# COMMUTATIVE PSEUDO BE-ALGEBRAS

L. C. CIUNGU

Abstract. The aim of this paper is to introduce the notion of commutative pseudo BE-algebras and investigate their properties. We generalize some results proved by A. Walendziak for the case of commutative BE-algebras. We prove that the class of commutative pseudo BE-algebras is equivalent to the class of commutative pseudo BCK-algebras. Based on this result, all results holding for commutative pseudo BCK-algebras also hold for commutative pseudo BE-algebras. For example, any finite commutative pseudo BE-algebra is a BE-algebra, and any commutative pseudo BE-algebra is a join-semilattice. Moreover, if a commutative pseudo BE-algebra is a meet-semilattice, then it is a distributive lattice. We define the pointed pseudo-BE algebras, and introduce and study the relative negations on pointed pseudo BE-algebras. Based on the relative negations we construct two closure operators on a pseudo BE-algebra. We also define relative involutive pseudo BE-algebras, we investigate their properties and prove equivalent conditions for a relative involutive pseudo BE-algebra. We define the relative Glivenko property for a relative good pseudo BE-algebra and show that any relative involutive pseudo BE-algebra has the relative Glivenko property.

## 1. Introduction

Developing algebraic models for non-commutative multiple-valued logics is a central topic in the study of fuzzy systems. Pseudo BE-algebra is such an algebraic structure. Starting from the systems of positive implicational calculus, weak systems of positive implicational calculus and BCI and BCK systems, in 1966 Y. Imai and K. Iséki introduced the BCK-algebras ([18]). BCK-algebras are also used in a dual form, with an implication $\to$ and with one constant element 1, that is the greatest element. Dual BCK-algebras were defined in [22]. BE-algebras have been defined in [21] as a generalization of BCK-algebras, and they have intensively been studied by many authors (see [28], [1], [30], [31], [33], [37]).
A *BE-algebra* is an algebra $(X, \to, 1)$ of the type $(2, 0)$ such that the following axioms are fulfilled for all $x, y, z \in X$:
$(BE_1)$ $x \to x = 1$,
$(BE_2)$ $x \to 1 = 1$,
$(BE_3)$ $1 \to x = x$,
$(BE_4)$ $x \to (y \to z) = y \to (x \to z)$.

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Commutative BE-algebras have been defined in [36] and investigated in [2], [32], [9]. A BE-algebra $X$ is said to be *commutative* if

$$(x \to y) \to y = (y \to x) \to x,$$

for all $x, y \in X$.

It was proved in [36] that any dual BCK-algebra is a BE-algebra and any commutative BE-algebra is a dual BCK-algebra. Pseudo BE-algebras were introduced in [4] as generalizations of BE-algebras and properties of these structures have recently been studied in [34]. Note that another generalization of a BE-algebra is the CI-algebra defined by B.L. Meng in [27]. Pseudo BCK-algebras were introduced by G. Georgescu and A. Iorgulescu in [17] as algebras with "two differences", a left- and right-difference, and with a constant element 0 as the least element. Nowadays pseudo BCK-algebras are used in a dual form, with two implications, $\to$ and $\rightsquigarrow$ and with one constant element 1, that is the greatest element. Thus such pseudo BCK-algebras are in the "negative cone" and are also called "left-ones". Commutative pseudo BCK-algebras were originally defined by G. Georgescu and A. Iorgulescu in [17] under the name of *semilattice-ordered pseudo BCK-algebras* and properties of these structures were investigated by J. Kühr in [24], [25].

In this paper we introduce the notion of a commutative pseudo BE-algebra and generalize some results proved by A. Walendziak ([36]) for the case of commutative BE-algebras. We prove that the class of commutative pseudo BE-algebras is equivalent to the class of commutative pseudo BCK-algebras. Based on this result, all results holding for commutative pseudo BCK-algebras also hold for commutative pseudo BE-algebras. For example, any finite commutative pseudo BE-algebra is a BE-algebra, and any commutative pseudo BE-algebra is a join-semilattice. Moreover, if a commutative pseudo BE-algebra is a meet-semilattice, then it is a distributive lattice. We define the pointed pseudo BE-algebras and we introduce and study the relative negations on pointed pseudo BE-algebras. Based on the relative negations we construct two closure operators on a pseudo BE-algebra. Relative involutive pseudo BE-algebras are defined, their properties are investigated, and equivalent conditions for a relative involutive pseudo BE-algebra are proved. We define the relative Glivenko property for a relative good pseudo BE-algebra and we show that any relative involutive pseudo BE-algebra has the relative Glivenko property.

## 2. **Preliminaries**

In this section we recall some basic notions and results regarding pseudo BCK-algebras and pseudo BE-algebras. We prove that any pseudo BCK-algebra is a pseudo BE-algebra.

**Definition 2.1.** [23] A structure $(X, \to, \rightsquigarrow, 1)$ of the type $(2, 2, 0)$ is a *pseudo BCK-algebra* (more precisely, *reversed left-pseudo BCK-algebra*) iff it satisfies the following identities and quasi-identity, for all $x, y, z \in X$:

$(psBCK_1)$ $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1;$
$(psBCK_2)$ $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1;$
$(psBCK_3)$ $1 \to x = x;$

$(psBCK_4)$ $1 \rightsquigarrow x = x$;
$(psBCK_5)$ $x \to 1 = 1$;
$(psBCK_6)$ $(x \to y = 1$ and $y \to x = 1)$ implies $x = y$.

The partial order $\leq$ is defined by $x \leq y$ iff $x \to y = 1$ (iff $x \rightsquigarrow y = 1$).
For details regarding pseudo BCK-algebras we refer the reader to [20], [13].
If there is an element 0 of a pseudo BCK-algebra $(X, \to, \rightsquigarrow, 1)$, such that $0 \leq x$ (i.e. $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in X$, then the pseudo BCK-algebra is said to be *bounded* and it is denoted by $(X, \to, \rightsquigarrow, 0, 1)$. In a bounded pseudo BCK-algebra $(X, \to, \rightsquigarrow, 0, 1)$ we define two negations:

$$x^- := x \to 0,\ x^\sim := x \rightsquigarrow 0,$$

for all $x \in X$.

**Proposition 2.2.** [19] *Let $(X, \to, \rightsquigarrow, 1)$ be a pseudo BCK-algebra. Then the following hold for all $x, y, z \in X$:*
(1) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(2) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(3) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(4) $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(5) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$.

**Proposition 2.3.** ([19]) *Let $(X, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo BCK-algebra. Then the following hold for all $x, y \in X$:*
(1) $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$;
(2) $x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim = x^{\sim -} \rightsquigarrow y^{\sim -}$.

