DISCRETE TOMOGRAPHY AND FUZZY INTEGER PROGRAMMING

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ABSTRACT. We study the problem of reconstructing binary images from four projections data in a fuzzy environment. Given the uncertainly projections, we want to find a binary image that respects as best as possible these projections. We provide an iterative algorithm based on fuzzy integer programming and linear membership functions.

1. Introduction

Discrete Tomography (DT) deals with the reconstruction of digital images from their projections. The projections of an image are defined by the number of pixels of the colors on each line. Digital images are most commonly represented by integer matrices. The problem of reconstructing monocolored images (black and white) from two projections is well known [14]. However, image processing in general and image reconstruction in particular is often characterized by uncertain and possibly inconsistent information. Since fuzzy programming is considered appropriate for solving real-world problems it seems reasonable to apply fuzzy programming methods to the reconstruction problem.

In this paper, we propose a fuzzy approach for reconstructing binary images, which, instead of reconstructing projections exactly as in the deterministic reconstruction problem, recovers them as much as possible.

The remainder of this paper is organized as follows. First, in Section 2, we briefly review fuzzy set theory and fuzzy programming and then, in Section 3, we present the problem of discrete tomography and image reconstruction. In Section 4, we discuss the application of fuzzy programming in image reconstruction and finally conclude the last section with a summary of our results.

2. Fuzzy Programming

Fuzzy linear programming (FLP) can be defined as linear programming with uncertain parameters and where a violation of constraints is permitted upto a degree. FLP was introduced to handle real-world problems when the available information is not exact. Verdegay [16] classified fuzzy programming into the following four categories: (1) a crisp objective and fuzzy constraints, (2) a fuzzy objective and crisp constraints, (3) a fuzzy objective and fuzzy constraints, (4) fuzzy parameters

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or coefficients. In this paper, we adopt the fuzzy linear program model with a crisp objective and fuzzy constraints.

There are several methods for solving FLP problems [19, 15, 5, 10]. The methods of converting FLP into conventional mathematical programming seem to be the main approaches of resolution.

Fuzzy integer linear programming (FILP) can be considered the fuzzy linear programming where the variables are restricted to integer values. It has been studied by some authors [13, 1] and solved with relevant algorithms. Herrera and Verdegay [7] proposed three models for dealing with FILP problems and then showed how to reduce each of them to a crisp integer linear programming problem.

The conventional model of FILP with a crisp objective and fuzzy constraints can be stated as:

$$P \quad \begin{cases} max\, cx \\ s.t. \\ Ax \precsim b \\ x_i \in \mathbb{N} \quad i = 1, \ldots, n \end{cases}$$

where A is a $m \times n$ constraints matrix. For $i = 1, \ldots, m$, we denote the ith row of A by A_i, and the ith constraint of P by $A_i x \precsim b_i$.

The symbol $\precsim$, a fuzzy version of $\leq$, is called "fuzzy less than or equal to". It means that constraints may be violated, but, depending on the constraint, these violations may have different degrees of importance.

To handle fuzzy constraints, we first convert them into crisp constraints using suitable membership functions. Various types of membership functions [18, 19] have been proposed. However, a linear membership function is usually considered. The effect of the nonlinearity of the membership functions will be accumulated, not into the structure of the constraints (which shall remain linear) but into the right hand side of the constraints. Therefore, we define a linear membership function for the ith constraint as follows:

$$\mu_i(x) = \begin{cases} 1 & if\, A_i x \leq b_i \\ 1 - \frac{A_i x - b_i}{d_i} & if\, b_i \leq A_i x \leq b_i + d_i \\ 0 & if\, A_i x \geq b_i + d_i \end{cases} \qquad (1)$$

where $d_i > 0$, $i = 1, \ldots, m$ are subjectively chosen constants of admissible violations. $\mu_i(x)$ is the degree to which x satisfies the ith constraint.

In a deterministic program, each admissible solution is either optimal or not. However, in a fuzzy program each feasible solution has a degree (membership) of optimality, i.e. the set of optimal solutions is a fuzzy set. Thus solving the fuzzy program P consists in determining the membership function, $\lambda(x)$, of the set of optimal solutions.

