obvious. On the other hand, by the definition of $\mathcal{N}^\star_x$, we have

$$\begin{aligned}\bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}^\star_y(V) &= \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \bigvee_{y\in W\subseteq V} \bigwedge_{z\in W} \mathcal{N}^z_{\lim}(W)\\ &\geqslant \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \bigwedge_{z\in V} \mathcal{N}^z_{\lim}(V)\\ &= \bigvee_{x\in V\subseteq U} \bigwedge_{z\in V} \mathcal{N}^z_{\lim}(V)\\ &= \mathcal{N}^\star_x(U).\end{aligned}$$

Therefore $\mathcal{N}_x^\star(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}_y^\star(V)$. □

Lemma 4.5. *Let* $(X, \lim)$ *be a pretopological* L*-ordered fuzzifying convergence space. Then* $\lim^\star : \mathcal{F}_L(X) \to L^X$ *defined by*

$$\forall \mathcal{F} \in \mathcal{F}_L(X),\ \forall x \in X,\ \lim^\star \mathcal{F}(x) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}_x^\star(U) \to \mathcal{F}(U)\Big)$$

is a topological L*-ordered fuzzifying convergence structure on* X.

Proof. (LYC1) We get from $\mathcal{N}_x^\star(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}_{\lim}^y(V) \leqslant \mathcal{N}_{\lim}^x(U) = \dot{x}(U)$ that

$$\lim^\star \dot{x}(x) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}_x^\star(U) \to \dot{x}(U)\Big) = \top.$$

(OLYC) Obvious.

For (OLYPC) and (OLYTC), we first check $\mathcal{N}_{\lim^\star}^x = \mathcal{N}_x^\star$. On one hand,

$$\begin{aligned}
\mathcal{N}_{\lim^\star}^x(U) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \Big(\lim^\star \mathcal{F}(x) \to \mathcal{F}(U)\Big) \\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \left(\bigwedge_{V\in 2^X} \Big(\mathcal{N}_x^\star(V) \to \mathcal{F}(V)\Big) \to \mathcal{F}(U)\right) \\
&\geqslant \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \Big((\mathcal{N}_x^\star(U) \to \mathcal{F}(U)) \to \mathcal{F}(U)\Big) \\
&\geqslant \mathcal{N}_x^\star(U).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{N}_{\lim^\star}^x(U) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L(X)} \Big(\lim^\star \mathcal{F}(x) \to \mathcal{F}(U)\Big) \\
&\leqslant \lim^\star \mathcal{N}_x^\star(x) \to \mathcal{N}_x^\star(U) = \top \to \mathcal{N}_x^\star(U) = \mathcal{N}_x^\star(U).
\end{aligned}$$

This shows $\mathcal{N}_{\lim^\star}^x = \mathcal{N}_x^\star$. Therefore, (OLYPC) follows from the definition of $\lim^\star$ and (OLYTC) follows from Lemma 4.4. □

Lemma 4.6. *If* $(X, \lim)$ *is a topological* L*-ordered fuzzifying convergence space, then* $\lim^\star = \lim$.

Proof. Since $(X, \lim)$ is a topological L-ordered fuzzifying convergence space, it follows that

$$\forall U \in 2^X,\ \mathcal{N}_x^\star(U) = \bigvee_{x\in V\subseteq U} \bigwedge_{y\in V} \mathcal{N}_{\lim}^y(V) = \mathcal{N}_{\lim}^x(U).$$

Then

$$\lim^\star \mathcal{F}(x) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}_x^\star(U) \to \mathcal{F}(U)\Big) = \bigwedge_{U\in 2^X} \Big(\mathcal{N}_{\lim}^x(U) \to \mathcal{F}(U)\Big) = \lim \mathcal{F}(x).$$

□

Theorem 4.7. *L-**OFYTC** is a bireflective subcategory of L-**OFYPC**.*

Proof. For a pretopological L-ordered fuzzifying convergence space $(X, \lim)$, we claim that $id_X : (X, \lim) \to (X, \lim^\star)$ is the L-**OFYTC**-bireflector.