**Remark 2.4.** Pseudo BCK-logic was defined by J. Kühr ([24, Definition 1.3.1]). Formulas of pseudo BCK-logic are built from propositional variables and the primitive connectives $\to$ and $\rightsquigarrow$. The axioms are the following formulas:
$(B_1)$ $(\varphi \to \psi) \to ((\psi \to \vartheta) \rightsquigarrow (\varphi \to \vartheta))$,
$(B_2)$ $(\varphi \rightsquigarrow \psi) \to ((\psi \rightsquigarrow \vartheta) \to (\varphi \rightsquigarrow \vartheta))$,
$(C_1)$ $(\varphi \to (\psi \rightsquigarrow \vartheta)) \to (\psi \rightsquigarrow (\varphi \to \vartheta))$,
$(C_2)$ $(\varphi \rightsquigarrow (\psi \to \vartheta)) \to (\psi \to (\varphi \rightsquigarrow \vartheta))$,
$(K_1)$ $\varphi \to (\psi \to \varphi)$,
$(K_2)$ $\varphi \rightsquigarrow (\psi \rightsquigarrow \varphi)$.
The inference rules are the following:
$(MP)$ $\frac{\varphi, \varphi \to \psi}{\psi}$, i.e. from $\varphi$ and $\varphi \to \psi$ we infer $\psi$,
$(IMP_1)$ $\frac{\varphi \to \psi}{\varphi \rightsquigarrow \psi}$, i.e. from $\varphi \to \psi$ we infer $\varphi \rightsquigarrow \psi$,
$(IMP_2)$ $\frac{\varphi \rightsquigarrow \psi}{\varphi \to \psi}$, i.e. from $\varphi \rightsquigarrow \psi$ we infer $\varphi \to \psi$.

Pseudo BE-algebras were introduced in [4] as generalizations of BE-algebras and properties of these structures have recently been studied in [34].

**Definition 2.5.** [4] A *pseudo BE-algebra* is an algebra $(X, \to, \rightsquigarrow, 1)$ of the type $(2, 2, 0)$ such that the following axioms are fulfilled for all $x, y, z \in X$:

$(psBE_1)$ $x \to x = x \rightsquigarrow x = 1$,
$(psBE_2)$ $x \to 1 = x \rightsquigarrow 1 = 1$,
$(psBE_3)$ $1 \to x = 1 \rightsquigarrow x = x$,
$(psBE_4)$ $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
$(psBE_5)$ $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

A pseudo BE-algebra is said to be *proper* if it is not a BE-algebra.

**Proposition 2.6.** [4] *Let $(X, \to, \rightsquigarrow, 1)$ be a pseudo BE-algebra. Then the following hold for all $x, y, z \in X$:*

(1) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(2) $x \to (y \rightsquigarrow x) = 1$ *and* $x \rightsquigarrow (y \to x) = 1$;
(3) $x \to (y \to x) = 1$ *and* $x \rightsquigarrow (y \rightsquigarrow x) = 1$;
(4) $x \to ((x \to y) \rightsquigarrow y) = 1$ *and* $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$.

We will refer to $(X, \to, \rightsquigarrow, 1)$ by its universe $X$.

**Example 2.7.** [4] Consider the set $X = \{a, b, c, 1\}$ and the operations $\to, \rightsquigarrow$ given by the following tables:

| $\to$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | 1 | 1 |
| $b$ | 1 | $a$ | 1 | $c$ |
| $c$ | 1 | $b$ | 1 | 1 |

| $\rightsquigarrow$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | 1 | 1 |
| $b$ | 1 | $c$ | 1 | $c$ |
| $c$ | 1 | $c$ | 1 | 1 |

.

Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra. Moreover, it is even a pseudo BCK-algebra.

**Example 2.8.** [6] Consider the set $X = \{1, a, b, c, d, e\}$ and the operations $\to, \rightsquigarrow$ given by the following tables:

| $\to$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | 1 | 1 | $c$ | $c$ | $d$ | 1 |
| $b$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $c$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $d$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $e$ | 1 | $a$ | $d$ | $d$ | $d$ | 1 |

| $\rightsquigarrow$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | 1 | 1 | $b$ | $c$ | $d$ | 1 |
| $b$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $c$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $d$ | 1 | $a$ | 1 | 1 | 1 | $e$ |
| $e$ | 1 | $a$ | $d$ | $d$ | $d$ | 1 |

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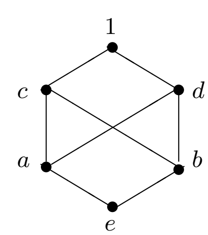
Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra. Since $b \to c = 1$ and $c \to b = 1$, but $b \neq c$, axiom $(psBCK_6)$ is not satisfied, hence $X$ is not a pseudo BCK-algebra.

**Example 2.9.** [12] Consider the structure $(X, \to, \rightsquigarrow, 1)$, where the operations $\to$ and $\rightsquigarrow$ on $X = \{1, a, b, c, d, e\}$ are defined as follows:

| $\to$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | 1 | 1 | $d$ | 1 | 1 | $d$ |
| $b$ | 1 | $c$ | 1 | 1 | 1 | $c$ |
| $c$ | 1 | $a$ | $d$ | 1 | $d$ | $a$ |
| $d$ | 1 | $c$ | $b$ | $c$ | 1 | $b$ |
| $e$ | 1 | 1 | 1 | 1 | 1 | 1 |

| $\rightsquigarrow$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | 1 | 1 | $c$ | 1 | 1 | $c$ |
| $b$ | 1 | $d$ | 1 | 1 | 1 | $d$ |
| $c$ | 1 | $d$ | $b$ | 1 | $d$ | $b$ |
| $d$ | 1 | $a$ | $c$ | $c$ | 1 | $a$ |
| $e$ | 1 | 1 | 1 | 1 | 1 | 1 |



Then $(X, \rightarrow, \rightsquigarrow, e, 1)$ is a bounded pseudo BE-algebra.

