TOWARDS THE THEORY OF L-BORNOLOGICAL SPACES

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Abstract. The concept of an L-bornology is introduced and the theory of L-bornological spaces is being developed. In particular the lattice of all L-bornologies on a given set is studied and basic properties of the category of L-bornological spaces and bounded mappings are investigated.

1. Introduction and Motivation

In 1968, that is just 3 years after L. Zadeh has published his fundamental work [19], where the concept of a fuzzy set was introduced, C.L. Chang [4] defined the concept of a fuzzy topological space and started to develop the theory of fuzzy topological spaces. Later J.A. Goguen [6], [7] extended the concept of a fuzzy set to an L-set and, respectively, the concept of a fuzzy topological space to an L-topological spaces where L is a complete distributive lattice: the new concepts reduce to the former ones in case $L = [0, 1]$, that is the unit interval viewed as a lattice. At present the theory of L-topological spaces is probably the best developed field of mathematics in the context of L-sets (in particular, of fuzzy sets) and is considered not only as a qualitative generalization of general topology, but also is a certian contribution to the last one, see e.g. [10], [18].

In [11] [12] S.T. Hu studied the problem of the possibility to define the concept of boundedness in a topological space. To do this he introduced a system of axioms which later gave rise to the concept of a bornology and a bornological space, see e.g. [9]. In a certain sense from the analytic point of view a bornological space can be viewed as a counterpart of a topological space if one is mainly interested in the property of boundedness of mappings and not in their property of continuity. At the first stage of research bornological structures were mainly considered on Banach, or more general on linear topological spaces, but later the reserach was extended to topological spaces without linear structure, see e.g. [2].

Although, as it was mentioned, bornological structures are in a certain, conditional, sense dual to topological structures, as far as we know, up to now there was no any

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reserach of bornological-type structures in the context of L-sets (fuzzy sets). (It is worth mentioning however, that recently some papers where property of boundedness in *fuzzy vestor spaces* and bounded mappings of such spaces was considered, see e.g. [14] and [16].) It is the aim of this work to introduce the concept of an L-bornology, where L is a complete lattice, and related concepts and to develop some aspects of the theory of L-bornological spaces. Our main interest here is the lattice properties of the family of all L-bornologies on a given set (Section 4) and in the categorical aspects of the theory of L-bornological spaces (Section 5).

2. **Prerequisities**

Although in this work we use standard basic terminology from lattice theory, theory of L-sets (or fuzzy sets) and the category theory, for the readers convenience we specify definitions of those terms related to these theories, which are crucial for the work.

2.1. **Lattice Theory.** The standard reference books for this subject are [3], [5]. A partially ordered set $(L, \leq)$ is called a lattice if for any $a, b \in L$ there exists infimum $a \wedge b \in L$ and supremum $a \vee b \in L$. A lattice L is called complete if every subset $M \subseteq L$ has supremum $\bigvee M \in L$ and infimum $\bigwedge M \in L$. In particular, every complete lattice has bottom and top elements denoted respectively by 0 and 1. A complete lattice is called infinitely distributive if

$$\bigvee\{a_i \mid i \in I\} \wedge b \ = \bigvee\{a_i \wedge b \mid i \in I\}$$

and

$$\bigwedge\{a_i \mid i \in I\} \vee b \ = \bigwedge\{a_i \vee b \mid i \in I\}$$

for any $\{a_i \mid i \in I\} \ \subseteq L$ and $b \in L$.

A subset $M \subseteq L$ is called a lower set if $M = \downarrow M$ where

$$\downarrow M = \{a \in L \mid \exists b \in M \text{ such that } a \leq b\}.$$

A lower set $M \in L$ is called an ideal if

$$a, b \in M \Longrightarrow a \vee b \in M.$$

2.2. **L-sets.** The references concerning this topic a reader can find in the following early publications [19], [6], [4], [17].

