

## ON THE RELATIONSHIPS BETWEEN TYPES OF $L$ -CONVERGENCE SPACES

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**ABSTRACT.** This paper focuses on the relationships between stratified  $L$ -convergence spaces, stratified strong  $L$ -convergence spaces and stratified levelwise  $L$ -convergence spaces. It has been known that: (1) a stratified  $L$ -convergence space is precisely a left-continuous stratified levelwise  $L$ -convergence space; and (2) a stratified strong  $L$ -convergence space is naturally a stratified  $L$ -convergence space, but the converse is not true generally. In this paper, a strong left-continuity condition for stratified levelwise  $L$ -convergence space is given. It is proved that a stratified strong  $L$ -convergence space is precisely a strongly left-continuous stratified levelwise  $L$ -convergence space. Then a sufficient and necessary condition for a stratified  $L$ -convergence space to be a stratified strong  $L$ -convergence space is presented.

### 1. Introduction

Stratified  $L$ -convergence spaces were first defined in [9] and then developed in a series of papers [6, 10, 11] for the case that the lattice  $L$  is a complete Heyting algebra (or a frame). Later, the theory of these spaces was generalized to the lattice context of complete residuated lattices [25]. In [22], the lattice situation was further extended to enriched cl-premonoids. Recently, under different lattice contexts, a type of  $L$ -convergence spaces is proposed from the viewpoint of fuzzy orders [2, 3, 12, 15, 16, 17, 18, 19, 20, 21]. It is proved that these spaces are slightly stronger than stratified  $L$ -convergence spaces, and thus are generally called stratified strong  $L$ -convergence spaces. In [2, 16], it is proved that a stratified  $L$ -convergence space need not to be strong. This leads to a natural question: Does there exist a sufficient and necessary condition that guarantees that a stratified  $L$ -convergence space is a stratified strong  $L$ -convergence space?

In the case of  $L$  being a frame, Flores and his co-workers [6] introduced the concept of stratified levelwise  $L$ -convergence space which can be regarded as an extension of the concept of probabilistic convergence space in [24]. They also proposed a so-called left-continuity condition and proved that the left-continuous stratified levelwise  $L$ -convergence spaces are precisely the stratified  $L$ -convergence spaces. This leads to another question: Can stratified strong  $L$ -convergence spaces be characterized by levelwise lattice-valued convergence spaces via some left-continuity condition?

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This article will focus on the above two questions. The contents are arranged as follows. Section 2 recalls some basic notions and notations used in this paper. Section 3 presents the levelwise characterization for the stratified strong  $L$ -convergence spaces via some left-continuity condition. Section 4 presents a sufficient and necessary condition for stratified  $L$ -convergence spaces to be stratified strong  $L$ -convergence spaces. Section 5 makes a conclusion.

## 2. Preliminaries

In this paper, if not otherwise specified,  $(L, *, 1)$  is always a complete residuated lattice. That is,  $L$  is a complete lattice with a top element 1 and a bottom element 0;  $*$  is a binary operation on  $L$  such that (i)  $(L, *, 1)$  is a commutative monoid; and (ii)  $*$  distributes over arbitrary joins. Since the binary operation  $*$  distributes over arbitrary joins, the mapping  $\alpha * (-) : L \rightarrow L$  has a right adjoint  $\alpha \rightarrow (-) : L \rightarrow L$  given by

$$\alpha \rightarrow \beta = \bigvee \{ \gamma \in L : \alpha * \gamma \leq \beta \}.$$

The binary operation  $\rightarrow$  is called the residuation with respect to  $*$ . We collect some basic properties of the binary operations  $*$  and  $\rightarrow$ .

**Proposition 2.1.** [1, 7] *Let  $(L, *, 1)$  be a complete residuated lattice. Then*

- (I1)  $0 * \alpha = 0$  and  $1 \rightarrow \alpha = \alpha$ ;
- (I2)  $\alpha \rightarrow \beta = 1 \Leftrightarrow \alpha \leq \beta$ ;
- (I3)  $\alpha * (\alpha \rightarrow \beta) \leq \beta$  and  $(\alpha \rightarrow \beta) * (\beta \rightarrow r) \leq \alpha \rightarrow r$ ;
- (I4)  $\alpha \rightarrow (\beta \rightarrow r) = (\alpha * \beta) \rightarrow r = \beta \rightarrow (\alpha \rightarrow r)$ ;
- (I5)  $\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$ ;
- (I6)  $(\bigvee_{j \in J} \alpha_j) \rightarrow \beta = \bigwedge_{j \in J} (\alpha_j \rightarrow \beta)$ ;
- (I7)  $\alpha \rightarrow (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (\alpha \rightarrow \beta_j)$ ;
- (I8)  $\alpha \leq \beta \Rightarrow \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$  and  $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$ .