**Proposition 2.10.** *Any pseudo BCK-algebra is a pseudo BE-algebra.*

*Proof.* Let $(X, \rightarrow, \rightsquigarrow, 1)$ be a pseudo BCK-algebra.
Applying $(psBCK_3)$ and $(psBCK_2)$ we have:

$$x \rightarrow x = 1 \rightarrow (x \rightarrow x) = (1 \rightsquigarrow 1) \rightarrow [(1 \rightsquigarrow x) \rightarrow (1 \rightsquigarrow x)] = 1.$$

Similarly by $(psBCK_4)$ and $(psBCK_1)$ we get:

$$x \rightsquigarrow x = 1 \rightsquigarrow (x \rightsquigarrow x) = (1 \rightarrow 1) \rightsquigarrow [(1 \rightarrow x) \rightsquigarrow (1 \rightarrow x)] = 1.$$

Hence $X$ satisfies condition $(psBE_1)$.
Obviously $(psBE_3)$ follows from $(psBCK_3)$ and $(psBCK_4)$.
Applying $(psBCK_3)$ and $(psBCK_1)$ we get:

$$x \rightsquigarrow [(x \rightarrow y) \rightsquigarrow y] = (1 \rightarrow x) \rightsquigarrow [(x \rightarrow y) \rightsquigarrow (1 \rightarrow y)] = 1.$$

Similarly by $(psBCK_4)$ and $(psBCK_2)$ we have:

$$x \rightarrow [(x \rightsquigarrow y) \rightarrow y] = (1 \rightsquigarrow x) \rightarrow [(x \rightsquigarrow y) \rightarrow (1 \rightsquigarrow y)] = 1.$$

If $x \rightarrow y = 1$ then $x \rightsquigarrow y = x \rightsquigarrow [(x \rightarrow y) \rightsquigarrow y] = 1$ and similarly, if $x \rightsquigarrow y = 1$ then $x \rightarrow y = x \rightarrow [(x \rightsquigarrow y) \rightarrow y] = 1$. It follows that $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$, that is $(psBE_5)$.
Axiom $(psBE_2)$ follows from $(psBCK_5)$ and $(psBE_5)$, while $(psBE_4)$ is a result of Proposition 2.2. We conclude that $(X, \rightarrow, \rightsquigarrow, 1)$ is a pseudo BE-algebra. $\square$

**Remark 2.11.** A revised version of the notion of a *pseudo-equality algebra* has recently been introduced in [15] as an algebra $(X, \wedge, \sim, \backsim, 1)$ of the type $(2, 2, 2, 0)$ such that the following axioms are fulfilled for all $x, y, z \in X$:

$(A_1)$ $(X, \wedge, 1)$ is a meet-semilattice with top element 1,
$(A_2)$ $x \sim x = x \backsim x = 1$,
$(A_3)$ $x \sim 1 = 1 \backsim x = x$,
$(A_4)$ $x \leq y \leq z$ implies $x \sim z \leq y \sim z$, $x \sim z \leq x \sim y$, $z \backsim x \leq z \backsim y$ and $z \backsim x \leq y \backsim x$,
$(A_5)$ $x \sim y \leq (x \wedge z) \sim (y \wedge z)$ and $x \backsim y \leq (x \wedge z) \backsim (y \wedge z)$,
$(A_6)$ $x \sim y \leq (z \sim x) \backsim (z \sim y)$ and $x \backsim y \leq (x \backsim z) \sim (y \backsim z)$,
$(A_7)$ $x \sim y \leq (x \sim z) \sim (y \sim z)$ and $x \backsim y \leq (z \backsim x) \backsim (z \backsim y)$.

It was proved in [15] that the structure $(X, \rightarrow, \rightsquigarrow, 1)$, where $x \rightarrow y = (x \wedge y) \sim x$, $x \rightsquigarrow y = x \backsim (x \wedge y)$ is a pseudo BCK-algebra. According to Proposition 2.10, $(X, \rightarrow, \rightsquigarrow, 1)$ is a pseudo BE-algebra.

A subset $F$ of a pseudo BE-algebra $X$ is called a *pseudo filter* of $X$ if it satisfies the following axioms:

$(pF_1)$ $1 \in F$,
$(pF_2)$ $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

A subset $F$ of $X$ is a pseudo filter if and only if it satisfies $(pF_1)$ and the axiom:
$(pF_2^{'})$ $x \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$.

Denote by $\mathcal{F}(X)$ the set of all pseudo filters of $X$.
A pseudo filter $F$ of $X$ is proper if $F \neq X$.
A *maximal* pseudo filter is a proper filter such that it is not included in any other proper pseudo filter. Denote by $\mathcal{M}(X)$ the set of all maximal pseudo filters of $X$.
For details regarding pseudo filters and congruence relations on a pseudo BE-algebra we refer the reader to [4], [34].

**Definition 2.12.** Let $X$ be a pseudo BE-algebra. Then $P \in \mathcal{F}(X)$ is called *prime* if and only if, for all $F_1, F_2 \in \mathcal{F}(X)$ such that $F_1 \cap F_2 \subseteq P$, we have $F_1 \subseteq P$ or $F_2 \subseteq P$. The set of all proper prime pseudo filters of $X$ is denoted by $\mathcal{P}(X)$ and called the *prime spectrum* of $X$.

**Example 2.13.** Let $(X, \rightarrow, \rightsquigarrow, 1)$ be the pseudo BE-algebra from Example 2.7. Then $\mathcal{F}(X) = \{\{1\}, \{1, b\}, X\}$, $\mathcal{P}(X) = \mathcal{M}(X) = \{\{1, b\}\}$.

## 3. Commutative Pseudo BE-algebras

In this section we introduce the notion of a commutative pseudo BE-algebra and we generalize some results proved by A. Walendziak ([36]) for the case of commutative BE-algebras. We prove that the class of commutative pseudo BE-algebras is equivalent to the class of commutative pseudo BCK-algebras. Based on this result, all results holding for commutative pseudo BCK-algebras also hold for commutative pseudo BE-algebras. For example, any finite commutative pseudo BE-algebra is a BE-algebra, and any commutative pseudo BE-algebra is a join-semilattice. Moreover, if a commutative pseudo BE-algebra is a meet-semilattice, then it is a distributive lattice.

**Definition 3.1.** A pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, 1)$ is said to be *commutative* if it satisfies the following conditions, for all $x, y \in X$:

$$(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x,$$
$$(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x.$$

**Example 3.2.** The pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, 1)$ from Example 2.7 is not commutative, since $(a \rightsquigarrow c) \rightarrow c = c \neq b = (c \rightsquigarrow a) \rightarrow a$.

**Example 3.3.** Let $(G, \vee, \wedge, \cdot, ^{-1}, e)$ be an $\ell$-group. On the negative cone $G^- = \{g \in G \mid g \leq e\}$ we define the operations $x \rightarrow y = y \cdot (x \vee y)^{-1}$, $x \rightsquigarrow y = (x \vee y)^{-1} \cdot y$. Then $(G^-, \rightarrow, \rightsquigarrow, e)$ is a commutative pseudo BE-algebra.