Given a set X and a lattice L, an L-subset of X (or an L-fuzzy subset of X in alternative terminology) is a mapping $A : X \to L$. Let L^X stand for the family of all L-subsets of X, that is

$$L^X = \{A : X \to L\}.$$

L^X becomes a lattice $(L^X, \leq)$ by pointwise extending the lattice structure from $(L, \leq)$ to $(L^X, \leq)$. Besides, $(L^X, \leq)$ is complete if $(L, \leq)$ is complete and $(L^X, \leq)$ is infinitely distributive whenever $(L, \leq)$ is infinitely distributive. Given L-sets

$A, B \in L^X$ their supremum (join) $A \vee B$ and their infimum (meet) $A \wedge B$ are interpreted respectively as their union and intersection.

Given $\alpha \in L$, let α_X denote the constant function $\alpha_X : X \to L$ with value α. Further, given a point $x \in X$ and $\lambda \in L, \lambda \neq 0$, by an L-point in X we call a mapping $x^\lambda : X \to L$ such that

$$x^\lambda(y) = \begin{cases} 0 & \text{if } y \neq x \\ \lambda & \text{if } y = x \end{cases}$$

In this case x is called the suport of the L-point x^λ and λ is its value.

If $A \subseteq X$, then $\mathbf{1}_A$ denotes the characteristic function of A, that is

$$\mathbf{1}_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

In particular $\mathbf{1}_{\{x\}} = x^1$ is an L-point with support x and value 1.

Given a mapping $f : X \to Y$ and an L-set $A \in L^X$ its image $f(A) \in L^Y$ is defined by

$$f(A)(y) = \begin{cases} \sup_{f(x)=y} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Given a mapping $f : X \to Y$ and an L-set $B \in L^Y$ its preimage $f^{-1}(B) \in L^X$ is defined by $f^{-1}(B)(x) = (B \circ f)(x)$.

One can easily verify that the images and preimages of L-sets in respect of union and intersection operations behave in the same way as images and preimages of ordinary sets.

2.3. **Category Theory.** We assume that the reader is familiar with such basic concepts of the category theory as an object and a morphism in a category, a subcategory of a category and a functor from one category into another. On the other hand such concepts as the initial lift of a source and the final lift of a sink in a pair of categories to be defined in the general setting would need much additional terminology. Therefore they will be explained in our concrete situation at the appropriate places. The standard reference for the terminology related to the category theory are the books [8] and [1].

3. Basic Definitions

Let $L = (L, \leq)$ be an infinitely distributive complete lattice whose bottom and top elements are 0 and 1 respectively and let X be a set.

Definition 3.1. An *L-bornology* on a set X is a family $\mathcal{B} \subseteq L^X$ such that

(1) $\bigvee\{B \mid B \in \mathcal{B}\} = 1_X$;
(2) $B \in \mathcal{B}, C \in L^X, C \leq B \Longrightarrow C \in \mathcal{B}$;
(3) $B_1, B_2 \in \mathcal{B} \Longrightarrow B_1 \vee B_2 \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called *an L-bornological space* and L-sets $B \in \mathcal{B}$ are called *bounded* in this space.
An L-bornology $\mathcal{B} \subseteq L^X$ will be called *a strict L-bornology* if it satisfies the following stronger version of the first axiom:

(1′) $\mathbf{1}_{\{x\}} \in \mathcal{B} \ \forall\, x \in X$.

Remark 3.2. Obviously $(1') \Longrightarrow (1)$. If $1 \in L$ is isolated, that is $1 \neq \sup M$ whenever $1 \notin M \subset L$, then $(1') \Longleftrightarrow (1)$. For example $(1') \Longrightarrow (1)$ in case $L = \mathbf{2}$ is the two-point lattice $\{0, 1\}$. On the other hand generally (for example in case $L = [0,1]$), $(1) \not\Longrightarrow (1')$.