For a set  $X$ , the set  $L^X$  of functions from  $X$  to  $L$  with the pointwise order becomes a complete lattice. Each element of  $L^X$  is called an  $L$ -subset (or a fuzzy subset) of  $X$ . For any  $\lambda \in L^X$ ,  $\mathcal{K} \subseteq L^X$  and  $\alpha \in L$ , we define the  $L$ -subsets  $\alpha * \lambda$ ,  $\alpha \rightarrow \lambda$ ,  $\bigvee \mathcal{K}$  and  $\bigwedge \mathcal{K}$  by  $(\alpha * \lambda)(x) = \alpha * \lambda(x)$ ,  $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$ ,  $(\bigvee \mathcal{K})(x) = \bigvee_{\mu \in \mathcal{K}} \mu(x)$  and  $(\bigwedge \mathcal{K})(x) = \bigwedge_{\mu \in \mathcal{K}} \mu(x)$ . Let  $\phi$  be a function, we define  $\phi_L^{\rightarrow} : L^X \rightarrow L^Y$  and  $\phi_L^{\leftarrow} : L^Y \rightarrow L^X$  [8] by  $\phi_L^{\rightarrow}(A)(y) = \bigvee_{\phi(x)=y} A(x)$  for  $A \in L^X$  and  $y \in Y$ , and  $\phi_L^{\leftarrow}(B) = B \circ \phi$  for  $B \in L^Y$ .

Let  $X$  be a set. A fuzzy partial order (or, an  $L$ -partial order) ([1, 26, 27, 28, 29]) on  $X$  is a function  $R : X \times X \rightarrow L$  such that

- (1)  $R(a, a) = 1$  for every  $a \in X$  (reflexivity);
- (2)  $R(a, b) = R(b, a) = 1$  implies that  $a = b$  for all  $a, b \in X$  (antisymmetry);
- (3)  $R(a, b) * R(b, c) \leq R(a, c)$  for all  $a, b, c \in X$  (transitivity).

The pair  $(X, R)$  is called an  $L$ -partially ordered set. An  $L$ -order-preserving function  $f : (X, R) \rightarrow (Y, S)$  between  $L$ -partially ordered sets is a function  $f : X \rightarrow Y$  such that  $R(a, b) \leq S(f(a), f(b))$  for all  $a, b \in X$ .

Let  $R : L^X \times L^X \longrightarrow L$  be a function defined by

$$R(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \rightarrow \mu(x)),$$

then  $R$  is an  $L$ -partial order on  $L^X$ . The value  $R(\lambda, \mu) \in L$  is the degree that  $\lambda$  is contained in  $\mu$ , so the fuzzy partially order  $R$  is called *fuzzy inclusion order* ([1]). The  $L$ -partially ordered set  $(L^X, R)$  is called the fuzzy powerset of  $X$ , denoted by  $[L^X]$  for short.

**Definition 2.2.** (Höhle and Šostak [8]) A stratified  $L$ -filter on a set  $X$  is a function  $\mathcal{F} : L^X \longrightarrow L$  such that for each  $\lambda, \mu \in L^X$  and each  $\alpha \in L$ ,

- (F1)  $\mathcal{F}(0) = 0, \mathcal{F}(1) = 1$ ; (F2)  $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$ ;
- (Fs)  $\mathcal{F}(\alpha * \lambda) \geq \alpha * \mathcal{F}(\lambda)$ .

The following examples of  $L$ -filters belong to the folklore, we list them here because the notations are needed.

**Example 2.3.** (1) For each point  $x$  in a set  $X$ , it is obvious that  $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$  is a stratified  $L$ -filter on  $X$ , called *the principal  $L$ -filter* generated by  $x$ .

(2) Let  $J$  be a non-empty set and  $\{\mathcal{F}_j | j \in J\}$  be a family of stratified  $L$ -filters on  $X$ , then  $\bigwedge_{j \in J} \mathcal{F}_j$  is also a stratified  $L$ -filter on  $X$ . Let  $\mathcal{F}_0$  denote the meet of all stratified  $L$ -filters on  $X$ , i.e., the smallest stratified  $L$ -filter on  $X$ .

(3) Let  $\phi : X \longrightarrow Y$  be a function and  $\mathcal{F}$  be a stratified  $L$ -filter on  $X$ . Then the function  $\phi^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$  defined by  $\lambda \mapsto \mathcal{F}(\lambda \circ \phi)$  is a stratified  $L$ -filter on  $Y$ , called the *image* of  $\mathcal{F}$  under  $\phi$ .

The set of stratified  $L$ -filters on a set  $X$  is denoted by  $\mathcal{F}_L^s(X)$ . There is a natural fuzzy partial order on  $\mathcal{F}_L^s(X)$  inherited from  $L^{(L^X)}$ . Precisely, if we let

$$[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G}) = [L^{L^X}](\mathcal{F}, \mathcal{G}) = \bigwedge_{\lambda \in L^X} (\mathcal{F}(\lambda) \rightarrow \mathcal{G}(\lambda))$$

for all  $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ , then  $[\mathcal{F}_L^s(X)]$  is an  $L$ -partially order. For simplicity, we use the symbol  $(\mathcal{F}, \mathcal{G})$  to denote the value  $[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G})$  below. We have an obvious lemma for the  $L$ -partial order  $[\mathcal{F}_L^s(X)]$ .