**Theorem 3.4.** *Any commutative pseudo BE-algebra is a pseudo BCK-algebra.*

*Proof.* Let $(X, \rightarrow, \rightsquigarrow, 1)$ be a commutative pseudo BE-algebra.
Axioms $(psBCK_3)$, $(psBCK_4)$ and $(psBCK_5)$ follow from $(psBE_3)$ and $(psBE_2)$, respectively.
Applying $(psBE_4)$ and Proposition 2.6 we have:

$$\begin{aligned}(x \rightarrow y) \rightsquigarrow [(y \rightarrow z) \rightsquigarrow (x \rightarrow z)] &= (x \rightarrow y) \rightsquigarrow [x \rightarrow ((y \rightarrow z) \rightsquigarrow z)] \\ &= (x \rightarrow y) \rightsquigarrow [x \rightarrow ((z \rightarrow y) \rightsquigarrow y)] \\ &= (x \rightarrow y) \rightsquigarrow [(z \rightarrow y) \rightsquigarrow (x \rightarrow y)] = 1.\end{aligned}$$

$$\begin{aligned}(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] &= (x \rightsquigarrow y) \to [x \rightsquigarrow ((y \rightsquigarrow z) \to z)] \\ &= (x \rightsquigarrow y) \to [x \rightsquigarrow ((z \rightsquigarrow y) \to y)] \\ &= (x \rightsquigarrow y) \to [(z \rightsquigarrow y) \to (x \rightsquigarrow y)] = 1.\end{aligned}$$

It follows that $(psBCK_1)$ and $(psBCK_2)$ are verified.

Obviously $(psBCK_3)$ and $(psBCK_4)$ follow from $(psBE_3)$, while $(psBCK_5)$ is in fact $(psBE_2)$.
If $x \to y = 1$ and $y \to x = 1$, we have $x = 1 \rightsquigarrow x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$, thus $(psBCK_6)$ is satisfied. We conclude that $(X, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra. □

**Theorem 3.5.** *The class of commutative pseudo BE-algebras is equivalent to the class of commutative pseudo BCK-algebras.*

*Proof.* It follows from Theorem 3.4 and Proposition 2.10. □

**Remark 3.6.** As a consequence of Theorem 3.5, all results holding for commutative pseudo BCK-algebras also hold for commutative pseudo BE-algebras. We recall some of these results:

(1) A pseudo BE-algebra $(X, \to, \rightsquigarrow, 1)$ is commutative if and only if $y \to x = y \rightsquigarrow x = 1$ implies $(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y = x$, for all $x, y \in X$ ([24, Lemma 4.1.4]).

(2) There are no proper finite commutative pseudo BE-algebras ([24, Corollary 4.1.6]). In the other words, every finite commutative pseudo BE-algebra is a commutative BE-algebra.

(3) If $(X, \to, \rightsquigarrow, 1)$ is a commutative pseudo BE-algebra then $(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$ ([13, Corollary 1.2]).

(4) If $(X, \to, \rightsquigarrow, 1)$ is a commutative pseudo BE-algebra then $(X, \vee, \to, \rightsquigarrow, 1)$ is a join-semilattice, where $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$ ([19]).

Note that generally, the underlying poset $(X, \leq)$ of a pseudo BE-algebra $(X, \to, \rightsquigarrow, 1)$ need not be a join-semilattice (see Example 2.9).

(5) For a commutative pseudo BE-algebra $(X, \to, \rightsquigarrow, 1)$, if $(X, \leq)$ is a meet-semilattice then it is a distributive lattice ([24, Corollary 4.1.9]).

(6) The prime spectrum $\mathcal{P}(X)$ of a commutative pseudo BE-algebra $(X, \to, \rightsquigarrow, 1)$ is equipped with the *spectral topology* (see (4) and [24]).

(7) Let us call *commutative pseudo BCK-logic* (*cpsBCK-logic for short*) the extension of pseudo BCK-logic associated to the class of commutative pseudo BCK-algebras. The commutative pseudo BE-algebras are models of the cpsBCK-logic.

**Remark 3.7.** Pseudo MV-algebras were introduced by G. Georgescu and A. Iorgulescu in [16], and independently by J. Rachůnek in [29]. It was proved that bounded commutative pseudo BCK-algebras are categorically isomorphic with pseudo MV-algebras (see [17], [19]). By Theorem 3.5, it follows that bounded commutative pseudo BE-algebras are also categorically isomorphic with pseudo MV-algebras.

## 4. **Relative Negations in Pseudo BE-algebras**

We define pointed pseudo BE-algebras, and we introduce and study relative negations on pointed pseudo BE-algebras. Based on relative negations we construct two closure operators on a pseudo BE-algebra. Relative involutive pseudo BE-algebras are defined, their properties are investigated, and equivalent conditions for a relative involutive pseudo BE-algebra are proved. We define the relative Glivenko property for a relative good pseudo BE-algebra and we show that any relative involutive pseudo BE-algebra has the relative Glivenko property.

**Definition 4.1.** A pseudo BE-algebra $(A, \to, \rightsquigarrow, 1)$ with a constant $a$ (which can denote any element) is called a *pointed pseudo BE-algebra* and it is denoted by $(A, \to, \rightsquigarrow, a, 1)$.

**Definition 4.2.** Let $(X, \to, \rightsquigarrow, a, 1)$ be a pointed pseudo BE-algebra. For all $x \in X$ we define: $x^{-_a} = x \to a$ and $x^{\sim_a} = x \rightsquigarrow a$ (called *negations relative to* $a$ or $a$-*relative negations*).

If $A$ is a bounded pseudo BE-algebra then we denote $x^{-_0}$ and $x^{\sim_0}$ by $x^-$ and $x^\sim$, respectively.

**Proposition 4.3.** *Let $(X, \to, \rightsquigarrow, a, 1)$ be a pointed pseudo BE-algebra. Then the following hold for all $x, y \in X$ :*
(1) $1^{-_a} = 1^{\sim_a} = a$;
(2) $x \leq x^{-_a\sim_a}$, $x \leq x^{\sim_a-_a}$;
(3) $a \leq x^{-_a\sim_a}$, $a \leq x^{\sim_a-_a}$.

