FUZZY MULTI-CRITERIA DECISION MAKING METHOD BASED ON FUZZY STRUCTURED ELEMENT WITH INCOMPLETE WEIGHT INFORMATION

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Abstract. The fuzzy structured element (FSE) theory is a very useful tool for dealing with fuzzy multi-criteria decision making (MCDM) problems by transforming the criterion value vectors of each alternative into the corresponding criterion function vectors. In this paper, some concepts related to function vectors are first defined, such as the inner product of two function vectors, the cosine of the included angle between two function vectors and the projection of a function vector on another. Then a method based on FSE is developed to solve fuzzy MCDM problems in which the criterion values take the form of general bounded closed fuzzy numbers and the criterion weight information is incomplete certain. In this method, the projections of criterion function vectors on the fuzzy ideal function point (FIFP) are used to rank all the alternatives and then select the most desirable one, and an optimization model is constructed to determine the weights of criteria according to the incomplete weight information. Finally, an example is given to illustrate the feasibility and effectiveness of the developed method.

1. Introduction

In many multi-criteria decision making (MCDM) problems, the decision information is usually imprecise or uncertain due to the complexity and uncertainty of objective things and the vagueness of human thinking. Accordingly, fuzzy set theory, introduced by Zadeh [44], is a very useful tool to be used to describe the imprecise or uncertain decision information. In the last decades, fuzzy MCDM has been receiving great attention from researchers [4, 7, 8, 20, 22, 24, 28, 32, 37, 40, 45]. A variety of fuzzy MCDM methods have been proposed, such as aggregation operator methods [3, 23, 24, 29, 30, 31, 32, 37, 38, 40, 46], TOPSIS (technique for order preference by similarity to ideal solution) method [2, 16, 35], compromise ratio method [21], GRA (grey related analysis) method [36, 47], expected value method [39], projection method [42], α cut set method [5, 6], etc. But in these methods, there still exist some shortcomings. First, some approaches precisely handle original fuzzy information, which is easy to result in loss of information and possibly leads to biased results [39, 42]. As a matter of fact, in the strict sense, these approaches have not drawn away from classic MCDM field. Second,

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sometimes, calculation process is troubled by parameter ergodicity problems. For example, the α cut set method [5, 6] requires the parameter α to take all values in $[0, 1]$, but it is actually not realistic. In addition, the comparison and sequencing of fuzzy numbers mainly rely on the relations of membership functions, but the formulas are complex, and moreover, some approaches for comparison between two fuzzy numbers do not satisfy the rational hypothesis of economic man [22].

The concept of fuzzy structured element (FSE) was introduced by Guo [9, 10]. In recent years, the FSE theory has made great progress [11-14], and has been applied in comparison of fuzzy numbers [15], fuzzy MCDM [25, 27], solution of fuzzy matrix games [43], multi-server fuzzy queues [48], fuzzy linear programming [26] and fuzzy linear differential systems [34]. Based on the homeomorphic property between bounded closed fuzzy number space and the family of bounded functions with the same monotone formal on $[-1, 1]$ [12, 13], the FSE theory can be used to handle fuzzy MCDM problems and some new fuzzy MCDM approaches were developed. For example, Liu and Guo [25] used the FSE theory to solve fuzzy multi-criteria group decision making problems, in which the criterion values, criterion weights and expert weights all take the form of triangular fuzzy numbers, and proposed two kinds of fuzzy compromised group decision making methods.

Among the aforementioned fuzzy MCDM methods, the criterion values generally take the form of interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, etc. However, in some situations, the criterion values may be expressed in general bounded closed fuzzy numbers. At present, regarding fuzzy MCDM problems where the criterion values take the form of general bounded closed fuzzy numbers, there is still little related research.

Moreover, in real MCDM process, it is difficult for the decision makers to give the precise values of criterion weights, or compare the importance degree between two criteria, making AHP (analytical hierarchy process), ANP (analytical network process), CNP (cognitive network process), etc., not capable of obtaining the weights of criteria. However, usually, the relation of weights can be given in the form of incomplete information, such as an interval for the criterion weights, an ordering between criteria or stating that a criterion is more important than another, difference between two criterion weights is greater than a real number, etc. [18, 19]. Thus, accordingly, incomplete information on criterion weights should be taken into account, and it is an interesting and important research issue.

In this paper, we shall focus on fuzzy MCDM problems in which the criterion values take the form of general bounded closed fuzzy numbers and the criterion weight information is incomplete certain, and develop a fuzzy MCDM method based on FSE with incomplete weight information. In order to do that, we organize the remainder of this paper as follows. Section 2 briefly reviews the FSE theory. Section 3 defines some concepts related to function vectors, such as the inner product of two function vectors, the cosine of the included angle between two function vectors and the projection of a function vector on another. Section 4 proposes a fuzzy MCDM method based on FSE with incomplete information on criterion weights. Section 5 gives an example to illustrate the validity of the proposed method. Finally, we conclude this paper in Section 6.

2. **Fuzzy Structured Element (FSE) Theory**

In the following, we briefly review the basic concepts and relations of FSE theory [9-13].

Definition 2.1. [9] Let R be the set of all real numbers, E be a fuzzy set in R, and $\mu_E(x)$ be the membership function of E. The fuzzy set E is called an FSE in R, if $\mu_E(x)$ satisfies the following conditions:

1) $\mu_E(0) = 1$;

2) $\mu_E(x)$ is monotone increasing and right continuous on $[-1, 0)$, and monotone decreasing and left continuous on $(0, 1]$;

3) $\mu_E(x) = 0$, $\forall x \in (-\infty, -1) \cup (1, +\infty)$.