Remark 3.3. Note that conditions (2) and (3) say that an L-bornology on a set X is just an ideal in the lattice L^X while condition (1) is specific for this ideal.

Definition 3.4. Given two L-bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ a mapping $f : X \to Y$ is called *bounded* if

$$\forall B \in \mathcal{B}_X \implies f(B) \in \mathcal{B}_Y.$$

Obviously if mappings $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ and $g : (Y, \mathcal{B}_Y) \to (Z, \mathcal{B}_Z)$ are bounded, then their composition $g \circ f : (X, \mathcal{B}_X) \to (Z, \mathcal{B}_Z)$ is bounded as well. Besides the identity mapping $id_X : (X, \mathcal{B}_X) \to (X, \mathcal{B}_X)$ is bounded. Hence L-bornological spaces and bounded mappings between them form a category which will be denoted $\mathbf{BORN}(L)$ and called *the category of L-bornological spaces*.

Remark 3.5. In case $L = \mathbf{2}$ is a two-point lattice the concept of a $\mathbf{2}$-bornological space is obviously equivalent to the classical concept of a bornological space [9] (of course if we interpret a set $A \subseteq X$ as its characteristic function $\mathbf{1}_A$) and the category $\mathbf{BORN}(\mathbf{2})$ is actually the category of bornological spaces and bounded mappings.

4. Lattice of L-bornologies on a Set

Let a lattice L and a set X be fixed and let $\mathfrak{B}(X, L)$ stand for the family of all L-bornologies on the set X. We introduce a partial order $\preceq$ on $\mathfrak{B}(X, L)$ by setting

$$\mathcal{B}_1 \preceq \mathcal{B}_2 \Longleftrightarrow \mathcal{B}_2 \subseteq \mathcal{B}_1,\ \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}(X, L).$$

In this case we say that L-bornology $\mathcal{B}_2$ is stronger than L-bornology $\mathcal{B}_1$.

Our next aim is to show that $(\mathfrak{B}(X, L), \preceq)$ is a complete lattice.

First we describe the strongest and the weakest L-bornologies on X. To describe the strongest one we introduce the following notation.

Given a set $A \subseteq X$ and a mapping $\lambda : A \to L_0$, where $L_0 = L \setminus \{0\}$ let

$$Pt(A, \lambda) = \bigvee \left\{ x^{\lambda(x)} \mid x \in A \right\}.$$

In other words λ is a mapping, assigning to each point $x \in A$ the value $\lambda(x) \in L_0$, thus creating a fuzzy point $x^{\lambda(x)}$. Further, let

$$\mathrm{Pt}(A) = \{ Pt(A, \lambda) \mid \lambda \in (L_0 \setminus \{1\})^A \}$$

and

$$\mathrm{Pt}_1(A) = \{Pt(A,\lambda) \mid \lambda \in L_0^A\}.$$

If 1 is isolated in L, then

$$\mathcal{B}_\top := \{\mathrm{Pt}_1(A) \mid A \subseteq X, |A| < \aleph_0\}$$

is the strongest L-bornology, as well as the strongest strict L-bornology, on X. In case 1 is not isolated in L, then

$$\mathcal{B}_\top := \{\mathrm{Pt}_1(A) \mid A \subseteq X, |A| < \aleph_0\}$$

is still the strongest strict L-bornology on X, while the strongest L-bornology is given by