*Proof.* (1) It follows from $(psBE_3)$.
(2) We apply Proposition 2.6(4).
(3) It is a consequence of Proposition 2.6(3). □

**Proposition 4.4.** *Let $(X, \to, \rightsquigarrow, a, 1)$ be a pointed commutative pseudo BE-algebra. Then the following hold for all $x, y \in X$:*

(1) $x \to y \leq y^{-_a} \rightsquigarrow x^{-_a}$ *and* $x \rightsquigarrow y \leq y^{\sim_a} \to x^{\sim_a}$;
(2) $x \leq y$ *implies* $y^{-_a} \leq x^{-_a}$ *and* $y^{\sim_a} \leq x^{\sim_a}$;
(3) $x \to y^{\sim_a} = y \rightsquigarrow x^{-_a}$ *and* $x \rightsquigarrow y^{-_a} = y \to x^{\sim_a}$;
(4) $x^{-_a\sim_a-_a} = x^{-_a}$ *and* $x^{\sim_a-_a\sim_a} = x^{\sim_a}$;
(5) $x \to y^{-_a\sim_a} = y^{-_a} \rightsquigarrow x^{-_a} = x^{-_a\sim_a} \to y^{-_a\sim_a}$ *and*
$x \rightsquigarrow y^{\sim_a-_a} = y^{\sim_a} \to x^{\sim_a} = x^{\sim_a-_a} \rightsquigarrow y^{\sim_a-_a}$;
(6) $x \to y^{\sim_a} = y^{\sim_a-_a} \rightsquigarrow x^{-_a} = x^{-_a\sim_a} \to y^{\sim_a}$ *and*
$x \rightsquigarrow y^{-_a} = y^{-_a\sim_a} \to x^{\sim_a} = x^{\sim_a-_a} \rightsquigarrow y^{-_a}$;
(7) $(x \to y^{\sim_a-_a})^{\sim_a-_a} = x \to y^{\sim_a-_a}$ *and* $(x \rightsquigarrow y^{-_a\sim_a})^{-_a\sim_a} = x \rightsquigarrow y^{-_a\sim_a}$;
(8) *if* $x \leq a$, *then* $x^{-_a\sim_a} = x^{\sim_a-_a} = a$;
(9) *if* $a \leq x$, *then* $x^{-_a\sim_a} = x^{\sim_a-_a} = x$;
(10) *if* $a \leq x, y$, *then* $x^{\sim_a} \to y = y^{-_a} \rightsquigarrow x$ *and* $x^{-_a} \rightsquigarrow y = y^{\sim_a} \to x$.

*Proof.* Since $(X, \to, \rightsquigarrow, a, 1)$ is a pseudo BCK-algebra, we apply the axioms and the properties of a pseudo BCK-algebra.
(1) It follows from $(psBCK_1)$ and $(psBCK_2)$ for $z = a$.

(2) From $x \leq y$, applying Proposition 2.2(3) we get $y \to a \leq x \to a$, so $y^{-_a} \leq x^{-a}$. Similarly $y^{\sim_a} \leq x^{\sim_a}$.

(3) By (1), Proposition 4.3(2) and Proposition 2.2(3), we get:

$$x \to y^{\sim_a} \leq y^{\sim_a -_a} \rightsquigarrow x^{-_a} \leq y \rightsquigarrow x^{-_a} \text{ and}$$
$$x \rightsquigarrow y^{-_a} \leq y^{-_a \sim_a} \to x^{\sim_a} \leq y \to x^{\sim_a}.$$

In the above inequalities we change $x$ and $y$ obtaining:
$y \to x^{\sim_a} \leq x \rightsquigarrow y^{-_a}$ and $y \rightsquigarrow x^{-_a} \leq x \to y^{\sim_a}$.
Thus $x \to y^{\sim_a} = y \rightsquigarrow x^{-_a}$ and $x \rightsquigarrow y^{-_a} = y \to x^{\sim_a}$.

(4) By Proposition 4.3(2) and (2) we get $x^{\sim_a -_a \sim_a} \leq x_a^{\sim}$ and $x^{-_a \sim -_a} \leq x^{-_a}$. By Proposition 4.3(2), replacing $x$ with $x^{\sim_a}$ and $x^{-_a}$ we get $x^{\sim_a} \leq x^{\sim_a -_a \sim_a}$ and $x^{-_a} \leq x^{-_a \sim_a -_a}$, respectively. Thus $x^{\sim_a -_a \sim_a} = x^{\sim_a}$ and $x^{-_a \sim_a -_a} = x^{-_a}$.

(5) By (3) we have: $y \rightsquigarrow x^{-_a} = x \to y^{\sim_a}$.
Replacing $y$ with $y^{-_a}$ we get: $y^{-_a} \rightsquigarrow x^{-_a} = x \to y^{-_a \sim_a}$.
Replacing $x$ by $x^{-_a \sim_a}$ in the last equality we get: $y^{-_a} \rightsquigarrow x^{-_a \sim_a -_a} = x^{-_a \sim_a} \to y^{-_a \sim_a}$. Hence applying (4) it follows that: $y^{-_a} \rightsquigarrow x^{-_a} = x^{-_a \sim_a} \to y^{-_a \sim_a}$.
Thus $x \to y^{-_a \sim_a} = y^{-_a} \rightsquigarrow x^{-_a} = x^{-_a \sim_a} \to y^{-_a \sim_a}$.
Similarly $x \rightsquigarrow y^{\sim_a -_a} = y^{\sim_a} \to x^{\sim_a} = x^{\sim_a -_a} \rightsquigarrow y^{\sim_a -_a}$.

(6) The assertions follow by replacing $y$ with $y^{\sim_a}$ and $y$ with $y^{-_a}$ respectively in (5) and then applying (4).

(7) Applying ($psBE_4$), Proposition 2.6(1) and (5) we have:

$$\begin{aligned} 1 &= (x \to y^{\sim_a -_a}) \rightsquigarrow (x \to y^{\sim_a -_a}) = x \to ((x \to y^{\sim_a -_a}) \rightsquigarrow y^{\sim_a -_a}) \\ &= x \to ((x \to y^{\sim_a -_a})^{\sim_a -_a} \rightsquigarrow y^{\sim_a -_a}) \\ &= (x \to y^{\sim_a -_a})^{\sim_a -_a} \rightsquigarrow (x \to y^{\sim_a -_a}). \end{aligned}$$

Hence $(x \to y^{\sim_a -_a})^{\sim_a -_a} \leq x \to y^{\sim_a -_a}$.
On the other hand, by Proposition 4.3(2) we have $x \to y^{\sim_a -_a} \leq (x \to y^{\sim_a -_a})^{\sim_a -_a}$, thus $(x \to y^{\sim_a -_a})^{\sim_a -_a} = x \to y^{\sim_a -_a}$. Similarly $(x \rightsquigarrow y^{-_a \sim_a})^{-_a \sim_a} = x \rightsquigarrow y^{-_a \sim_a}$.

(8) From $x \leq a$ we get $1 = a^{-_a} \leq x^{-_a}$, hence $x^{-_a \sim_a} \leq a$. Applying Proposition 4.3(3), it follows that $x^{-_a \sim_a} = a$. Similarly $x^{\sim_a -_a} = a$.

(9) It is a consequence of Remark 3.6(1).