Definition 2.2. [9] A FSE E is called regular if its membership function $\mu_E(x)$ satisfies the following conditions:

1) $\mu_E(x) > 0$, $\forall x \in (-1, 1)$;

2) $\mu_E(x)$ is strictly increasing and continuous on $[-1, 0)$, and strictly decreasing and continuous on $(0, 1]$.

Theorem 2.3. [9] *Let E be an FSE in R, whose membership function is $\mu_E(x)$. If $f(x)$ is a monotone bounded function on $[-1, 1]$, then $f(E)$ is a bounded closed fuzzy number in R, and the membership function of $f(E)$ is $\mu_E(f^{-1}(x))$, where $f^{-1}(x)$ is the rotational symmetry function of $f(x)$ (if $f(x)$ is a continuous and strictly monotone function, then $f^{-1}(x)$ is the inverse function of $f(x)$).*

Theorem 2.4. [9] *For any bounded closed fuzzy number $\tilde{A}$ and a given regular FSE E, there exists a monotone bounded function $f(x)$ on $[-1, 1]$ such that $\tilde{A} = f(E)$.*

For example, consider a triangular fuzzy number $\tilde{A} = (a^L, a^M, a^U)$, whose membership function is defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a^L)/(a^M - a^L), \ a^L \leqslant x \leqslant a^M \\ (a^U - x)/(a^U - a^M), \ a^M < x \leqslant a^U \\ 0, \ \text{otherwise} \end{cases} \tag{1}$$

Let E be a regular FSE with the membership function $\mu_E(x)$, where

$$\mu_E(x) = \begin{cases} 1 + x, \ \ -1 \leqslant x \leqslant 0 \\ 1 - x, \ 0 < x \leqslant 1 \\ 0, \ \text{otherwise} \end{cases} \tag{2}$$

If

$$f(x) = \begin{cases} (a^M - a^L)x + a^M, \ \ -1 \leqslant x \leqslant 0 \\ (a^U - a^M)x + a^M, \ 0 < x \leqslant 1 \end{cases} \tag{3}$$

then

$$f^{-1}(x) = \begin{cases} (x - a^M)/(a^M - a^L), \ a^L \leqslant x \leqslant a^M \\ (x - a^M)/(a^U - a^M), \ a^M < x \leqslant a^U \end{cases}$$

$$\begin{aligned}\mu_E(f^{-1}(x)) &= \begin{cases} 1+(x-a^M)/(a^M-a^L), \ a^L \leqslant x \leqslant a^M \\ 1-(x-a^M)/(a^U-a^M), \ a^M < x \leqslant a^U \\ 0, \ \text{otherwise} \end{cases} \\ &= \begin{cases} (x-a^L)/(a^M-a^L), \ a^L \leqslant x \leqslant a^M \\ (a^U-x)/(a^U-a^M), \ a^M < x \leqslant a^U \\ 0, \ \text{otherwise} \end{cases} \\ &= \mu_{\tilde{A}}(x)\end{aligned}$$

thus, by Theorem 2.3 and Theorem 2.4, we have

$$\tilde{A} = f(E)$$

That is, for any triangular fuzzy number $\tilde{A}$ and a given regular FSE E, there exists a monotone bounded function $f(x)$ on $[-1,1]$ such that $\tilde{A} = f(E)$.

Definition 2.5. [12, 13] Let $f(x)$ and $g(x)$ be two monotone functions. If they are both decreasing or increasing functions on $[-1,1]$, then $f(x)$ and $g(x)$ are called the same monotone formal functions.

For convenience, let $N_C(R)$ be the set of all bounded closed fuzzy numbers, and $B_{[-1,1]}$ be the set of all bounded functions with the same monotone formal on $[-1,1]$.

Definition 2.6. [12] Given a regular FSE E, the mapping H_E defined by

$$H_E : B_{[-1,1]} \to N_C(R)$$

$$f \in B_{[-1,1]} \to H_E(f) = f(E) \in N_C(R)$$

is called the fuzzy functional on $B_{[-1,1]}$ derived from the FSE E.

From Theorem 2.3 and Theorem 2.4, we know that H_E is a one-to-one mapping between $B_{[-1,1]}$ and $N_C(R)$. As such, Guo [12] proposed the homeomorphic property between the bounded closed fuzzy number space and the family of bounded functions with the same monotone formal on $[-1,1]$. Thus, the related study of bounded closed fuzzy numbers can be transformed into that of the corresponding bounded functions with the same monotone formal on $[-1,1]$.

3. Inner Product and Projection of Function Vectors

Consider that, by using the FSE theory to deal with fuzzy MCDM problems, the criterion value vectors of each alternative should be transformed into the corresponding criterion function vectors. Thus, in this section, based on the concept of inner product of two functions [1], we propose some new concepts related to function vectors, which will be needed in the following analysis.

Definition 3.1. [1] If f and g are two integrable functions on $[a,b]$, then the integral

$$\int_a^b f(x)g(x)dx \tag{4}$$

is called the inner product of f and g, and is denoted by $\langle f, g\rangle$. The nonnegative number $\langle f, f\rangle^{1/2}$, denoted by $\|f\|$, is called the norm of f.

The integral in (4) resembles the sum $\sum\limits_{k=1}^{n} x_k y_k$ which defines the dot product of two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. The function values $f(x)$ and $g(x)$ in (4) play the role of the components x_k and y_k, and integration takes the place of summation. The norm of f is analogous to the length or module of a vector.

The basic properties of inner products and norms are described in the next theorem.