**Lemma 2.4.** Let  $\phi : X \longrightarrow Y$  be a function. Then  $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), (\mathcal{F}, \mathcal{G}) \leq (\phi^{\Rightarrow}(\mathcal{F}), \phi^{\Rightarrow}(\mathcal{G}))$ .

**Definition 2.5.** (Flores and Mohapatra and Richardson [6] for  $L$  a frame) A collection  $\bar{q} = (q_\alpha)_{\alpha \in L}$ , where  $q_\alpha : \mathcal{F}_L^s(X) \longrightarrow \mathcal{P}(X)$ , is called a *stratified levelwise  $L$ -convergence structure* on  $X$  if it satisfies:

- (LL1)  $[x] \xrightarrow{q_\alpha} x, \mathcal{F}_0 \xrightarrow{q_0} x$  for each  $x \in X$ ,
- (LL2)  $\mathcal{G} \geq \mathcal{F} \xrightarrow{q_\alpha} x$  implies  $\mathcal{G} \xrightarrow{q_\alpha} x$ ,
- (LL3)  $\mathcal{F} \xrightarrow{q_\alpha} x$  implies  $\mathcal{F} \xrightarrow{q_\beta} x$  whenever  $\beta \leq \alpha$ .

The notation,  $\mathcal{F} \xrightarrow{q_\alpha} x$ , means that  $x \in q_\alpha(\mathcal{F})$ . The pair  $(X, \bar{q})$  is called a *stratified levelwise  $L$ -convergence space*.

A function  $\phi : X \rightarrow Y$  between stratified levelwise  $L$ -convergence spaces  $(X, \bar{q})$  and  $(Y, \bar{q}')$  is said to be continuous if  $\mathcal{F} \xrightarrow{q_\alpha} x$  implies  $\phi \Rightarrow (\mathcal{F}) \xrightarrow{q'_\alpha} \phi(x)$  for each  $\mathcal{F} \in \mathcal{F}_L^s(X), \alpha \in L, x \in X$ . The category of stratified levelwise  $L$ -convergence spaces and continuous functions is denoted by  $L\text{-SLC}$ .

**Definition 2.6.** (Jäger [9] and Yao [25]) A stratified  $L$ -convergence structure on a set  $X$  is a function  $\lim : \mathcal{F}_L^s(X) \rightarrow L^X$  satisfying

- (LC1)  $\lim[x](x) = 1$  for every  $x \in X$ ; and
- (LC2)  $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), \mathcal{F} \leq \mathcal{G} \implies \lim \mathcal{F} \leq \lim \mathcal{G}$ .

The pair  $(X, \lim)$  is called a stratified  $L$ -convergence space.

A function  $\phi : X \rightarrow Y$  between stratified  $L$ -convergence spaces  $(X, \lim)$  and  $(Y, \lim')$  is said to be continuous if  $\lim \mathcal{F}(x) \leq \lim' \phi \Rightarrow (\mathcal{F})(\phi(x))$  for each  $\mathcal{F} \in \mathcal{F}_L^s(X)$  and each  $x \in X$ . The category of stratified  $L$ -convergence spaces and continuous functions is denoted by  $L\text{-SC}$ .

In [16], a notion of stratified strong  $L$ -convergence space (called stratified  $L$ -ordered convergence space in [2]) was proposed, and it was defined as a function  $\lim : \mathcal{F}_L^s(X) \rightarrow L^X$  satisfying (LC1) and

- (LC2')  $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), (\mathcal{F}, \mathcal{G}) \leq \lim \mathcal{F} \rightarrow \lim \mathcal{G}$ , or equivalent,  $(\mathcal{F}, \mathcal{G}) * \lim \mathcal{F} \leq \lim \mathcal{G}$ . That is,  $\lim$  is  $L$ -order-preserving.

The full subcategory of  $L\text{-SC}$  consisting of strong objects is denoted by  $L\text{-SSC}$ .

Except for the above three kinds of lattice-valued convergence spaces, there are many other kinds of lattice-valued convergence spaces [4, 5, 13, 14, 23] are discussed by many researchers.

### 3. A Levelwise Characterization for Stratified Strong $L$ -convergence Space Via Strong Left-continuity Condition

In [6], for  $L$  a frame, a stratified levelwise  $L$ -convergence space  $(X, \bar{q})$  is said to be *left-continuous* if  $\mathcal{F} \xrightarrow{\alpha} x$  iff there exists  $A \subseteq L$  such that  $\bigvee A = \alpha$  and  $\mathcal{F} \xrightarrow{q_\beta} x$  for each  $\beta \in A$ . The category of left-continuous stratified levelwise  $L$ -convergence spaces is proved to be isomorphic to the category of stratified  $L$ -convergence spaces. Observing the proof in [6], we find that its left-continuity condition does not work in the general lattice context. But if we make a slight modification, that is, replacing  $\bigvee A = \alpha$  by  $\bigvee A \geq \alpha$ , it will do. The proof is similar to that in [6], thus, it is omitted.