(10) By replacing $x$ with $x^{\sim_a}$ and $y$ with $y^{-_a}$ in (3) and applying (9) we get $x^{\sim_a} \to y = y^{-_a} \rightsquigarrow x$. By interchanging $x$ and $y$ we obtain $x^{-_a} \rightsquigarrow y = y^{\sim_a} \to x$. □

**Proposition 4.5.** *Let* $(X, \to, \rightsquigarrow, a, 1)$ *be a pointed commutative pseudo BE-algebra. Then the maps* $\gamma_1, \gamma_2 : A \longrightarrow A$, *defined by* $\gamma_1(x) = x^{-_a \sim_a}$, $\gamma_2(x) = x^{\sim_a -_a}$ *for all* $x \in A$, *are closure operators on* $X$.

*Proof.* We have to prove that $\gamma_i$ $(i \in \{1, 2\})$ satisfy the following conditions for all $x, y \in X$:
$(C_1)$ $x \leq \gamma_i(x)$ (extensive);
$(C_2)$ $x \leq y$ implies $\gamma_i(x) \leq \gamma_i(y)$ (monotone);
$(C_3)$ $\gamma_i(\gamma_i(x)) = \gamma_i(x)$ (idempotent).
Indeed condition $(C_1)$ follows from Proposition 4.3(2).

If $x \leq y$, applying Proposition 4.4(2) twice, we get $(C_2)$.
From Proposition 4.4(4) we have:

$$\gamma_1(\gamma_1(x)) = x^{-_a \sim_a -_a \sim_a} = x^{-_a \sim_a} = \gamma_1(x) \text{ and}$$
$$\gamma_2(\gamma_2(x)) = x^{\sim_a -_a \sim_a -_a} = x^{\sim_a -_a} = \gamma_2(x),$$

for all $x \in X$. Thus condition $(C_3)$ is satisfied.

We conclude that $\gamma_1$, $\gamma_2$ are closure operators on $X$. □

**Definition 4.6.** Let $(X, \rightarrow, \rightsquigarrow, a, 1)$ be a pointed pseudo BE-algebra. Then:

(1) $X$ is called *a-relative good* (*a-good*, for short) if $x^{-_a \sim_a} = x^{\sim_a -_a} = x$ for all $x \in X$.
(2) An element $x \in A$ is said to be *a-regular* if $x^{-_a \sim_a} = x^{\sim_a -_a} = x$. Denote by $\mathrm{Reg}_a(X)$ the set of all $a$-regular elements of $X$.
(3) $X$ is said to be *a-relative involutive* (or briefly, *a-involutive*) if $X = \mathrm{Reg}_a(X)$.

**Example 4.7.** (1) Let $(X, \rightarrow, \rightsquigarrow, 1)$ be the pseudo BE-algebra from Example 2.7. Then the pointed pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, x, 1)$ is $x$-good for $x \in \{1, b, c\}$.

(2) If $(X, \rightarrow, \rightsquigarrow, 1)$ is the pseudo BE-algebra from Example 2.8 then the pointed pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, x, 1)$ is $x$-good for all $x \in X$.

(3) The pointed pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, e, 1)$ from Example 2.9 is $e$-involutive.

**Remark 4.8.** As a consequence of Remark 3.6(3), every pointed commutative pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, a, 1)$ is $a$-good.

**Example 4.9.** If $(X, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded commutative pseudo BE-algebra then according to Proposition 4.4(9) we have $x^{-_0 \sim_0} = x^{\sim_0 -_0} = x$ for all $x \in X$, that is $X$ is 0-involutive.

**Lemma 4.10.** *In any commutative a-involutive pseudo BE-algebra $(X, \rightarrow, \rightsquigarrow, a, 1)$, the following hold for all $x, y \in X$ :*

(1) $x \rightarrow y = y^{-_a} \rightsquigarrow x^{-_a}$ *and* $x \rightsquigarrow y = y^{\sim_a} \rightarrow x^{\sim_a}$;
(2) $x^{\sim_a} \rightarrow y = y^{-_a} \rightsquigarrow x$ *and* $x^{-_a} \rightsquigarrow y = y^{\sim_a} \rightarrow x$.

*Proof.* (1) It is a consequence of Proposition 4.4(5).
(2) It follows from Proposition 4.4(5) replacing $x$ with $x^{\sim_a}$ and $x^{-_a}$, respectively. □

**Theorem 4.11.** *If $(X, \rightarrow, \rightsquigarrow, a, 1)$ is a pointed commutative pseudo BE-algebra then the following are equivalent:*

$(a)$ *$X$ is a-involutive;*
$(b)$ $x \rightarrow y = y^{-_a} \rightsquigarrow x^{-_a}$ *and* $x \rightsquigarrow y = y^{\sim_a} \rightarrow x^{\sim_a}$*, for all* $x, y \in X$;
$(c)$ $x^{\sim_a} \rightarrow y = y^{-_a} \rightsquigarrow x$ *and* $x^{-_a} \rightsquigarrow y = y^{\sim_a} \rightarrow x$*, for all* $x, y \in X$;
$(d)$ $x^{-_a} \leq y$ *implies* $y^{\sim_a} \leq x$ *and* $x^{\sim_a} \leq y$ *implies* $y^{-_a} \leq x$*, for all* $x, y \in X$.

*Proof.* $(a) \Rightarrow (b)$ follows by Lemma 4.10(1).
$(b) \Rightarrow (c)$ By Proposition 4.4(5) we have: $x^{\sim_a} \rightarrow y^{-_a \sim_a} = y^{-_a} \rightsquigarrow x^{\sim_a -_a}$.
Applying $(b)$ we get: $x^{\sim_a} \rightarrow y = y^{-_a} \rightsquigarrow x^{\sim_a -_a}$ and $y^{-_a} \rightsquigarrow x = x^{\sim_a} \rightarrow y^{-_a \sim_a}$.
Thus $x^{\sim_a} \rightarrow y = y^{-_a} \rightsquigarrow x$. Similarly $x^{-_a} \rightsquigarrow y = y^{\sim_a} \rightarrow x$.