Theorem 3.2. [1] *If f, g and h are integrable on $[a, b]$, and if λ is real, then we have:*

1) $\langle f, g\rangle = \langle g, f\rangle$ (commutativity)
2) $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ (linearity)
3) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$ (associativity)
4) $\|\lambda f\| = |\lambda|\,\|f\|$ (homogeneity)
5) $\|\langle f, g\rangle\| \leqslant \|f\|\,\|g\|$ (Cauchy-Schwarz inequality)
6) $\|f + g\| \leqslant \|f\| + \|g\|$ (triangle inequality)

Definition 3.3. [1] If f and g are two integrable functions on $[a, b]$, then the norm $\|f - g\|$ is called the distance between f and g, denoted by

$$d(f, g) = \|f - g\| = \left(\int_a^b (f(x) - g(x))^2 dx\right)^{1/2} \tag{5}$$

It is easy proved that the distance measure $d(f, g)$ satisfies the following properties for all integrable functions f, g and h on $[a, b]$:

1) $d(f, f) = 0$;
2) $d(f, g) > 0$ if $f \neq g$;
3) $d(f, g) = d(g, f)$;
4) $d(f, g) \leqslant d(f, h) + d(h, g)$.

Based on Definition 3.1, in what follows, we define some concepts related to function vectors, such as the inner product of two function vectors, the cosine of the included angle between two function vectors, and the projection of one function vector on another.

Definition 3.4. If $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ and $\mathbf{g} = (g_1, g_2, \cdots, g_n)$ are two function vectors, where f_j and g_j are integrable functions on $[a, b]$ $(j = 1, 2, \cdots, n)$, then we call

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_a^b \left(\sum_{j=1}^{n} f_j(x) g_j(x)\right) dx \tag{6}$$

the inner product of $\mathbf{f}$ and $\mathbf{g}$.

Furthermore, we call

$$\|\mathbf{f}\| = \langle \mathbf{f}, \mathbf{f} \rangle^{1/2} = \left(\int_a^b \left(\sum_{j=1}^n f_j^2(x) \right) dx \right)^{1/2} \tag{7}$$

the module of $\mathbf{f}$.

Similar to Theorem 3.2, by (6), the following theorem holds.

Theorem 3.5. *If* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{h}$ *are three function vectors, and if* λ *is real, then we have the following:*

1) $\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle$ *(commutativity)*
2) $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$ *(linearity)*
3) $\langle \lambda \mathbf{f}, \mathbf{g} \rangle = \lambda \langle \mathbf{f}, \mathbf{g} \rangle$ *(associativity)*
4) $\|\lambda \mathbf{f}\| = |\lambda| \, \|\mathbf{f}\|$ *(homogeneity)*
5) $\|\langle \mathbf{f}, \mathbf{g} \rangle\| \leqslant \|\mathbf{f}\| \, \|\mathbf{g}\|$ *(Cauchy-Schwarz inequality)*
6) $\|\mathbf{f} + \mathbf{g}\| \leqslant \|\mathbf{f}\| + \|\mathbf{g}\|$ *(triangle inequality)*

Definition 3.6. If $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ and $\mathbf{g} = (g_1, g_2, \cdots, g_n)$ are two function vectors, where f_j and g_j are integrable functions on $[a, b]$ $(j = 1, 2, \cdots, n)$, then the cosine of the included angle between $\mathbf{f}$ and $\mathbf{g}$, denoted by $\cos(\mathbf{f}, \mathbf{g})$, is defined as:

$$\cos(\mathbf{f}, \mathbf{g}) = \frac{\langle \mathbf{f}, \mathbf{g} \rangle}{\|\mathbf{f}\| \, \|\mathbf{g}\|} \tag{8}$$

Obviously, the cosine of the included angle between and satisfies the following properties.

1) $\cos(\mathbf{f}, \mathbf{g}) = \cos(\mathbf{g}, \mathbf{f})$;
2) $\cos(\mathbf{f}, \mathbf{g}) = 1$ if and only if $\mathbf{f} = \mathbf{g}$.

From (8), we know that the greater the value of $\cos(\mathbf{f}, \mathbf{g})$, the more similar the direction of $\mathbf{f}$ and $\mathbf{g}$.

Definition 3.7. If $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ and $\mathbf{g} = (g_1, g_2, \cdots, g_n)$ are two function vectors, where f_j and g_j are integrable functions on $[a, b]$ $(j = 1, 2, \cdots, n)$, then the projection of $\mathbf{f}$ on $\mathbf{g}$, denoted by $\mathrm{Prj}_{\mathbf{g}}\mathbf{f}$, is defined as follows:

$$\mathrm{Prj}_{\mathbf{g}}\mathbf{f} = \|\mathbf{f}\| \cos(\mathbf{f}, \mathbf{g}) = \frac{\langle \mathbf{f}, \mathbf{g} \rangle}{\|\mathbf{g}\|} \tag{9}$$

As we know, a vector is composed of direction and module [41]. However, $\cos(\mathbf{f}, \mathbf{g})$ only reflects the similarity degree of the direction of $\mathbf{f}$ and $\mathbf{g}$. From (9), we can further know that the projection of $\mathbf{f}$ on $\mathbf{g}$ can measure the similarity degree between $\mathbf{f}$ and $\mathbf{g}$ from the global point of view. The greater the value $\mathrm{Prj}_{\mathbf{g}}\mathbf{f}$, the closer $\mathbf{f}$ to $\mathbf{g}$.

4. Fuzzy MCDM Method Based on FSE with Incomplete Weight Information

In this section, based on the FSE theory, we propose a method to handle fuzzy MCDM problems, in which the criterion values take the form of general bounded closed fuzzy numbers and the information on criterion weights is incomplete certain.