$$\mathcal{B}_\top := \{\mathrm{Pt}(A) \mid A \subseteq X, |A| < \aleph_0\}$$

The weakest L-bornology on X is given by

$$\mathcal{B}_\perp = L^X.$$

Now, to show that $\mathfrak{B}(X,L)$ is a complete lattice let

$$\mathfrak{B}_0 = \{\mathcal{B}_i \mid i \in I\} \subseteq \mathfrak{B}(X,L),$$

and let

$$\mathcal{B}_0 = \bigcap_{i\in I} \mathcal{B}_i.$$

One can easily notice that $\mathcal{B}_0$ is an L-bornology whenever all $\mathcal{B}_i, i \in I$, are L-bornologies. Besides if all $\mathcal{B}_i, i \in I$ are strict L-bornologies, then $\mathcal{B}_0$ is also strict. Hence $\mathcal{B}_0 =: \bigvee \mathfrak{B}_0$, that is $\mathcal{B}_0$ is the supremum of the family $\mathfrak{B}_0$ in the partially ordered set $\mathfrak{B}(X,L)$ of (strict) L-bornologies and hence $\mathfrak{B}(X,L)$ is a complete join-semilattice.

Since $\mathfrak{B}(X,L)$ is a complete join semilattice, to show that $\mathfrak{B}(X,L)$ is a complete lattice it is sufficient to show that finite meets exist in $\mathfrak{B}(X,L)$. However we prefer to give here an effective description of the infimum $\bigwedge \mathfrak{B}_0$ for an arbitrary family $\mathfrak{B}_0 \subseteq \mathfrak{B}(X,L)$. Namely, given a family $\mathfrak{B}_0 = \{\mathcal{B}_i \mid i \in I\} \subseteq \mathfrak{B}(X,L)$ let $\mathcal{D} = \bigcup_{i\in I} \mathcal{B}_i$, that is $\mathcal{D}$ is the union of all sets belonging to some $\mathcal{B}_i$:

$$\mathcal{D} = \{D \in L^X \mid \exists i \in I \text{ such that } D \in \mathcal{B}_i\}.$$

Obviously $\mathcal{D}$ satisfies condition (1) (respectively condition (1′)) whenever at least one $\mathcal{B}_i$ satisfies (1) (respectively (1′)), and $\mathcal{D}$ satisfies condition (2) whenever all $\mathcal{B}_i$ satisfy (2). Hence $\mathcal{D}$ is a base for an L-bornology $< \mathcal{D} >$ which is obtained by taking all finite suprema of L-sets belonging to $\mathcal{D}$:

$$< \mathcal{D} >= \left\{ \bigvee_{k=1}^{n} D_k \mid D_k \in \mathcal{D}, n \in \mathbb{N} \right\}.$$

From its construction it is clear that the L-bornology thus defined is less than or equal every L-bornology $\mathcal{B}_i, i \in I$ in the lattice $(\mathfrak{B}(X,L), \preceq)$ and besides it is the largest one with this property. Hence $\bigwedge \mathcal{B}_0 :=< \mathcal{D} >$ is the infimum of the family $\mathfrak{B}_0$ in the lattice $(\mathfrak{B}(X,L), \preceq)$.

To show that $\mathfrak{B}(X,L)$ is infinitely distributive let $\{\mathcal{B}_i \mid i \in I\} \subseteq \mathfrak{B}(X,L)$ and $\mathcal{A} \in \mathfrak{B}(X,L)$. We have to show that

$$\left(\bigvee_{i\in I} \mathcal{B}_i\right) \wedge \mathcal{A} \;=\; \bigvee_{i\in I} (\mathcal{B}_i \wedge \mathcal{A})\,.$$

The inequality

$$\left(\bigvee_{i\in I} \mathcal{B}_i\right) \wedge \mathcal{A} \;\geq\; \bigvee_{i\in I} (\mathcal{B}_i \wedge \mathcal{A})$$

holds in any lattice. Conversely, L-sets belonging to the left hand side L-bornology are of the form $B \vee A$ where $B \in \bigvee_{i\in I} \mathcal{B}_i = \bigcap_{i\in I} \mathcal{B}_i$, and $A \in \mathcal{A}$. Hence $B \in \mathcal{B}_i$ for every $i \in I$ and therefore $B \vee A \in \mathcal{B}_i \vee \mathcal{A}$ for every $i \in I$. However this just means that $B \vee A \in \bigvee_{i\in I} (\mathcal{B}_i \wedge \mathcal{A})$.