We denote by $X_i(\alpha) = \{x \in R^n | \mu_i(x) \geq \alpha\}$ the $\alpha-$cut associated with the ith constraint of P. It is easy to verify that $X_i(\alpha) = \{x \in R^n | A_i x \leq b_i + (1 - \alpha) d_i\}$. We write $X_i(\alpha) = \{x \in R^n | \mu_i(x) \geq \alpha\}$ and, for simplicity, put $\alpha = 1 - \theta$.

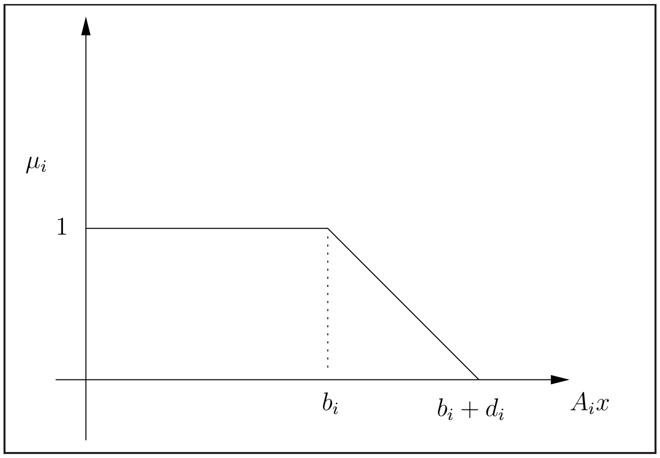

FIGURE 1. Membership of the Constraints

Compared with the number of methods of solving fuzzy linear programs, the literature on fuzzy integer programming [4, 7] is not abundant. Herrera and Verdegay [7] reduce the program P to the auxiliary parametric program, P_θ, where the variables belong to the ordinary set $X(\alpha)$.

$$P_\theta \quad \begin{cases} max\, cx \\ s.t. \\ A_i x \leq b_i + \theta d_i,\ i = 1, \ldots, m \\ x_i \in \mathbb{N} \quad i = 1, \ldots, n \end{cases}$$

Many approaches have been proposed for solving P_θ [2, 11]. In practice, we want to find the optimal solution of P_θ, for each θ or α, and also obtain the interval of θ for which this optimal solution remains optimal.

We denote by $S(\theta) = \{x \in R^n | x \text{ optimal for } P_\theta\}$. Then $S(\theta)$ is the set of optimal solutions for P_θ.

According to Orlovski [12], a fuzzy optimal solution of P is the fuzzy set defined by the membership function

$$\lambda(x) = \begin{cases} \sup_{x \in S(\theta)} \alpha & if\, x \in \cup_\theta S(\theta) \\ 0 & elsewhere \end{cases} \tag{2}$$

3. **Discrete Tomography and Deterministic Reconstruction**

A 2D discrete set is a finite subset of the integer lattice $\mathbb{Z}^2$ defined up to a translation. Discrete sets are most commonly represented by binary matrices or binary images. We suppose that the vertical axis is up-down directed and that the upper left corner is the position $(1, 1)$.

A lattice direction is a vector $v = (a, b) \in \mathbb{Z}^2 \backslash (0, 0)$ such that a and b are coprime. A lattice line in direction v is a line that is parallel to v and passes through at least one point of $\mathbb{Z}^2$. We denote by $\mathcal{L}_v$ the set of lattice lines in direction v. The projection of the discrete set A in the direction v is defined by the following function:

$\mathcal{P}_A^v : \mathcal{L}_v \to \mathbb{N}$ such that $\mathcal{P}_A^v(l) = |(i,j) \in \mathbb{Z}^2 : (i,j) \in A \cap l|$ for every lattice line l in direction v. In this work, we will focus mainly on the set of lattice directions $\{(1,0),(0,1),(1,1),(-1,1)\}$. In practice, the projections of a discrete set A (binary matrix) of size $m \times n$ in these directions are often represented by the vectors $H = (h_1,\dots,h_m)$, $V = (v_1,\dots,v_n)$, $D = (d_1,\dots,d_{m+n-1})$ and $A = (a_1,\dots,a_{m+n-1})$ respectively called the horizontal, the vertical, the diagonal and the antidiagonal projections. The elements of these vectors are:

$h_i = \sum_{j=1}^{n} A_{ij}, i = 1,\dots,m$ the number of ones on row i.
$v_j = \sum_{i=1}^{m} A_{ij}, j = 1,\dots,n$ the number of ones on column j.
$d_k = \sum_{m-i+j=k} A_{ij}, k = 1,\dots,m+n-1$ the number of ones on line $m-i+j=k$.
$a_k = \sum_{i+j=k+1} A_{ij}, k = 1,\dots,m+n-1$ the number of ones on line $i+j=k+1$.