For a fuzzy MCDM problem, let $X = \{X_1, X_2, \cdots, X_m\}$ be the set of m alternatives, and $I = \{I_1, I_2, \cdots, I_n\}$ be the set of n criteria. Assume that the characteristics of the alternatives X_i with respect to the criteria I_j are represented by the general bounded closed fuzzy numbers $\tilde{a}_{ij}$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$. Let $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$ be the fuzzy decision matrix.

In general, the decision makers need to determine the importance degrees of n criteria I_j $(j = 1, 2, \cdots, n)$. Suppose that the criterion weight information provided by the decision makers is incomplete certain, and may be presented in the following forms [18, 19], for $j \neq q$:

Form 1. A weak ranking: $w_j \geqslant w_q$;

Form 2. A strict ranking: $w_j - w_q \geqslant \eta_j$, $0 \leqslant \eta_j \leqslant 1$;

Form 3. A ranking of differences: $w_j - w_q \geqslant w_k - w_l$, $q \neq k \neq l$;

Form 4. A ranking with multiples: $w_j \geqslant \eta_j w_q$, $0 \leqslant \eta_j \leqslant 1$;

Form 5. An interval form: $\eta_j \leqslant w_j \leqslant \eta_j + \xi_j$, $0 \leqslant \eta_j < \eta_j + \xi_j \leqslant 1$.

For convenience, let Ω contained in R^n be the set of positive real vectors that fulfill the incomplete relative criterion weight information given by the decision makers.

To measure all criteria in dimensionless units, we need to normalize the decision matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$. Let I^b be the set of all benefit criteria and I^c be the set of all cost criteria. If $\tilde{a}_{ij}$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$ are triangular fuzzy numbers, then we can use the following formulas [39] to transform the decision matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n} = ((a_{ij}^L, a_{ij}^M, a_{ij}^U))_{m\times n}$ into the corresponding normalized decision matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{m\times n} = ((b_{ij}^L, b_{ij}^M, b_{ij}^U))_{m\times n}$:

$$\begin{cases} b_{ij}^L = a_{ij}^L \Big/ \max\limits_i \{a_{ij}^U\} \\ b_{ij}^M = a_{ij}^M \Big/ \max\limits_i \{a_{ij}^M\} \\ b_{ij}^U = \left\{ a_{ij}^U \Big/ \max\limits_i \{a_{ij}^L\} \right\} \wedge 1 \end{cases}, I_j \in I^b \tag{10}$$

$$\begin{cases} b_{ij}^L = \min\limits_i \{a_{ij}^L\} \Big/ a_{ij}^U \\ b_{ij}^M = \min\limits_i \{a_{ij}^M\} \Big/ a_{ij}^M \\ b_{ij}^U = \left\{ \min\limits_i \{a_{ij}^U\} \Big/ a_{ij}^L \right\} \wedge 1 \end{cases}, I_j \in I^c \tag{11}$$

The FSE employed in the transformation is common to all the elements of the matrix, and depends solely on the shape of the membership functions. To deal with the fuzzy MCDM problems using the FSE theory, we need to choose a regular FSE E, and transform the normalized decision matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{m\times n}$ into the corresponding bounded function matrix $\tilde{\mathbf{F}} = (f_{ij})_{m\times n}$, where f_{ij} are the bounded functions with same monotone formal on $[-1, 1]$ such that $\tilde{b}_{ij} = f_{ij}(E)$ $(i = 1, 2, \cdots, m;$ $j = 1, 2, \cdots, n)$.

For the convenience of depiction, based on $\tilde{\mathbf{F}} = (f_{ij})_{m\times n}$, we denote the i-th alternative X_i by

$$\mathbf{X}_i = (f_{i1}, f_{i2}, \cdots, f_{in}) \tag{12}$$

Let $f_j^+ = \max\limits_i \{f_{ij}\}$ $(j = 1, 2, \cdots, n)$, then we call

$$\mathbf{X}^+ = (f_1^+, f_2^+, \cdots, f_n^+) \tag{13}$$

the fuzzy ideal function point (FIFP).

If the information on criterion weights is completely known, that is, the weights $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T (w_j \geqslant 0, \sum\limits_{j=1}^{n} w_j = 1)$ of the criteria I_j $(j = 1, 2, \cdots, n)$ can be determined in advance, then, by (6) and (7), we denote

$$\begin{aligned}\|\mathbf{X}_i\|_{\mathbf{w}} &= \langle (w_1 f_{i1}, w_2 f_{i2}, \cdots, w_n f_{in}), ((w_1 f_{i1}, w_2 f_{i2}, \cdots, w_n f_{in})) \rangle \\ &= \left(\int_{-1}^{1} \left(\sum_{j=1}^{n} w_j^2 f_{ij}^2(x) \right) dx \right)^{1/2}\end{aligned} \tag{14}$$

as the weighted module of $\mathbf{X}_i$,

$$\begin{aligned}\left\|\mathbf{X}^+\right\|_{\mathbf{w}} &= \langle (w_1 f_1^+, w_2 f_2^+, \cdots, w_n f_n^+), (w_1 f_1^+, w_2 f_2^+, \cdots, w_n f_n^+) \rangle \\ &= \left(\int_{-1}^{1} \left(\sum_{j=1}^{n} w_j^2 \left(f_j^+(x) \right)^2 \right) dx \right)^{1/2}\end{aligned} \tag{15}$$

as the weighted module of $\mathbf{X}^+$, and

$$\begin{aligned}\langle \mathbf{X}_i, \mathbf{X}^+ \rangle_{\mathbf{w}} &= \langle (w_1 f_{i1}, w_2 f_{i2}, \cdots, w_n f_{in}), (w_1 f_1^+, w_2 f_2^+, \cdots, w_n f_n^+) \rangle \\ &= \int_{-1}^{1} \left(\sum_{j=1}^{n} w_j^2 f_{ij}(x) f_j^+(x) \right) dx\end{aligned} \tag{16}$$

as the weighted inner product of $\mathbf{X}_i$ and $\mathbf{X}^+$.