For this it suffices to prove:

(1) $(X, \lim^\star)$ is a topological L-ordered fuzzifying convergence space.
(2) $id_X : (X, \lim) \to (X, \lim^\star)$ is continuous.
(3) For each topological L-ordered fuzzifying convergence space $(Y, \lim_Y)$ and each mapping $f : X \to Y$, the continuity of $f : (X, \lim) \to (Y, \lim_Y)$ implies the continuity of $f : (X, \lim^\star) \to (Y, \lim_Y)$.

By Lemmas 4.1, 4.5 and 4.6, (1)–(3) are trivial and omitted. □

Corollary 4.8. *The category L-**OFYTC** is topological over **Set**.*

5. Relations Between SL-OGCS (SL-GCS) and L-OFYC (L-FYC)

The aim of this section is to investigate the relations between L-ordered fuzzifying convergence structures and stratified L-ordered convergence structures.

Theorem 5.1. *Let* $\lim$ *be an L-ordered fuzzifying convergence structure on X and define $Lim^{\lim} : \mathscr{F}_L^s(X) \to L^X$ as follows:*

$$\forall \mathscr{F} \in \mathscr{F}_L^s(X),\ x \in X,\ Lim^{\lim}\mathscr{F}(x) = \lim \mathcal{F}^{\mathscr{F}}(x).$$

Then $Lim^{\lim}$ is a stratified L-ordered convergence structure on X.

Proof. (LGC1) By Lemma 3.3 (2), we have

$$Lim^{\lim}[x](x) = \lim \mathcal{F}^{[x]}(x) = \lim \dot{x}(x) = \top.$$

(OLGC) Take any $\mathscr{F}$, $\mathscr{G} \in \mathscr{F}_L^s(X)$. Then

$$\begin{aligned} \mathcal{S}(Lim^{\lim}\mathscr{F}, Lim^{\lim}\mathscr{G}) &= \mathcal{S}(\lim \mathcal{F}^{\mathscr{F}}, \lim \mathcal{F}^{\mathscr{G}}) \\ &\geqslant \mathcal{S}_{\mathcal{F}}(\mathcal{F}^{\mathscr{F}}, \mathcal{F}^{\mathscr{G}}) \qquad \text{(by (OLYC))} \\ &= \bigwedge_{U \in 2^X} (\mathcal{F}^{\mathscr{F}}(U) \to \mathcal{F}^{\mathscr{G}}(U)) \\ &= \bigwedge_{U \in 2^X} (\mathscr{F}(\chi_U) \to \mathscr{G}(\chi_U)) \\ &\geqslant \bigwedge_{A \in L^X} (\mathscr{F}(A) \to \mathscr{G}(A)) \\ &= \mathcal{S}_{\mathscr{F}}(\mathscr{F}, \mathscr{G}). \end{aligned}$$

□

Corollary 5.2. *If* $\lim$ *is an L-fuzzifying convergence structure on X, then $Lim^{\lim}$ is a stratified L-generalized convergence structure on X.*

Theorem 5.3. *If $f : (X, \lim_1) \to (Y, \lim_2)$ is continuous with respect to L-fuzzifying (L-ordered fuzzifying) convergence structures $\lim_1$ and $\lim_2$, then $f : (X, Lim^{\lim_1}) \to (Y, Lim^{\lim_2})$ is continuous with respect to stratified L-generalized (L-ordered) convergence structures $Lim^{\lim_1}$ and $Lim^{\lim_2}$.*

Proof. Since $f:(X,\lim_1)\to(Y,\lim_2)$ is continuous, we have

$$\forall\mathcal{F}\in\mathcal{F}_L(X),\ \forall x\in X,\ \lim_1\mathcal{F}(x)\leqslant\lim_2 f^{\Rightarrow}(\mathcal{F})(f(x)).$$

Then for each $\mathscr{F}\in\mathscr{F}_L^s(X)$ and $x\in X$, we have

$$\begin{aligned} Lim^{\lim_1}\mathscr{F}(x) &= \lim_1\mathcal{F}^{\mathscr{F}}(x)\\ &\leqslant \lim_2 f^{\Rightarrow}(\mathcal{F}^{\mathscr{F}})(f(x))\\ &= \lim_2\mathcal{F}^{f^{\Rightarrow}(\mathscr{F})}(f(x)) \quad \text{(by Lemma 4.3 (4))}\\ &= Lim^{\lim_2} f^{\Rightarrow}(\mathscr{F})(f(x)). \end{aligned}$$