In this section, we shall present a strong left-continuity condition for stratified levelwise  $L$ -convergence spaces and prove the category of strongly left-continuous stratified levelwise  $L$ -convergence spaces is isomorphic to the category of stratified strong  $L$ -convergence spaces.

**Definition 3.1.** A stratified levelwise  $L$ -convergence space  $(X, \bar{q})$  is said to be *strongly left-continuous* if  $\mathcal{F} \xrightarrow{q_\alpha} x$  iff there exists  $A \subseteq L$  and  $\mathcal{G}_\beta \xrightarrow{q_\beta} x$  for each  $\beta \in A$  such that  $\bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$ .

**Remark 3.2.** Note that the if part of strong left-continuity is obvious. Thus, when examining it, we only need to check its only if part. In addition, it is easily seen that strong left-continuity implies left-continuity.

The full subcategory of  $L$ -**SLC** consisting of strongly left-continuous objects is denoted by  $L$ -**SLSLC**. We prove below that  $L$ -**SLSLC** is isomorphic to  $L$ -**SSC**.

**Proposition 3.3.** *Let  $(X, \bar{q})$  be a stratified levelwise  $L$ -convergence space. Then the pair  $(X, \lim_{\bar{q}})$  is a stratified strong  $L$ -convergence space, where*

$$\lim_{\bar{q}} : \mathcal{F}_L^s(X) \longrightarrow L^X, \quad \lim_{\bar{q}} \mathcal{F}(x) = \bigvee \{ \alpha * (\mathcal{G}, \mathcal{F}) \mid \mathcal{G} \stackrel{q_\alpha}{\rightarrow} x \}.$$

*Proof.* **(LC1):** By **(LL1)**,  $[x] \stackrel{q_\alpha}{\rightarrow} x$  for each  $\alpha \in L, x \in X$ . Thus

$$\lim_{\bar{q}} [x](x) \geq \bigvee \{ \alpha \mid \alpha \in L \} = 1.$$

**(LC2')**: For each  $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), x \in X$ , we have

$$\begin{aligned} (\mathcal{F}, \mathcal{G}) * \lim_{\bar{q}} \mathcal{F}(x) &= (\mathcal{F}, \mathcal{G}) * \bigvee \{ \alpha * (\mathcal{H}, \mathcal{F}) \mid \mathcal{H} \stackrel{q_\alpha}{\rightarrow} x \} \\ &= \bigvee \{ \alpha * (\mathcal{F}, \mathcal{G}) * (\mathcal{H}, \mathcal{F}) \mid \mathcal{H} \stackrel{q_\alpha}{\rightarrow} x \} \\ &\leq \bigvee \{ \alpha * (\mathcal{H}, \mathcal{G}) \mid \mathcal{H} \stackrel{q_\alpha}{\rightarrow} x \} = \lim_{\bar{q}} (\mathcal{G})(x). \end{aligned}$$

□

**Remark 3.4.** Taking  $\mathcal{G} \equiv \mathcal{F}$ ,  $\lim_{\bar{q}}$  appeared in [6]. In this case,  $\lim_{\bar{q}}$  is only a stratified  $L$ -convergence space.

**Proposition 3.5.** *Let  $(X, \bar{q})$  and  $(Y, \bar{q}')$  be stratified levelwise  $L$ -convergence spaces. If  $\phi : (X, \bar{q}) \longrightarrow (Y, \bar{q}')$  is continuous, then so is  $\phi : (X, \lim_{\bar{q}}) \longrightarrow (Y, \lim_{\bar{q}'})$ .*

*Proof.* Let  $\mathcal{F} \in \mathcal{F}_L^s(X), x \in X$ . Then

$$\begin{aligned} \lim_{\bar{q}'} \phi^{\Rightarrow}(\mathcal{F})(\phi(x)) &= \bigvee \{ \alpha * (\mathcal{H}, \phi^{\Rightarrow}(\mathcal{F})) \mid \mathcal{H} \stackrel{q'_\alpha}{\rightarrow} \phi(x) \} \\ &\geq \bigvee \{ \alpha * (\phi^{\Rightarrow}(\mathcal{G}), \phi^{\Rightarrow}(\mathcal{F})) \mid \mathcal{G} \stackrel{q_\alpha}{\rightarrow} x \} \\ &\geq \bigvee \{ \alpha * (\mathcal{G}, \mathcal{F}) \mid \mathcal{G} \stackrel{q_\alpha}{\rightarrow} x \} = \lim_{\bar{q}} \mathcal{F}(x). \end{aligned}$$

Where the first inequality holds for the continuity of  $\phi$  and the second inequality follows from Lemma 2.4. □

**Proposition 3.6.** *Let  $(X, \lim)$  be a stratified strong  $L$ -convergence space, then the pair  $(X, \overline{q^{\lim}})$  is a strongly left-continuous stratified levelwise  $L$ -convergence space, where*

$$\forall \alpha \in L, \mathcal{F} \stackrel{(q^{\lim})_\alpha}{\rightarrow} x \iff \lim \mathcal{F}(x) \geq \alpha.$$

*Proof.* In the case of  $L$  being a frame, for a stratified  $L$ -convergence space  $(X, \lim)$ , it is proved in [6] that  $(X, \overline{q^{\lim}})$  is a stratified levelwise  $L$ -convergence space. So, it suffices to prove the strong left-continuity of  $\overline{q^{\lim}}$ .