By (8), we introduce the weighted cosine of the included angle between $\mathbf{X}_i$ and $\mathbf{X}^+$ as:

$$\cos(\mathbf{X}_i, \mathbf{X}^+)_{\mathbf{w}} = \frac{\langle \mathbf{X}_i, \mathbf{X}^+ \rangle_{\mathbf{w}}}{\|\mathbf{X}_i\|_{\mathbf{w}} \|\mathbf{X}^+\|_{\mathbf{w}}} \tag{17}$$

Consider that $\cos(\mathbf{X}_i, \mathbf{X}^+)_{\mathbf{w}}$ only reflects the similarity degree of the direction of $\mathbf{X}_i$ and $\mathbf{X}^+$, thus, in order to measure the similarity degree between $\mathbf{X}_i$ and $\mathbf{X}^+$ from the global point of view, by (9), we then introduce a formula of weighted projection of $\mathbf{X}_i$ on $\mathbf{X}^+$:

$$\mathrm{Prj}_{\mathbf{X}^+} \mathbf{X}_i = \frac{\langle \mathbf{X}_i, \mathbf{X}^+ \rangle_{\mathbf{w}}}{\|\mathbf{X}^+\|_{\mathbf{w}}} = \frac{\int_{-1}^{1} \left(\sum\limits_{j=1}^{n} w_j^2 f_{ij}(x) f_j^+(x) \right) dx}{\left(\int_{-1}^{1} \left(\sum\limits_{j=1}^{n} w_j^2 \left(f_j^+(x) \right)^2 \right) dx \right)^{1/2}} \tag{18}$$

Obviously, the greater the value $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$, the closer $\mathbf{X}_i$ to $\mathbf{X}^+$, and thus, the closer the alternative $\mathbf{X}_i$ to the FIFP $\mathbf{X}^+$, that is, the better the alternative $\mathbf{X}_i$ is. Therefore, according to the weighted projections $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$, we can rank the alternatives $\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$, and then select the most desirable one.

4.1. **An Optimization Model for Determining Criterion Weights.** In the situations where the information on criterion weights provided by the decision makers is incomplete certain, in the following, we construct an optimization model to determine the weights of criteria according to the incomplete weight information.

On the one hand, for the criterion I_j, by (5), the deviation degree between the i-th alternative $\mathbf{X}_i$ and the FIFP $\mathbf{X}^+$ can be expressed as:

$$d_{ij}(\mathbf{X}_i, \mathbf{X}^+) = \left(\int_{-1}^{1} w_j^2 \big(f_j^+(x) - f_{ij}(x)\big)^2 dx \right)^{1/2} \tag{19}$$

Furthermore, for all alternatives $\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$, let $D_j(\mathbf{w})$ be the overall deviation between $\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$ and $\mathbf{X}^+$ with respect to the criterion I_j, we have

$$D_j(\mathbf{w}) = \sum_{i=1}^{m} d_{ij}(\mathbf{X}_i, \mathbf{X}^+) \tag{20}$$

To obtain the optimal criterion weights, one of the purposes is to minimize the sum of the overall deviation for all criteria I_j $(j = 1, 2, \cdots, n)$, then we can establish the following optimization model:

$$\begin{aligned} \text{Minimize: } \sum_{j=1}^{n} D_j(\mathbf{w}) &= \sum_{j=1}^{n}\sum_{i=1}^{m} \left(\int_{-1}^{1} w_j^2 \big(f_j^+(x) - f_{ij}(x)\big)^2 dx \right)^{1/2} \\ \text{Subject to: } & \mathbf{w} \in \Omega \\ & \sum_{j=1}^{n} w_j = 1 \\ & w_j \geqslant 0, j = 1, 2, \cdots, n \end{aligned} \tag{21}$$

On the other hand, considering that the real weights of different criteria are random variables, we can utilize the criterion weights' entropy $H(\mathbf{w})$ to describe the uncertainty of the criterion weight vector $\mathbf{w}$, which is defined as follows [33]:

$$H(\mathbf{w}) = -\sum_{j=1}^{n} w_j \ln w_j \tag{22}$$

Similarly, to obtain the optimal criterion weights, the other purpose is to maximize the criterion weights' entropy $H(\mathbf{w})$ [17]. To achieve this we can construct the following mathematical model:

$$\begin{aligned} \text{Maximize : } H(\mathbf{w}) = & -\sum_{j=1}^{n} w_j \ln w_j \\ \text{Subject to: } & \mathbf{w} \in \Omega \end{aligned}$$

$$\sum_{j=1}^{n} w_j = 1$$
$$w_j \geqslant 0, j = 1, 2, \cdots, n \tag{23}$$

In order to accomplish the above two purposes, we construct the following mathematical model:

$$\text{Minimize: } \vartheta \sum_{j=1}^{n} \sum_{i=1}^{m} \left(\int_{-1}^{1} w_j^2 \big(f_j^+(x) - f_{ij}(x)\big)^2 dx \right)^{1/2} + (1-\vartheta) \sum_{j=1}^{n} w_j \ln w_j$$
$$\text{Subject to: } \mathbf{w} \in \Omega$$
$$\sum_{j=1}^{n} w_j = 1$$
$$w_j \geqslant 0, j = 1, 2, \cdots, n \tag{24}$$

where ϑ $(0 < \vartheta < 1)$ is the balancing coefficient, which plays a role of balance between the purposes (21) and (23).