We conclude by collecting the information obtained above in the following

Theorem 4.1. *$\mathfrak{B}(X,L)$ is a complete infinitely distributive lattice. Its top and bottom elements are respectively $\mathfrak{B}_\top$ and $\mathfrak{B}_\perp$. Given a family $\mathfrak{B}_0 = \{\mathcal{B}_i \mid i \in I\}$ its supremum is $\bigcap_{i\in I} \mathcal{B}_i$ and its infimum is the family $< \mathcal{D} >$ of all finite joins of L-sets of the family*

$$\mathcal{D} = \bigcup \{B_i \mid B_i \in \mathcal{B}_i, i \in I\} \subseteq L^X.$$

5. Category Born (L)

5.1. **Final Structures in BORN(L).** Given an L-bornological space $(X, \mathcal{B}_X)$, a set Y and a mapping $f : (X, \mathcal{B}_X) \to Y$ we define the final L-bornological structure on Y as follows.

Let $Y' = f(X) \subseteq Y$. Further, let

$$\mathcal{C} = \{f(B_X) \mid B_X \in \mathcal{B}_X\}.$$

We show that $\mathcal{C}$ is an L-bornology on the set Y'. Indeed

$$\bigvee \{f(B_X) \mid B_X \in \mathcal{B}_X\} = f\left(\bigvee \{B_X \mid B_X \in \mathcal{B}_X\}\right) = f(X) = Y',$$

and hence $\mathcal{C}$ satisfies condition (1). Besides, $\mathcal{C}$ satisfies $(1')$ whenever $\mathcal{B}_X$ is a strict L-bornology. Indeed, given $y \in Y'$ let $x \in X$ be such that $f(x) = y$. Then $\mathbf{1}_{\{x\}} \in \mathcal{B}_X$ and hence $f(\mathbf{1}_x) = \mathbf{1}_{\{f(x)\}} \in \mathcal{C}$.

The family $\mathcal{C}$ satisfies (2). Indeed, let $C \in \mathcal{C}$. Then $C = f(B)$ for some $B \in \mathcal{B}_X$. Let now $D \leq C$, then $f^{-1}(D) \leq f^{-1}(f(B)) \leq B$, and hence $f^{-1}(D) \in \mathcal{B}_X$. Thus it is sufficient to notice that, by surjectivity of $f : X \to Y'$, $f(f^{-1}(D)) = D$ for all $D \in L^{Y'}$:

$$f(f^{-1}(D)(y) = \sup_{f(x)=y} f^{-1}(D)(x) = \sup_{f(x)=y} D(f(x)) = D(y)\ \forall y \in Y'.$$

The family $\mathcal{C}$ satisfies (3). Indeed, let $C_1, C_2 \in \mathcal{C}$. Then $f^{-1}(C_1) = B_1, f^{-1}(C_2) = B_2 \in \mathcal{B}_X$, and hence $f(B_1 \vee B_2) = f(B_1) \vee f(B_2) = C_1 \vee C_2 \in \mathcal{C}$.

Thus $\mathcal{C}$ is an L-bornology on Y'. We extend it to an L-bornology on Y as follows: Given $C \in \mathcal{C}$ let

$$\tilde{C}(y) = \begin{cases} C(y) & \text{if } y \in Y' \\ 0 & \text{if } y \notin Y' \end{cases}$$

Further, let

$$\mathcal{B}_Y = \{\tilde{C} \mid C \in \mathcal{C}\} \cup \{\tilde{C} \vee D \mid C \in \mathcal{C},\ D \in \mathrm{Pt}(A),\ A \subseteq Y \setminus Y', |A| < \aleph_0\}.$$

In other words, $\mathcal{B}_Y$ is defined as the family of all L-sets $E : L \to Y$ which are either of the form $\tilde{C}$ for some $C \in \mathcal{C}$ or are obtained as a join $\tilde{C} \vee D$ for some $C \in \mathcal{C}$ and some L-set $D : (Y \setminus Y') \to L$ with a finite support.