Assume that there exists  $A \subseteq L$  and  $\mathcal{G}_\beta \stackrel{(q^{\lim})_\beta}{\rightarrow} x$  for each  $\beta \in A$  such that  $\bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$ . Then by  $\mathcal{G}_\beta \stackrel{(q^{\lim})_\beta}{\rightarrow} x \iff \lim \mathcal{G}_\beta(x) \geq \beta$  and **(LC2')** we have

$$\lim \mathcal{F}(x) \geq \lim \mathcal{G}_\beta(x) * (\mathcal{G}_\beta, \mathcal{F}), \forall \beta \in A.$$

So,

$$\lim \mathcal{F}(x) \geq \bigvee_{\beta \in A} (\lim \mathcal{G}_\beta(x) * (\mathcal{G}_\beta, \mathcal{F})) \geq \bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha.$$

This means  $\mathcal{F} \xrightarrow{(q^{\text{lim}})^\alpha} x$  as desired.  $\square$

**Proposition 3.7.** *Let  $(X, \text{lim})$  and  $(Y, \text{lim}')$  be stratified strong  $L$ -convergence spaces. If  $\phi : (X, \text{lim}) \rightarrow (Y, \text{lim}')$  is continuous, then so is  $\phi : (X, \overline{q^{\text{lim}}}) \rightarrow (Y, \overline{q^{\text{lim}'}})$ .*

*Proof.* Suppose that  $\mathcal{F} \xrightarrow{(q^{\text{lim}})^\alpha} x$ , i.e.,  $\lim \mathcal{F}(x) \geq \alpha$ . By the continuity of  $\phi$ , we get  $\text{lim}' \phi^{\Rightarrow}(\mathcal{F})(\phi(x)) \geq \lim \mathcal{F}(x) \geq \alpha$ . So,  $\phi^{\Rightarrow}(\mathcal{F}) \xrightarrow{(q^{\text{lim}'})^\alpha} \phi(x)$ .  $\square$

$$\text{We define } \begin{cases} \Psi : L\text{-SLSLC} \rightarrow L\text{-SSC}, \\ \Psi(\phi) = \phi, \\ \Psi(X, \overline{q}) = (X, \text{lim}_{\overline{q}}). \end{cases} \quad \begin{cases} \Delta : L\text{-SSC} \rightarrow L\text{-SLSLC}, \\ \Delta(\phi) = \phi, \\ \Delta(X, \text{lim}) = (X, \overline{q^{\text{lim}}}). \end{cases}$$

Then  $\Psi$  and  $\Delta$  are all concrete functors. The following theorem shows that  $\Delta$  is an isomorphism, i.e.,  $L\text{-SLSLC}$  is isomorphic to  $L\text{-SSC}$ .

**Theorem 3.8.**  $\Psi \circ \Delta = \text{id}_{L\text{-SSC}}$ ,  $\Delta \circ \Psi = \text{id}_{L\text{-SLSLC}}$ .

*Proof.* (1) For  $(X, \text{lim}) \in L\text{-SSC}$ , we check that  $\Psi \circ \Delta(\text{lim}) = \text{lim}$ .

Indeed,  $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$ ,  $\forall x \in X$ .

$$\begin{aligned} \lim_{\overline{q^{\text{lim}}}} \mathcal{F}(x) &= \bigvee \{ \alpha * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \xrightarrow{(q^{\text{lim}})^\alpha} x \} \geq \bigvee \{ \alpha * (\mathcal{F}, \mathcal{F}) | \mathcal{F} \xrightarrow{(q^{\text{lim}})^\alpha} x \} \\ &= \bigvee \{ \alpha | \lim \mathcal{F}(x) \geq \alpha \} = \lim \mathcal{F}(x) \end{aligned}$$

Conversely, let

$$\lim_{\overline{q^{\text{lim}}}} \mathcal{F}(x) = \bigvee \{ \alpha * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \xrightarrow{(q^{\text{lim}})^\alpha} x \} = \beta.$$

By the strong left-continuity of  $\overline{q^{\text{lim}}}$ , we have  $\mathcal{F} \xrightarrow{(q^{\text{lim}})^\beta} x$ , i.e.,  $\lim \mathcal{F}(x) \geq \beta = \lim_{\overline{q^{\text{lim}}}} \mathcal{F}(x)$ . Thus  $\lim \mathcal{F}(x) = \lim_{\overline{q^{\text{lim}}}} \mathcal{F}(x)$  as desired.

(2) For  $(X, \overline{q}) \in L\text{-SLSLC}$ , we check that  $\Delta \circ \Psi(\overline{q}) = \overline{q}$ .