By solving (24), we derive the optimal solution $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$, and then we can utilize (18) to calculate the weighted projections $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$.

4.2. **A Fuzzy MCDM Procedure Based on FSE with Incomplete Weight Information.** Based on the analysis above, in the following, for the fuzzy MCDM problems with incomplete information on criterion weights, we develop a practical procedure to rank all the alternatives and then select the most desirable one.

Step 1: Normalize the fuzzy decision matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$ into the corresponding decision matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{m\times n}$.

Step 2: Choose a suitable regular FSE E to transform the normalized decision matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{m\times n}$ into the corresponding bounded function matrix $\tilde{\mathbf{F}} = (f_{ij})_{m\times n}$ such that $\tilde{b}_{ij} = f_{ij}(E)$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$. Denote the alternatives by $\mathbf{X}_i = (f_{i1}, f_{i2}, \cdots, f_{in})$ $(i = 1, 2, \cdots, m)$ and the FIFP by $\mathbf{X}^+ = (f_1^+, f_2^+, \cdots, f_n^+)$.

Step 3: Utilize (24) to construct a mathematical model and derive the optimal criterion weights $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$.

Step 4: Utilize (18) to calculate the weighted projections $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$.

Step 5: Rank the alternatives X_i $(i = 1, 2, \cdots, m)$ according to the values $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, \cdots, m)$, and then select the most desirable one.

5. Illustrative Example

In this section, we use a fuzzy MCDM problem of selecting the best investment alternative [23] to illustrate the application of the proposed method. It is worth pointing out that the criterion values of all alternatives in the example are given with triangular fuzzy numbers and the only reason to do so is to facilitate the calculus but the approach can employ any shape of fuzzy number.

To develop a new product, four investment alternatives X_1, X_2, X_3 and X_4 are drafted. In assessing the potential contribution of each alternative, four criteria are considered, I_1 (investment amount), I_2 (expected net present value), I_3 (venture profit value), and I_4 (risk loss value). Among the four criteria, I_2 and I_3 are benefit criteria, I_1 and I_4 are cost criteria. Suppose that the information on criterion weights is $\Omega = \left\{\mathbf{w} | \mathbf{w} = (w_1, w_2, w_3, w_4)^T,\ 0.15 \leqslant w_1 \leqslant 0.35,\ 0.20 \leqslant w_2 \leqslant 0.30,\ 0.10 \leqslant w_3 \leqslant 0.30,\ 0.20 \leqslant w_4 \leqslant 0.40,\ \sum\limits_{j=1}^{4} w_j = 1\right\}$, and the decision makers represent the characteristics of the alternatives X_i $(i = 1, 2, 3, 4)$ by the triangular fuzzy numbers $\tilde{a}_{ij}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ with respect to the criteria I_j $(j = 1, 2, 3, 4)$, listed in Table 1 (unit: ten thousand CNY (China Yuan)).

Based on the above information, give the most desirable investment alternative.

alternative	I_1	I_2	I_3	I_4
X_1	(5,6,7)	(3,4,5)	(4,5,6)	(0.4,0.5,0.6)
X_2	(9,10,11)	(5,5.5,6)	(5,5.5,6)	(1.4,1.7,2)
X_3	(5,5.5,6)	(4,4.5,5)	(3,3.5,4)	(0.8,0.9,1)
X_4	(8,9,10)	(3,3.5,4)	(3,3.5,4)	(0.5,0.6,0.7)

TABLE 1. Decision Matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{4\times4}$

5.1. **Decision Making Steps.** To get the best investment alternative, we will proceed as follows:

Step 1: Utilize (10) and (11) to normalize the decision matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{4\times4}$ into the corresponding decision matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{4\times4}$ (Table 2).

alternative	I_1	I_2	I_3	I_4
X_1	(0.71,0.92,1.00)	(0.50,0.73,1.00)	(0.67,0.91,1.00)	(0.67,1.00,1.00)
X_2	(0.45,0.55,0.67)	(0.83,1.00,1.00)	(0.83,1.00,1.00)	(0.20,0.29,0.43)
X_3	(0.83,1.00,1.00)	(0.67,0.82,1.00)	(0.50,0.64,0.80)	(0.40,0.56,0.75)
X_4	(0.50,0.61,0.75)	(0.50,0.64,0.80)	(0.50,0.64,0.80)	(0.57,0.83,1.00)

TABLE 2. Normalized Decision Matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{4\times4}$

Step 2: Utilize (2) and (3) to transform the normalized decision matrix $\tilde{\mathbf{B}}$ into the corresponding bounded function matrix $\tilde{\mathbf{F}} = (f_{ij})_{4\times4}$ (Table 3). Based on $\tilde{\mathbf{F}} = (f_{ij})_{4\times4}$, we denote the i-th alternative X_i by $\mathbf{X}_i = (f_{i1}, f_{i2}, f_{i3}, f_{i4})$ $(i = 1, 2, 3, 4)$, and the FIFP by $\mathbf{X}^+ = (f_1^+, f_2^+, f_3^+, f_4^+)$, where

$$f_1^+(x) = f_2^+(x) = f_3^+(x) = \begin{cases} 0.17x + 1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$$

$$f_4^+(x) = \begin{cases} 0.33x + 1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$$