Therefore, $f:(X,Lim^{\lim_1})\to(Y,Lim^{\lim_2})$ is continuous. □

By Theorems 5.1 and 5.3, we obtain a concrete functor $\mathbb{F}$: L-**OFYC**$\to$$L$-**OGCS** by

$$\mathbb{F}:(X,\lim)\mapsto(X,Lim^{\lim}) \text{ and } f\mapsto f.$$

Theorem 5.4. *Let Lim be a stratified L-ordered convergence structure on X and define $\lim^{Lim}:\mathcal{F}_L(X)\to L^X$ as follows:*

$$\forall\mathcal{F}\in\mathcal{F}_L(X),\ \forall x\in X,\ \lim^{Lim}\mathcal{F}(x)=Lim\mathscr{F}^{\mathcal{F}}(x).$$

Then $\lim^{Lim}$ is an L-ordered fuzzifying convergence structure on X.

Proof. (LYC1) By Lemma 3.1 (2), we have

$$\lim^{Lim}\dot{x}(x)=Lim\mathscr{F}^{\dot{x}}(x)=Lim[x](x)=\top.$$

(OLYC) Take any $\mathcal{F},\ \mathcal{G}\in\mathcal{F}_L(X)$. Then

$$\begin{aligned} \mathcal{S}(\lim^{Lim}\mathcal{F},\lim^{Lim}\mathcal{G}) &= \mathcal{S}(Lim\mathscr{F}^{\mathcal{F}},Lim\mathscr{F}^{\mathcal{G}})\\ &\geqslant \mathcal{S}_{\mathscr{F}}(\mathscr{F}^{\mathcal{F}},\mathscr{F}^{\mathcal{G}}) \qquad \text{(by (OLGC))}\\ &= \bigwedge_{A\in L^X}(\mathscr{F}^{\mathcal{F}}(A)\to\mathscr{F}^{\mathcal{G}}(A))\\ &= \bigwedge_{A\in L^X}\left(\bigvee_{a\in L}a\wedge\mathcal{F}(A_{[a]})\to\bigvee_{b\in L}b\wedge\mathcal{G}(A_{[b]})\right)\\ &\geqslant \bigwedge_{A\in L^X}\bigwedge_{a\in L}\big(a\wedge\mathcal{F}(A_{[a]})\to a\wedge\mathcal{G}(A_{[a]})\big)\\ &\geqslant \bigwedge_{A\in L^X}\bigwedge_{a\in L}\big(\mathcal{F}(A_{[a]})\to\mathcal{G}(A_{[a]})\big) \quad \text{(by Lemma 2.1 (H4))}\\ &\geqslant \bigwedge_{U\in 2^X}(\mathcal{F}(U)\to\mathcal{G}(U))\\ &= \mathcal{S}_{\mathcal{F}}(\mathcal{F},\mathcal{G}). \end{aligned}$$

□

Corollary 5.5. *If Lim is a stratified L-generalized convergence structure on X, then $\lim^{Lim}$ is an L-fuzzifying convergence structure on X.*

Theorem 5.6. *If $f:(X,Lim_1)\to(Y,Lim_2)$ is continuous with respect to stratified L-generalized (L-ordered) convergence structures Lim_1 and Lim_2, then $f:(X,\lim^{Lim_1})\to(Y,\lim^{Lim_2})$ is continuous with respect to L-fuzzifying (L-ordered fuzzifying) convergence structures $\lim^{Lim_1}$ and $\lim^{Lim_2}$.*

Proof. Since $f:(X,Lim_1)\to(Y,Lim_2)$ is continuous, we have

$$\forall\mathscr{F}\in\mathscr{F}_L^s(X),\ \forall x\in X,\ Lim_1\mathscr{F}(x)\leqslant Lim_2 f^{\Rightarrow}(\mathscr{F})(f(x)).$$