It is easy to notice that $\mathcal{B}_Y$ is an L-bornology on Y, and besides the strongest one for which the mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is bounded. Thus $\mathcal{B}_Y$ is the final L-bornology for $f : (X, \mathcal{B}_X) \to Y$. To obtain the final strict L-bornology, we obviously have to use $\mathrm{Pt}_1(A)$ instead of $\mathrm{Pt}(A)$.

Let now an L-bornological space $(Z, \mathcal{B}_Z)$, a bounded mapping $g : (X, \mathcal{B}_X) \to (Z, \mathcal{B}_Z)$ and a mapping $h : Y \to Z$ such that $g = h \circ f$ be given. Then the mapping $h : (Y, \mathcal{B}_Y) \to (Z, \mathcal{B}_Z)$ is bounded. Indeed if $B_Y \in \mathcal{B}_Y$, then there exist $B_X \in \mathcal{B}_X$ and a finite set $A \subseteq Y \setminus Y'$ such that $B_Y \le f(B_X) \vee \mathbf{1}_A$. Then

$$h(B_Y) \le h(f(B_X)) \vee h(\mathbf{1}_A) = g(B_X) \vee \mathbf{1}_{h(A)} \in \mathcal{B}_Z,$$

and hence $h(B_Y) \in \mathcal{B}_Z$. We conclude from here that $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is the final lift of $f : (X, \mathcal{B}_X) \to Y$. Referring now to Theorem 4.1 we can extend this result to a family of mappings:

Theorem 5.1. *Let $(X_i, \mathcal{B}_i)$, $i \in I$, be a family of L-bornological spaces and let Y be a set. Given a sink, that is a family of mappings,*

$$\{f_i : (X_i, \mathcal{B}_i) \to Y \ \mid\ i \in I\},$$

there exists a unique final lift

$$\{f_i : (X_i, \mathcal{B}_i) \to (Y, \mathcal{B}_Y) \ \mid\ i \in I\}.$$

Indeed, let $f_i : (X_i, \mathcal{B}_i) \to (Y, C_i)$ be the final lift for f_i. Further, let $\mathcal{B}_Y = \bigwedge_{i \in I} \mathcal{C}_i$ be the infimum of the family of L-bornologies $\mathcal{C}_i$ in $\mathfrak{B}(X, L)$ constructed in Theorem 4.1. From its construction it is easy to conclude that $\{f_i : (X_i, \mathcal{B}_i) \to (Y, \mathcal{B}_Y) \ \mid\ i \in I\}$ is indeed the final lift of the sink $\{f_i : (X_i, \mathcal{B}_i) \to Y \ \mid\ i \in I\}$. The uniqueness of the lift is obvious.

To illustrate applications of this theorem we describe the construction of co-product of L-bornological spaces.

Let $\{(X_i, \mathcal{B}_i) \mid i \in I\}$ be a family of L-bornological spaces and $X = \oplus_{i \in I} X_i$ be the disjoint sum of the sets X_i. Further, let $\{q_i : (X_i, \mathcal{B}_i) \to X,\ i \in I\}$ be the family of inclusions. Then the co-product of the family $\{(X_i, \mathcal{B}_i) \mid i \in I\}$ is the pair $(X, \mathcal{B})$ where $\mathcal{B}$ is the final L-bornology for this family of mappings. Efficiently it can be described as the family of all finite joins:

$$\bigvee_{k=1}^{n} \bar{B}_{i_k},\ B_{i_k} \in \mathcal{B}_{i_k}, n \in \mathbb{N}$$

where $\bar{B}_i : X \to L$ is defined by

$$\bar{B}_i(x) = \begin{cases} B_i(x) & \text{if } x \in X_i \\ 0 & \text{if } x \notin X_i \end{cases}$$