In fact,

$$\mathcal{F} \xrightarrow{(q^{\text{lim}_{\overline{q}}})^\alpha} x \iff \lim_{\overline{q}} \mathcal{F}(x) = \bigvee \{ \beta * (\mathcal{G}, \mathcal{F}) | \mathcal{G} \xrightarrow{q^\beta} x \} \geq \alpha \iff \mathcal{F} \xrightarrow{q^\alpha} x.$$

Where the second equivalence holds by the strong left-continuity of  $(X, \overline{q})$ .  $\square$

**Theorem 3.9.**  $L\text{-SLSLC}$  (and thus  $L\text{-SSC}$ ) is reflective in  $L\text{-SLC}$ .

*Proof.* Given  $(X, \overline{q}) \in L\text{-SLC}$ , define  $(X, \overline{Lq})$  as follows:  $\mathcal{F} \xrightarrow{(Lq)^\alpha} x$  iff there exists  $A \subseteq L$  and  $\mathcal{G}_\beta \xrightarrow{q^\beta} x$  for each  $\beta \in A$  such that  $\bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$ . It is not difficult to check that  $(X, \overline{Lq}) \in L\text{-SLC}$ . Next, we prove that it is strongly left-continuous.

Suppose that there exists  $A \subseteq L$  and  $\mathcal{G}_\beta \xrightarrow{(Lq)\beta} x$  for each  $\beta \in A$  such that  $\bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$ . We fix  $\beta \in A$ . Because  $\mathcal{G}_\beta \xrightarrow{(Lq)\beta} x$ , then there exists  $A_\beta \subseteq L$  and  $\mathcal{H}_\gamma \xrightarrow{q\gamma} x$  for each  $\gamma \in A_\beta$  such that  $\bigvee_{\gamma \in A_\beta} (\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta)) \geq \beta$ . Denoting  $B = \bigcup \{A_\beta | \beta \in A\}$ , then

$$\begin{aligned} \bigvee_{\gamma \in B} (\gamma * (\mathcal{H}_\gamma, \mathcal{F})) &= \bigvee_{\beta \in A} \bigvee_{\gamma \in A_\beta} (\gamma * (\mathcal{H}_\gamma, \mathcal{F})) \\ &\geq \bigvee_{\beta \in A} \bigvee_{\gamma \in A_\beta} (\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta) * (\mathcal{G}_\beta, \mathcal{F})) \\ &= \bigvee_{\beta \in A} ((\mathcal{G}_\beta, \mathcal{F}) * (\bigvee_{\gamma \in A_\beta} (\gamma * (\mathcal{H}_\gamma, \mathcal{G}_\beta)))) \\ &\geq \bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha. \end{aligned}$$

Thus  $\mathcal{F} \xrightarrow{(Lq)\alpha} x$  as desired.

We check below that  $\text{id}_X: (X, \bar{q}) \longrightarrow (X, L\bar{q})$  is the  $L$ -**SLSLC**-reflection.

Since  $\mathcal{F} \xrightarrow{q\beta} x$  implies  $\mathcal{F} \xrightarrow{(Lq)\beta} x$ , thus  $\text{id}_X$  is obvious continuous. Assume that  $(Y, \bar{p}) \in L$ -**SLSLC**, and  $\phi: (X, \bar{q}) \longrightarrow (Y, \bar{p})$  is continuous. Next, we check that  $\phi: (X, L\bar{q}) \longrightarrow (Y, \bar{p})$  is also continuous. Let  $\mathcal{F} \xrightarrow{(Lq)\alpha} x$ . There exists  $A \subseteq L$  and  $\mathcal{G}_\beta \xrightarrow{q\beta} x$  for each  $\beta \in A$  such that  $\bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha$ . Because  $\phi: (X, \bar{q}) \longrightarrow (Y, \bar{p})$  is continuous, thus  $\forall \beta \in A, \phi^{\Rightarrow}(\mathcal{G}_\beta) \xrightarrow{p\beta} \phi(x)$ . So, by Lemma 2.4

$$\bigvee_{\beta \in A} (\beta * (\phi^{\Rightarrow}(\mathcal{G}_\beta), \phi^{\Rightarrow}(\mathcal{F}))) \geq \bigvee_{\beta \in A} (\beta * (\mathcal{G}_\beta, \mathcal{F})) \geq \alpha.$$

Therefore,  $\phi^{\Rightarrow}(\mathcal{F}) \xrightarrow{p\alpha} x$  by the strong left-continuity of  $(Y, \bar{p})$ .  $\square$

#### 4. A Sufficient and Necessary Condition Such That Stratified $L$ -convergence Spaces to be Stratified Strong $L$ -convergence Spaces

It is known that a stratified  $L$ -convergence structure needs not to be strong [2, 16]. In the following, we shall give a sufficient and necessary condition such that stratified  $L$ -convergence spaces to be strong. Our idea comes from the following equivalence [28]: A function  $f: L^X \longrightarrow L^Y$  is  $L$ -order-preserving iff

- (1)  $f$  is order-preserving function, i.e.,  $\lambda \leq \mu \implies f(\lambda) \leq f(\mu)$ ;
- (2) For each  $\alpha \in L, \lambda \in L^X, f(\alpha * \lambda) \geq \alpha * f(\lambda)$ .