Then for each $\mathcal{F}\in\mathcal{F}_L(X)$ and $x\in X$, we have

$$\begin{array}{rcl}\lim^{Lim_1}\mathcal{F}(x) & = & Lim_1\mathscr{F}^{\mathcal{F}}(x)\\ & \leqslant & Lim_2 f^{\Rightarrow}(\mathscr{F}^{\mathcal{F}})(f(x))\\ & = & Lim_2\mathscr{F}^{f^{\Rightarrow}(\mathcal{F})}(f(x)) \quad \text{(by Lemma 4.1 (4))}\\ & = & \lim^{Lim_2} f^{\Rightarrow}(\mathcal{F})(f(x)).\end{array}$$

Therefore, $f:(X,\lim^{Lim_1})\to(Y,\lim^{Lim_2})$ is continuous. □

By Theorems 5.4 and 5.6, we obtain a concrete functor $\mathbb{G}$: SL-**OGCS**→L-**OFYC** by

$\mathbb{G}:(X,Lim)\mapsto(X,\lim^{Lim})$ and $f\mapsto f$.

Theorem 5.7. *Let* lim *be an L-fuzzifying (L-ordered fuzzifying) convergence structure on X and Lim a stratified L-generalized (L-ordered) convergence structure on X. Then $\lim^{Lim^{\lim}}=\lim$ and $Lim^{\lim^{Lim}}\leqslant Lim$.*

Proof. By Theorem 3.5 , we have

$$\lim^{Lim^{\lim}}\mathcal{F}(x)=Lim^{\lim}\mathscr{F}^{\mathcal{F}}(x)=\lim\mathcal{F}^{\mathscr{F}^{\mathcal{F}}}(x)=\lim\mathcal{F}(x)$$

and

$$Lim^{\lim^{Lim}}\mathscr{F}(x)=\lim^{Lim}\mathcal{F}^{\mathscr{F}}(x)=Lim\mathscr{F}^{\mathcal{F}^{\mathscr{F}}}(x)\leqslant Lim\mathscr{F}(x).$$

□

Theorem 5.8. *The category L-***OFYC** *can be embedded in the category SL-***OGCS** *as a coreflective full subcategory.*

Proof. For a stratified L-ordered convergence space (X,Lim), we claim that the L-**OFYC**-coreflector is given by

$$id_X:\mathbb{F}\circ\mathbb{G}(X,Lim)=(X,Lim^{\lim^{Lim}})\to(X,Lim).$$

In fact:

Step 1: $id_X:\mathbb{F}\circ\mathbb{G}(X,Lim)=(X,Lim^{\lim^{Lim}})\to(X,Lim)$ is continuous. By Theorem 5.7, it is obvious.

Step 2: The continuity of $f:\mathbb{F}(Y,\lim_Y)=(Y,Lim^{\lim_Y})\to(X,Lim)$ implies the continuity of $f:(Y,\lim_Y)\to\mathbb{G}(X,Lim)=(X,\lim^{Lim})$. Since $f:\mathbb{F}(Y,\lim_Y)\to(X,Lim)$ is continuous, we have

$$\forall\mathscr{F}\in\mathscr{F}_L^s(X),\ \forall y\in Y,\ Lim^{\lim_Y}\mathscr{F}(y)\leqslant Lim f^{\Rightarrow}(\mathscr{F})(f(y)).$$

Then for each $\mathcal{F} \in \mathcal{F}_L(X)$, it follows that

$$\begin{aligned} \lim_Y \mathcal{F}(y) &= \lim^{Lim^{\lim_Y}} \mathcal{F}(y) \quad \text{(by Theorem 5.7)} \\ &= Lim^{\lim_Y} \mathscr{F}^{\mathcal{F}}(y) \\ &\leqslant Lim f^{\Rightarrow}(\mathscr{F}^{\mathcal{F}})(f(y)) \\ &= Lim \mathscr{F}^{f^{\Rightarrow}(\mathcal{F})}(f(y)) \\ &= \lim^{Lim}(f^{\Rightarrow}(\mathcal{F}))(f(y)). \end{aligned}$$

Thus, $f : (Y, \lim_Y) \to \mathbb{G}(X, Lim)$ is continuous. Therefore, the conclusion holds. □

Corollary 5.9. *The category L-**FYC** can be embedded in the category SL-**GCS** as a coreflective full subcategory.*

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