Step 3: Utilize (24) to construct the following mathematical model:

$$\text{Minimize: } \vartheta(0.5405w_1^2 + 0.2380w_2^2 + 0.9097w_3^2 + 1.0429w_4^2)^{0.5} + (1-\vartheta)\sum_{j=1}^{3} w_j \ln w_j$$

$$\text{Subject to: } 0.15 \leqslant w_1 \leqslant 0.35$$

alternative	I_1	I_2
X_1	$\begin{cases} 0.21x+0.92, & -1 \leqslant x \leqslant 0 \\ 0.08x+0.92, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.23x+0.73, & -1 \leqslant x \leqslant 0 \\ 0.27x+0.73, & 0 < x \leqslant 1 \end{cases}$
X_2	$\begin{cases} 0.10x+0.55, & -1 \leqslant x \leqslant 0 \\ 0.12x+0.55, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.17x+1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$
X_3	$\begin{cases} 0.17x+1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.15x+0.82, & -1 \leqslant x \leqslant 0 \\ 0.18x+0.82, & 0 < x \leqslant 1 \end{cases}$
X_4	$\begin{cases} 0.11x+0.61, & -1 \leqslant x \leqslant 0 \\ 0.14x+0.61, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.14x+0.64, & -1 \leqslant x \leqslant 0 \\ 0.16x+0.64, & 0 < x \leqslant 1 \end{cases}$

alternative	I_3	I_4
X_1	$\begin{cases} 0.24x+0.91, & -1 \leqslant x \leqslant 0 \\ 0.09x+0.91, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.33x+1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$
X_2	$\begin{cases} 0.17x+1.00, & -1 \leqslant x \leqslant 0 \\ 1.00, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.09x+0.29, & -1 \leqslant x \leqslant 0 \\ 0.14x+0.29, & 0 < x \leqslant 1 \end{cases}$
X_3	$\begin{cases} 0.14x+0.64, & -1 \leqslant x \leqslant 0 \\ 0.16x+0.64, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.16x+0.56, & -1 \leqslant x \leqslant 0 \\ 0.19x+0.56, & 0 < x \leqslant 1 \end{cases}$
X_4	$\begin{cases} 0.14x+0.64, & -1 \leqslant x \leqslant 0 \\ 0.16x+0.64, & 0 < x \leqslant 1 \end{cases}$	$\begin{cases} 0.26x+0.83, & -1 \leqslant x \leqslant 0 \\ 0.17x+0.83, & 0 < x \leqslant 1 \end{cases}$

TABLE 3. Bounded function matrix $\tilde{\mathbf{F}} = (f_{ij})_{4\times 4}$

$$0.20 \leqslant w_2 \leqslant 0.30$$
$$0.10 \leqslant w_3 \leqslant 0.30$$
$$0.20 \leqslant w_4 \leqslant 0.40$$
$$w_1 + w_2 + w_3 + w_4 = 1$$

Let $\vartheta = 0.5$, by solving this model, we obtain the optimal criterion weights:

$$\mathbf{w} = (0.2626, 0.3000, 0.2240, 0.2134)^T$$

Step 4: Utilize (18) to calculate the weighted projections $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, 3, 4)$:

$$\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_1 = 0.6010,\ \mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_2 = 0.5256,$$
$$\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_3 = 0.5603,\ \mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_4 = 0.4778.$$

Step 5: Rank the alternatives X_i $(i = 1, 2, 3, 4)$ according to the values $\mathrm{Prj}_{\mathbf{X}^+}\mathbf{X}_i$ $(i = 1, 2, 3, 4)$:

$$X_1 \succ X_3 \succ X_2 \succ X_4.$$

Therefore, the most desirable investment alternative is X_1.

5.2. **Comparative Analysis and Discussion.** 1) Validity of the proposed method

To verify the validity of the method proposed in this paper, we adopt several other methods to illustrate this example, including TOPSIS method proposed by Jahanshahloo et al. [16], compromise ratio (CR) method by Li [21], GRA method by Wei [36], expected value (EV) method by Xu [39], aggregation operator (AO) method by Lin [23] and projection method by Yang [42]. The decision results are listed in Table 4.

method	X_1	X_2	X_3	X_4
TOPSIS method [16]	0.2123	0.1892	0.1968	0.1702
CA method [21] ($p = 2$)	1	$0.5932 - 0.4409\varepsilon$	$0.6970 - 0.2069\varepsilon$	0
GRA method [36] ($\rho = 0.5$)	0.5912	0.4665	0.5587	0.3782
EV method [39]	0.8453	0.7129	0.7641	0.6721
AO method [23]	0.3600	0.1300	0.2700	0.2400
projection method [42]	0.5068	0.4912	0.5010	0.5084

method	ranking result
TOPSIS method [16]	$X_1 \succ X_3 \succ X_2 \succ X_4$
CA method [21] ($p = 2$)	$X_1 \succ X_3 \succ X_2 \succ X_4$
GRA method [36] ($\rho = 0.5$)	$X_1 \succ X_3 \succ X_2 \succ X_4$
EV method [39]	$X_1 \succ X_3 \succ X_2 \succ X_4$
AO method [23]	$X_1 \succ X_3 \succ X_4 \succ X_2$
projection method [42]	$X_4 \succ X_1 \succ X_3 \succ X_2$

Table 4. Decision Results Obtained by Several Other Methods

From Table 4, we know that, our proposal, TOPSIS method by Jahanshahloo et al. [16], CR method by Li [21], GRA method by Wei [36] and EV method by Xu [39] have the same ranking results. This verifies the validity of our proposal. In addition, by using AO method by Lin [23], the ranking result of the alternatives is $X_1 \succ X_3 \succ X_4 \succ X_2$. Although the ranking result change in the last two positions, the most desirable investment alternative is also X_1.