5.2. **Initial Structures.** Let $f : X \to (Y, \mathcal{B}_Y)$. We are looking for the weakest L-bornology $\mathcal{B}_X$ on X such that the mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is bounded. Let

$$\mathcal{C} := \{f^{-1}(B) \mid B \in \mathcal{B}_Y\}.$$

Note that $\mathcal{C}$ satisfies condition (1) whenever $\mathcal{B}_Y$ satisfies condition (1):

$$\bigvee \mathcal{C} = \bigvee \{f^{-1}(B) \mid B \in \mathcal{B}_Y\ \} = f^{-1}\left(\bigvee \{B \mid B \in \mathcal{B}_Y\}\right) = f^{-1}(Y) = X.$$

Note also that $\mathcal{C}$ satisfies (3), since $\mathcal{B}_Y$ satisfies (3). Thus to obtain an L-bornology on X we have to saturate the family $\mathcal{C}$ by all L-subsets of L-sets contained in $\mathcal{C}$, in other words to define $\mathcal{B}_X$ as the lower set of $\mathcal{C}$ in the lattice L^X:

$$\mathcal{B}_X = \downarrow \mathcal{C} = \{A \mid A \in L^X, \exists C \in \mathcal{C} \text{ such that } A \leq C\}.$$

From the construction it is clear that $\mathcal{B}_X$ is the smallest L-bornology on X in the lattice $(\mathfrak{B}(X, L), \preceq)$ for which f is bounded. Besides, note that

$$\mathbf{1}_{\{x\}} \leq f^{-1}(\mathbf{1}_{\{f(x)\}})$$

and hence $\mathcal{B}_X$ is a strict L-bornology whenever $\mathcal{B}_Y$ is a strict L-bornology.

Let now an L-bornological space $(Z, \mathcal{B}_Z)$, a bounded mapping $g : (Z, \mathcal{B}_Z) \to (Y, \mathcal{B}_Y)$ and a mapping $h : (Z, \mathcal{B}_Z) \to X$ be given such that $g = f \circ h$. Then the mapping $h : (Z, \mathcal{B}_Z) \to (X, \mathcal{B}_X)$ is bounded. Indeed, if $B_Z \in \mathcal{B}_Z$, then we have

$$(f \circ h)(B_Z) = g(B_Z) \in \mathcal{B}_Y$$

(by boundedness of g), and hence

$$h(B_Z) \leq f^{-1}(B_Y) \in \mathcal{B}_X, \text{ that is } h(B_Z) \in \mathcal{B}_X.$$

However this means that $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is the initial lift of $f : X \to (Y, \mathcal{B}_Y)$. The uniqueness of the lift is obvious.

Let now X be a set, $(Y_i, \mathcal{B}_i), i \in I$ be a family of L-bornological spaces and

$$f_i : X \to (Y_i, \mathcal{B}_i),\ i \in I,$$

a family of mappings. Further, let $\mathcal{C}_i$ be the initial L-bornology on the set X for a given mapping $f_i : X \to (Y_i, \mathcal{B}_i)$, and let

$$\mathcal{B}_X = \bigvee \{\mathcal{C}_i \mid i \in I\}$$

be the supremum of the family $\{\mathcal{C}_i \mid i \in I\}$ in the lattice $\mathfrak{B}(X, L)$.

Referring now to Theorem 4.1 we obtain

Theorem 5.2. *Let* $(Y_i, \mathcal{B}_i)$, $i \in I$, *be a family of* L*-bornological spaces, and* X *a set. Given a source, that is a family of mappings,* $\{f_i : X \to (Y_i, \mathcal{B}_i) \mid i \in I\}$, *there exists a unique initial lift* $\{f_i : (X_i, \mathcal{B}_i) \to (Y, \mathcal{B}_Y) \mid i \in I\}$.

To illustrate applications of this theorem we describe the construction of product for L-bornological spaces.