$(c) \Rightarrow (d)$ If $x^{-_a} \leq y$ then $x^{-_a} \rightsquigarrow y = 1$. Applying $(c)$ we get $y^{\sim_a} \to x = 1$, that is $y^{\sim_a} \leq x$. Similarly $x^{\sim_a} \leq y$ implies $y^{-_a} \leq x$.
$(d) \Rightarrow (a)$ Consider $x, y \in A$. From $x^{-_a} \leq x^{-_a}$ and $(d)$ we have $x^{-_a \sim_a} \leq x$, thus $x^{-_a \sim_a} = x$. Similarly $x^{\sim_a -_a} = x$. It follows that $X$ is $a$-involutive. □

**Theorem 4.12.** *If $(X, \to, \rightsquigarrow, a, 1)$ is a commutative $a$-good pseudo BE-algebra then the following are equivalent:*
$(a)$ $(x^{-_a \sim_a} \to x)^{-_a \sim_a} = (x^{-_a \sim_a} \rightsquigarrow x)^{-_a \sim_a} = 1$ *for all* $x \in X$;
$(b)$ $(x^{-_a \sim_a} \to x)^{\sim_a} = (x^{-_a \sim_a} \rightsquigarrow x)^{-_a} = a$ *for all* $x \in X$;
$(c)$ $(x \to y)^{-_a \sim_a} = x \to y^{-_a \sim_a}$ *and* $(x \rightsquigarrow y)^{-_a \sim_a} = x \rightsquigarrow y^{-_a \sim_a}$ *for all* $x \in X$.

*Proof.* $(a) \Leftrightarrow (b)$ is straightforward.
$(a) \Rightarrow (c)$ By Proposition 4.4(1), (5) we have $x \to y \leq y^{-_a} \rightsquigarrow x^{-_a} = x \to y^{-_a \sim_a}$ and applying Proposition 4.4(7) we get

$$(x \to y)^{-_a \sim_a} \leq (x \to y^{-_a \sim_a})^{-_a \sim_a} \leq x \to y^{-_a \sim_a}.$$

Taking into consideration Proposition 2.2(5), it follows that

$$y^{-_a \sim_a} \to y \leq (x \to y^{-_a \sim_a}) \to (x \to y).$$

But $x \to y \leq (x \to y)^{-_a \sim_a}$, hence

$$(x \to y^{-_a \sim_a}) \to (x \to y) \leq (x \to y^{-_a \sim_a}) \to (x \to y)^{-_a \sim_a}.$$

Applying $(a)$ and Proposition 4.4(7), we have:

$$\begin{aligned} 1 = (y^{-_a \sim_a} \to y)^{-_a \sim_a} &\leq [(x \to y^{-_a \sim_a}) \to (x \to y)^{-_a \sim_a}]^{-_a \sim_a} \\ &= (x \to y^{-_a \sim_a}) \to (x \to y)^{-_a \sim_a}. \end{aligned}$$

It follows that $(x \to y^{-_a \sim_a}) \to (x \to y)^{-_a \sim_a} = 1$, thus $x \to y^{-_a \sim_a} \leq (x \to y)^{-_a \sim_a}$.
We conclude that $(x \to y)^{-_a \sim_a} = x \to y^{-_a \sim_a}$.
Similarly $(x \rightsquigarrow y)^{-_a \sim_a} = x \rightsquigarrow y^{-_a \sim_a}$.
$(c) \Rightarrow (a)$ Applying $(c)$ we have:

$$(x^{-_a \sim_a} \to x)^{-_a \sim_a} = x^{-_a \sim_a} \to x^{-_a \sim_a} = 1 \text{ and}$$
$$(x^{-_a \sim_a} \rightsquigarrow x)^{-_a \sim_a} = x^{-_a \sim_a} \rightsquigarrow x^{-_a \sim_a} = 1.$$

□

**Definition 4.13.** If an $a$-good pseudo BE-algebra satisfies the equivalent conditions from Theorem 4.12 then we say that it has the *$a$-relative Glivenko property* (*$a$-Glivenko property*, for short).

**Example 4.14.** (1) Any $a$-involutive pseudo BE-algebra has the $a$-Glivenko property.
(2) The pointed pseudo BE-algebra $(X, \to, \rightsquigarrow, e, 1)$ from Example 2.9 has the $e$-Glivenko property.

**Remark 4.15.** Glivenko showed that a proposition is classically demonstrable if and only if its double negation is intuitionistically demonstrable. In the other words, the classical propositional logic can be interpreted in intuitionistic propositional logic. Versions of both logical and algebraic formulations of Glivenko's property were intensively studied. An important motivation of these studies is that Glivenko's property is used to prove the existence of probabilities (states) on different algebras of fuzzy logics. Algebraic versions of this property have been investigated for the commutative residuated structures by Cignoli and Torrens [7], [8]. Given an algebra $(A, \to, ^-)$ with the implication $\to$ and negation $^-$, Glivenko's property is formulated in algebraic terms as follows: $(a \to b)^{--} = a \to b^{--}$. In [11], [12] this property has been extended to other fuzzy structures, while in [38], [14] it was investigated for $\mathrm{FL}_e$-algebras and $\mathrm{FL}_w$-algebras in the context of relative negations.

## 5. Concluding Remarks

We think that this paper could contribute to the study of pseudo BE-algebras and to the development of a theory of probabilities on these structures. Since these topics are of current interest we suggest further directions of research:

(1) States on BE-algebras have been studied in [5], while the states on pseudo BE-algebras have been investigated in [35]. Generalized Bosbach and Riečan states on pseudo BCK-algebras were studied in [12], and it was proved that, in the case of pseudo BCK-algebras having the Glivenko property the two kind of states coincide. Starting from the notions of pointed pseudo BE-algebras and relative involutive pseudo BE-algebras, one can define and investigate the (generalized) states as well as the internal states on pseudo BE-algebras in the context of relative negations. The Glivenko property plays an important role in the study of these states (see [38], [14]).

(2) The notion of a measure on a pseudo BCK-algebra $(X, \to, \rightsquigarrow, 1)$ was introduced in [10] as a function $m : X \longrightarrow [0, \infty)$ such that $m(x \to y) = m(x \rightsquigarrow y) = m(y) - m(x)$, whenever $y \leq x$. Similarly one can define and study the measures on pseudo BE-algebras.

(3) As we mentioned in Remark 3.6(6), the prime spectrum $\mathcal{P}(X)$ of a commutative pseudo BE-algebra $(X, \to, \rightsquigarrow, 1)$ is equipped with the spectral topology. This topology could be investigated on an arbitrary pseudo BE-algebra.

(4) Pseudo-valuations on BE-algebras were defined and studied in [26], and based on a pseudo-valuation a pseudo-metric was constructed. These notions and results can be extended to the case of pseudo BE-algebras, and based on the corresponding pseudo-metric one can define and investigate a topology on pseudo BE-algebras.

(5) A. Borumand Saeid studied in [3] the notion of *Smarandache weak BE-algebra*, as a BE-algebra $X$ in which there exists a proper subset $Q$ of $X$ such that: $(S_1)$ $1 \in Q$ and $\mid Q \mid \geq 2$; $(S_2)$ $Q$ is a CI-algebra under the operation of $X$. Another topic of research could be to define and investigate the notion of Smarandache weak pseudo BE-algebra.