The axiom **(LC2')** means that  $\lim$  is  $L$ -order-preserving. Analogizing the above equivalence, it seems **(LC2')** can be split into **(LC2)** and the following condition:

$$\text{For each } \alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X), \lim(\alpha * \mathcal{F}) \geq \alpha * \lim \mathcal{F}.$$

Unfortunately,  $\alpha * \mathcal{F}$  is not a stratified  $L$ -filter in general. We use

$$[\alpha * \mathcal{F}] = \bigwedge \{ \mathcal{G} \in \mathcal{F}_L^s(X) | \mathcal{G} \geq \alpha * \mathcal{F} \} = \bigwedge \{ \mathcal{G} \in \mathcal{F}_L^s(X) | \alpha \leq (\mathcal{F}, \mathcal{G}) \}$$

to replace it. Note that the set  $\{\mathcal{G} \in \mathcal{F}_L^s(X) | \mathcal{G} \geq \alpha * \mathcal{F}\}$  is not empty since  $\mathcal{F} \geq \alpha * \mathcal{F}$ . Intuitively,  $[\alpha * \mathcal{F}]$  is the coarsest stratified  $L$ -filter finer than the function  $\lambda \mapsto \alpha * \mathcal{F}(\lambda)$ . In this way, we obtain the condition

**(LC2'')** For all  $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X), \lim[\alpha * \mathcal{F}] \geq \alpha * \lim \mathcal{F}$ .

**Theorem 4.1.** *Let  $(X, \lim)$  be a stratified  $L$ -convergence space, then*

$$\mathbf{(LC2')} \iff \mathbf{(LC2)} + \mathbf{(LC2'')}.$$

*Proof.* **(LC2')  $\implies$  (LC2'')**. For each  $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^s(X)$ , by  $\alpha * \mathcal{F} \leq [\alpha * \mathcal{F}]$  we have  $(\mathcal{F}, [\alpha * \mathcal{F}]) \geq \alpha$ . So by **(LC2')**

$$\alpha * \lim \mathcal{F} \leq (\mathcal{F}, [\alpha * \mathcal{F}]) * \lim \mathcal{F} \leq \lim[\alpha * \mathcal{F}].$$

Conversely,  $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ , let  $\alpha = (\mathcal{F}, \mathcal{G})$ . Then  $\alpha * \mathcal{F} \leq \mathcal{G}$ . Thus by **(LC2)** and **(LC2'')**

$$(\mathcal{F}, \mathcal{G}) * \lim \mathcal{F} = \alpha * \lim \mathcal{F} \leq \lim[\alpha * \mathcal{F}] \leq \lim \mathcal{G}. \quad \square$$

Note that for  $L = \{0, 1\}$ , the axiom **(LC2'')** simply states that

$$\lim[0 * \mathcal{F}] \geq 0 * \lim \mathcal{F} = 0, \quad \lim \mathcal{F} \geq \lim \mathcal{F}.$$

Therefore, in this case, we have the equivalence **(LC2)  $\iff$  (LC2'')**.

The next two examples show that this equivalence is not always true for  $L \neq \{0, 1\}$ .

**Example 4.2.** [2, 17] Let  $L$  be the linearly ordered frame  $(\{0, a, 1\}, \wedge, 1)$  with  $0 < a < 1$ . Assume  $X = \{x, y\}$ .

For each  $\mathcal{F} \in \mathcal{F}_L^s(X)$  and  $z \in X$ , let  $\lim \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} \geq [z]; \\ 0, & \text{others.} \end{cases}$ . Then  $(X, \lim)$  satisfies the axiom **(LC2)**. Let  $\mathcal{F}_x(\lambda) = \begin{cases} 1, & \lambda = 1_X; \\ a, & \lambda(x) = 1, \lambda(y) \neq 1; \\ a, & \lambda(x) = a; \\ 0, & \lambda(x) = 0. \end{cases}, \forall \lambda \in L^X$ .

Then  $\mathcal{F}_x$  is a stratified  $L$ -filter on  $X$ . It is easily seen that  $\mathcal{F}_x = [a \wedge [x]]$  and  $[x] \not\leq \mathcal{F}_x$ . So,

$$a \wedge \lim[x](x) = a \not\leq 0 = \lim[a \wedge [x]](x).$$

That means  $(X, \lim)$  does not satisfy **(LC2'')**.