2) Comparing with several other methods

In our proposal and several other methods, each decision procedure is straightforward and easy to implement on a computer. However, the definition of distance measure in TOPSIS method by Jahanshahloo et al. [16], the formula for the comparison between two fuzzy numbers in AO method by Lin [42], the compromise coefficient ε $(0 \leqslant \varepsilon \leqslant 1)$ in CR method by Li [21] and the identification coefficient ρ $(0 \leqslant \rho \leqslant 1)$ in GRA method by Wei [36] maybe influence the ranking results of the alternatives. EV method by Xu [39] and projection method by Yang [42] maybe result in loss of the original fuzzy information. For example, by using the projection method by Yang [42], the ranking result of the alternatives is $X_4 \succ X_1 \succ X_3 \succ X_2$, and there are changes in all positions. This is because the projection method by Yang [42] did not make full use of the information on membership functions of fuzzy numbers. Our proposal utilizes the FSE theory to transform the criterion value vectors of each alternative into the corresponding criterion function vectors, and utilizes the weighted projections of these function vectors on the FIFP to rank all the alternatives and then select the most desirable one, which can reflect the similarity degree between two alternatives from the global point of view and avoid losing the original decision information and comparing any two fuzzy numbers, and thus ensure the effectiveness of this method.

In addition, Our proposal, GRA method by Wei [36] and AO method by Lin [23] can derive the weights of criteria according to the incomplete information on criterion weights. GRA method by Wei [36] constructed a optimization model by using the largest degree of grey relation from the positive-ideal solution and the smallest degree of grey relation from the negative-ideal solution, and derived the weights of criteria $\mathbf{w} = (0.3500, 0.3000, 0.1500, 0.2000)^T$. AO method by Lin [23] established a linear programming model by using Hausdauff distance to measure the deviation degrees between two triangular fuzzy numbers, and obtained the optimal criterion weights $\mathbf{w} = (0.3000, 0.2000, 0.1000, 0.4000)^T$. Our proposal uses the overall deviation between all alternatives and the FIFP and the randomness of criterion weights to construct an optimization model, and determines the optimal criterion weights $\mathbf{w} = (0.2626, 0.3000, 0.2240, 0.2134)^T$. The models in GRA method by Wei [36] and AO method by Lin [23] are not complicated , but these two models did not take the randomness of criterion weights into account.

3) Exploring the influences of balancing coefficient ϑ on the decision making result

In order to illustrate the influences of balancing coefficient ϑ $(0 < \vartheta < 1)$ on the decision making result, we adopt the different values of ϑ, the ranking results are shown in Table 5.

The balancing coefficient ϑ plays a role of balance between the purposes (21) and (23). In general, ϑ can influence the ranking results of the alternatives. The decision makers can select the value of ϑ according to actual decision making problems, and further adopt the different values of ϑ for exploring the influences of ϑ on the ranking results of the alternatives. For a non-expert decision maker, he may adopt $\vartheta = 0.5$. In this paper, we adopt $\vartheta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.92, 0.94, 0.96, 0.98$, and obtain that all the ranking results are the same (Table 5).

ϑ	optimal criterion weight vectors	ranking results
$\vartheta = 0.1$	$\mathbf{w} = (0.2522, 0.2574, 0.2462, 0.2442)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.2$	$\mathbf{w} = (0.2545, 0.2662, 0.2419, 0.2374)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.3$	$\mathbf{w} = (0.2564, 0.2772, 0.2363, 0.2301)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.4$	$\mathbf{w} = (0.2582, 0.2909, 0.2296, 0.2213)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.6$	$\mathbf{w} = (0.2717, 0.3000, 0.2207, 0.2076)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.7$	$\mathbf{w} = (0.2827, 0.3000, 0.2165, 0.2008)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.8$	$\mathbf{w} = (0.2913, 0.3000, 0.2087, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.9$	$\mathbf{w} = (0.3014, 0.3000, 0.1986, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.92$	$\mathbf{w} = (0.3036, 0.3000, 0.1964, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.94$	$\mathbf{w} = (0.3059, 0.3000, 0.1941, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.96$	$\mathbf{w} = (0.3084, 0.3000, 0.1916, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$
$\vartheta = 0.98$	$\mathbf{w} = (0.3110, 0.3000, 0.1890, 0.2000)^T$	$X_1 \succ X_3 \succ X_2 \succ X_4$

Table 5. Influences of Balancing Coefficient ϑ on the Decision Making Result

6. **Conclusions**

The FSE theory is suitable to deal with fuzzy MCDM problems by transforming the criterion value vectors of each alternative into the corresponding criterion function vectors. In this paper, we have defined some concepts related to function vectors, such as the inner product of two function vectors, the cosine of the included angle between two function vectors and the projection of a function vector on another. Based on these, we have proposed a method based on FSE for solving fuzzy MCDM problems, in which the criterion values take the form of general bounded closed fuzzy numbers and the criterion weight information is incomplete certain. This method utilizes the weighted projections of the criterion function vectors on the FIFP to rank all the alternatives and then select the most desirable one, which is straightforward and easy to implement on a computer and can avoid losing or distorting the original decision information and comparing any two fuzzy numbers. Considering that the information on criterion weights is incomplete certain, we have established an optimization model to determine the optimal criterion weights. This model not only thinks about the overall deviation between all alternatives and the FIFP, but also takes the randomness of criterion weights into account. We have also given a practical example to illustrate the feasibility and effectiveness of the developed method. This paper enriches and develops the FSE theory and fuzzy MCDM theory. Aggregation operators, similarity measures and correlation measures of fuzzy numbers based on the FSE theory are our further work in the future.

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