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## References

[1] S. S. Ahn and K. S. So, *On ideals and upper sets in BE-algebras*, Scientiae Mathematicae Japonicae, **68(2)** (2008), 279-285.

[2] S. S. Ahn, Y. H. Kim and J. M. Ko, *Filters in commutative BE-algebras*, Communications of the Korean Mathematical Society, **27(2)** (2012), 233-242.

[3] A. Borumand Saeid, *Smarandache BE-algebras*, Education Publisher, Columbus, Ohio, USA, (2013).

[4] R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar and R. Ameri, *On pseudo BE-algebras*, Discussiones Mathematicae General Algebra and Applicationes, **33(1)** (2013), 95-108.

[5] R. A. Borzooei, A. Borumand Saeid and R. Ameri, *States on BE-algebras*, Kochi Journal of Mathematics, **9(1)** (2014), 27-42.

[6] R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar and R. Ameri, *Distributive pseudo BE−algebras*, Fasciculi Mathematici, **54(1)** (2015), 21-39.

[7] R. Cignoli and A. Torrens, *Glivenko like theorems in natural expansions of BCK-logic*, Mathematical Logic Quaterly, **50(2)** (2004), 111-125.

[8] R. Cignoli and A. Torrens, *Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation* $2(x^2) = (2x)^2$, Studia Logica, **83(1-3)** (2006), 157-181.

[9] Z. Ciloğlu and Y. Ceven, *Commutative and bounded BE-algebras*, Hindawi Publishing Corporation, **2013(1)** (2013), Article ID 473714.

[10] L. C. Ciungu and A. Dvurečenskij, *Measures, states and de Finetti maps on pseudo-BCK algebras*, Fuzzy Sets and Systems, **161(22)** (2010), 2870-2896.

[11] L. C. Ciungu, G. Georgescu and C. Muresan, *Generalized Bosbach states: part I*, Archive for Mathematical Logic, **52(3-4)** (2013), 335-376.

[12] L. C. Ciungu and J. Kühr, *New probabilistic model for pseudo-BCK algebras and pseudo-hoops*, Journal of Multiple-Valued Logic and Soft Computing, **20(3-4)** (2013), 373-400.

[13] L. C. Ciungu, *Non-commutative multiple-valued logic algebras*, Springer, Cham, Heidelberg, New York, Dordrecht, London, 2014.

[14] L. C. Ciungu, *Relative negations in non-commutative fuzzy structures*, Soft Computing, **18(1)** (2014), 15-33.

[15] A. Dvurečenskij and O. Zahiri, *Pseudo equality algebras: revision*, Soft Computing, doi: 10.1007/s00500-015-1888-x, (2015).

[16] G. Georgescu and A. Iorgulescu, *Pseudo MV-algebras*, Multiple-Valued Logic, **6(1-2)** (2001), 95-135.

[17] G. Georgescu and A. Iorgulescu, *Pseudo-BCK algebras: An extension of BCK-algebras*, Proceedings of DMTCS'01: Combinatorics, Computability and Logic, Springer, London, (2001), 97-114.

[18] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV*, Proceedings of the Japan Academy, **42(1)** (1966), 19-22.

[19] A. Iorgulescu, *Classes of pseudo-BCK algebras - Part I*, Journal of Multiple-Valued Logic and Soft Computing, **12(1-2)** (2006), 71-130.

[20] A. Iorgulescu, *Algebras of logic as BCK-algebras*, ASE Ed., Bucharest, 2008.

[21] H. S. Kim and Y. H. Kim, *On BE-algebras*, Scientiae Mathematicae Japonicae, **66(1)** (2007), 113-116.

[22] K. H. Kim and Y. H. Yon, *Dual BCK-algebra and MV-algebra*, Scientiae Mathematicae Japonicae, **66(2)** (2007), 247-254.

[23] J. Kühr, *Pseudo-BCK semilattices*, Demonstratio Mathematica, **40(3)** (2007), 495-516.

[24] J. Kühr, *Pseudo-BCK algebras and related structures*, Habilitation thesis, Palacký University in Olomouc, 2007.

[25] J. Kühr, *Commutative pseudo BCK-algebras*, Southeast Asian Bulletin of Mathematics, **33(3)** (2009), 451-475.

[26] K. J. Lee, *Pseudo-valuations on BE-algebras*, Applied Mathematical Sciences, **7(125)** (2013), 6199-6207.

[27] B. L. Meng, *CI-algebras*, Scientiae Mathematicae Japonicae, **71(1)** (2010), 11-17.

[28] B. L. Meng, *On filters in BE-algebras*, Scientiae Mathematicae Japonicae, **71(2)** (2010), 201-207.

[29] J. Rachůnek, *A non-commutative generalization of MV-algebras*, Czechoslovak Mathematical Journal, **52(127)** (2002), 255-273.
[30] A. Rezaei and A. Borumand Saeid, *On fuzzy subalgebras of BE-algebras*, Afrika Matematika, **22(2)** (2011), 115-127.
[31] A. Rezaei and A. Borumand Saeid, *Some results in BE-algebras*, Annals of Oradea University - Mathematics Fascicola, **XIX(1)** (2012), 33-44.
[32] A. Rezaei and A. Borumand Saeid, *Commutative ideals in BE-algebras*, Kyungpook Mathematical Journal, **52(4)** (2012), 483-494.
[33] A. Rezaei, A. Borumand Saeid and R. A. Borzooei, *Relation between Hilbert algebras and BE-algebras*, Applications and Applied Mathematics, **8(2)** (2013), 573-584.
[34] A. Rezaei, A. Borumand Saeid, A. Radfar and R. A. Borzooei, *Congruence relations on pseudo BE-algebras*, Annnals of the University of Craiova, Mathematics and Computer Sciences Series, **41(2)** (2014), 166-176.
[35] A. Rezaei, L. C. Ciungu and A. Borumand Saeid, *States on pseudo BE-algebras*, submitted.
[36] A. Walendziak, *On commutative BE-algebras*, Scientiae Mathematicae Japonicae, **69(2)** (2009), 281-284.
[37] A. Walendziak, *On normal filters and congruence relations in BE-algebras*, Commentationes Mathematicae, **52(2)** (2012), 199-205.
[38] H. Zhou and B. Zhao, *Generalized Bosbach and Riečan states based on relative negations in residuated lattices*, Fuzzy Sets and Systems, **187(1)** (2012), 33-57.

L. C. Ciungu, Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, Iowa 52242-1419, Usa
*E-mail address*: `lavinia-ciungu@uiowa.edu`