**Example 4.3.** Let  $L, X$  and  $\mathcal{F}_z$  be defined as the above example. For each  $\mathcal{F} \in \mathcal{F}_L^s(X)$  and  $z \in X$ , let

$$\lim \mathcal{F}(z) = \begin{cases} 1, & \mathcal{F} = \mathcal{F}_z \text{ or } \mathcal{F} = [z]; \\ 0, & \text{others.} \end{cases}$$

When  $\alpha = 0, 1$ , the axiom **(LC2'')** holds obviously. For  $\alpha = a$ , it suffices to consider the case of  $\mathcal{F} = \mathcal{F}_z, [z]$ . It is easily seen that  $[a \wedge [z]] = [a \wedge \mathcal{F}_z] = \mathcal{F}_z$ . So,

$$\lim[a \wedge \mathcal{F}](z) = 1 \geq a = a \wedge \lim \mathcal{F}(z).$$

This means that  $\lim$  satisfies the axioms **(LC2'')**. Let

$$\mathcal{F}_1(\lambda) = \begin{cases} 0, & \lambda(x) = 0; \\ 1, & \text{others.} \end{cases}, \forall \lambda \in L^X.$$



Then  $\mathcal{F}_1$  a stratified  $L$ -filter and  $\mathcal{F}_1 \geq \mathcal{F}_x$ . However,  $\lim \mathcal{F}_1(x) = 0 < 1 = \lim \mathcal{F}_x(x)$ . That means  $\lim$  does not satisfy the axiom **(LC2)**.

In [2], for  $L$  being a frame, it is proved that  $L$ -SSC is a reflective subcategory of  $L$ -SC. Indeed, we observe easily that this result also holds in the general lattice context.

Let  $E_{\lim}(X)$  be the set of all stratified strong  $L$ -convergence structures on  $X$ . For  $\lim, \lim' \in E_{\lim}(X)$ , we define  $\lim \leq \lim'$  iff  $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \lim \mathcal{F} \leq \lim' \mathcal{F}$ . Then it is easily seen that for each non-void subset  $\{\lim_i \mid i \in I\}$  of  $E_{\lim}(X)$ , we have  $\bigwedge_{i \in I} \lim_i \in E_{\lim}(X)$ . The least element of  $(E_{\lim}(X), \leq)$  is defined as  $\lim \mathcal{F}(x) = ([x], \mathcal{F})$ , and the largest element of  $(E_{\lim}(X), \leq)$  is defined as  $\lim \mathcal{F}(x) \equiv 1$ . Thus  $(E_{\lim}(X), \leq)$  forms a complete lattice.

For each  $(X, \lim) \in L$ -SC, the  $L$ -SSC-reflection [2] is given by  $\text{id}_X : (X, \lim) \rightarrow (X, \lim^*)$ , where  $\lim^* = \bigwedge \{\lim' \in E_{\lim}(X) \mid \lim' \geq \lim\}$ . The following proposition shows that  $\lim_*$  can be constructed by  $\lim$  itself.

**Proposition 4.4.** *Let  $(X, \lim)$  be a stratified  $L$ -convergence space. Then the function  $\overline{\lim} : \mathcal{F}_L^s(X) \rightarrow L^X$  defined by*

$$\overline{\lim} \mathcal{F} = \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} ((\mathcal{G}, \mathcal{F}) * \lim \mathcal{G})$$

is the least stratified strong  $L$ -convergence structure on  $X$  larger than or equal to  $\lim$ . So,  $\lim^* = \overline{\lim}$ .

*Proof.* **(LC1)** For each  $x \in X$ ,  $\overline{\lim}[x](x) \geq \lim[x](x) * ([x], [x]) = 1$ .

**(LC2')** For all  $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ ,

$$\begin{aligned} \overline{\lim} \mathcal{F} * (\mathcal{F}, \mathcal{G}) &= \bigvee_{\mathcal{H} \in \mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{F}) * (\mathcal{F}, \mathcal{G})) \\ &\leq \bigvee_{\mathcal{H} \in \mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{G})) = \overline{\lim} \mathcal{G}. \end{aligned}$$

That  $\overline{\lim} \geq \lim$  is obvious. For all  $\lim' \in E_{\lim}(X)$  with  $\lim' \geq \lim$ , we have

$$\overline{\lim} \mathcal{F} = \bigvee_{\mathcal{H} \in \mathcal{F}_L^s(X)} (\lim \mathcal{H} * (\mathcal{H}, \mathcal{F})) \leq \bigvee_{\mathcal{H} \in \mathcal{F}_L^s(X)} (\lim' \mathcal{H} * (\mathcal{H}, \mathcal{F})) \stackrel{\text{(LC2')}}{\leq} \lim' \mathcal{F}. \quad \square$$

In the proof of  $\overline{\lim} \in L$ -SSC, **(LC2)** is not used. Thus one can construct a stratified strong  $L$ -convergence structure from a function  $\lim$  only with **(LC1)**.

## 5. Conclusions

This paper focuses on the relationships between three types of  $L$ -convergence spaces: stratified  $L$ -convergence spaces, stratified strong  $L$ -convergence spaces and stratified levelwise  $L$ -convergence spaces. The main results are summarized as the following two: (1) a levelwise characterization for stratified strong  $L$ -convergence

spaces via strong left-continuity condition is obtained; (2) a sufficient and necessary condition such that a stratified  $L$ -convergence space is a stratified strong  $L$ -convergence space is obtained.

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