Let $\{(X_i, \mathcal{B}_i) \mid i \in I\}$ be a family of L-bornological spaces and $X = \prod_{i \in I} X_i$ be the product of the sets X_i. Further, let $\{p_i : X \to (X_i, \mathcal{B}_i) \mid i \in I\}$ be the family of projections. Then the product of the family $\{(X_i, \mathcal{B}_i) \mid i \in I\}$ is the pair $(X, \mathcal{B})$ where $\mathcal{B}$ is the initial L-bornology for the family $\{p_i : X \to (X_i, \mathcal{B}_i), \mid i \in I\}$. Efficiently it can be described as the family $\mathcal{B} = \downarrow \mathcal{C}$ where

$$\mathcal{C} = \{\prod_{i \in I} B_i \mid B_i \in \mathcal{B}_i\}$$

and $\downarrow \mathcal{C}$ is the lower set of $\mathcal{C}$, that is

$$\downarrow \mathcal{C} = \{A \in L^X \mid \exists B \in \mathcal{C} \text{ such that } A \leq B\}.$$

From Theorems 5.1 and 5.2 we obtain the following fundamental fact about the category of L-bornological spaces:

Theorem 5.3. *The category of* L*-bornological spaces and bounded mappings is topological over the category* **SET** *of sets and mappings with respect to the forgetful functor*

$$\mathfrak{F} : \mathbf{BORN}(L) \to \mathbf{SET}.$$

In case $L = \{0, 1\}$ is the two-point lattice we get the corollary for classical (that is crisp) bornological spaces:

Corollary 5.4. *The category of bornological spaces and bounded mappings is topological over the category* **SET** *with respect to the forgetful functor*

$$\mathfrak{F} : \mathbf{BORN} \to \mathbf{SET}.$$

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References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and concrete categories*, John Wiley & Sons, New York, 1990.

[2] G. Beer, *Metric bornologies and Kuratowska-Painleve convergence to the empty set*, Journal of Convex Analysis, **8** (2001), 273-289.

[3] G. Birkhoff, *Lattice theory*, AMS Providence, RI, 1995.

[4] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, Cambridge, 2003.

[6] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967) 145-174.

[7] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **18** (1973), 734-742.

[8] H. Herrlich and G. E. Strecker, *Category theory*, Heldermann Verlag, Berlin, 1987.

[9] H. Hogbe-Nlend, *Bornology and functional analysis*, Math. Studies, North-Holland, Amsterdam, **26** (1977).
[10] U. Höhle and A. Šostak, *Axiomatics for fixed-based fuzzy topologies*, Chapter 3 in [8].
[11] S. T. Hu, *Boundedness in a topological space*, *J. Math. Pures Appl.*, **28** (1949), 287-320.
[12] S. T. Hu, *Introduction to general topology*, Holden-Day, San-Francisko, 1966.
[13] B. Hutton, *Normality in fuzzy topologicl spaces*, J. Math. Anal. Appl., **50** (1975), 74-79.
[14] S. Q. Jiang and C. H. Yan, *Fuzzy bounded sets and totally fuzzy bounded sets in I-topological vector spaces*, Iranian Journal of Fuzzy Systems, **6(3)** (2009), 73-90.
[15] U. Höhle and S. E. Rodabaugh, eds., *Mathematics of fuzzy sets: logic, topology and measure theory*, Handbook Series, Kluwer Acad. Publ., **3** (1999).
[16] A. Narayaanan, S. Vijayabalaji and N. Thillaigovitidan, *Intuitionistic fuzzy linear bounded operators*, Iranian Journal of Fuzzy Systems, **4(1)** (2007), 89-93.
[17] C. V. Negoita and D. A. Ralescu, *Application of fuzzy sets to system analysis*, John Wiley & Sons, New York, 1975.
[18] S. E. Rodabaugh, *Powerset operator foundations for poslat fuzzy set theories and topologies*, Chapter 2 in [15].
[19] L. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

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