The reconstruction problem of a binary image from its orthogonal projections is defined as follows: given two vectors H and V, decide whether there is at least one binary image whose horizontal projection is described by H and whose vertical projection is described by V. It is well known that this problem is polynomial [14]. However, the reconstruction problem of a binary images from three or more directions is NP-complete [6].

A deterministic formulation of the reconstruction problem of a binary images from the horizontal, the vertical, the diagonal and the antidiagonal projections (H,V,D,A) is as follows: The binary decision variable x_{ij} is equal to the value of the cell (i,j).

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = h_i, \quad i = 1,\dots,m \\ \sum_{i=1}^{m} x_{ij} = v_j, \quad j = 1,\dots,n \\ \sum_{m-i+j=k} x_{ij} = d_k, \quad k = 1,\dots,m+n-1 \\ \sum_{i+j=k+1} x_{ij} = a_k, \quad k = 1,\dots,m+n-1 \\ x_{ij} \in \{0,1\} \end{cases}$$

The four constraints guarantee the satisfaction of the four projections.

In general, the tomography reconstruction with a limited number of projections, appears as a highly underdetermined ill-posed problem. The projections data generated are initially noisy. A variety of deterministic binary images reconstruction methods have been considered in the literature, each using different noisy model and additional constraints[8, 3, 9]. To handle the reconstruction with errors in measurement and noise, we introduce the following equivalent integer linear program, called best-inner- fit (BIF), which is better suited to handle vagueness [17].

$$BIF \quad \begin{cases} max \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \\ \sum_{j=1}^{n} x_{ij} \le h_i, \quad i = 1,\dots,m \\ \sum_{i=1}^{m} x_{ij} \le v_j, \quad j = 1,\dots,n \\ \sum_{m-i+j=k} x_{ij} \le d_k, \quad k = 1,\dots,m+n-1 \\ \sum_{i+j=k+1} x_{ij} \le a_k, \quad k = 1,\dots,m+n-1 \\ x_{ij} \in \{0,1\} \end{cases}$$

4. **Combining Fuzzy Programming and Discrete Tomography**

We shall consider the reconstruction problem of images from four projections. The deterministic formulation of this problem has been presented in the previous section. The Fuzzy formulation problem is as follows:

$$FP \quad \begin{cases} max \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \\ \sum_{j=1}^{n} x_{ij} \precsim h_i, \quad i=1,\ldots,m \\ \sum_{i=1}^{m} x_{ij} \precsim v_j, \quad j=1,\ldots,n \\ \sum_{m-i+j=k} x_{ij} \precsim d_k, \quad k=1,\ldots,m+n-1 \\ \sum_{i+j=k+1} x_{ij} \precsim a_k, \quad k=1,\ldots,m+n-1 \\ x_{ij}\in\{0,1\} \end{cases}$$

To solve FP, we start by fuzzfying the constraints and choosing a membership function [7]. For all the constraints we suppose that the membership is linear and has the form (1). The permitted violations for constraints are either randomly chosen or fixed constants. Once the violations are fixed, we build the auxiliary parametric program:

$$FP_\theta \quad \begin{cases} max \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \\ \sum_{j=1}^{n} x_{ij} \leq h_i+\theta dh_i, \quad i=1,\ldots,m \\ \sum_{i=1}^{m} x_{ij} \leq v_j+\theta dv_j, \quad j=1,\ldots,n \\ \sum_{(m-i)+j=k} x_{ij} \leq d_k+\theta dd_k, \quad k=1,\ldots,m+n-1 \\ \sum_{i+j=k+1} x_{ij} \leq a_k+\theta da_k, \quad k=1,\ldots,m+n-1 \\ x_{ij}\in\{0,1\} \end{cases}$$

where dh_i and dv_j are the violations on row i and column j and dd_k and da_k are the violations on lines $m-i+j=k$ and $i+j=k+1$ respectively.

To find the fuzzy optimal solution to the program FP, we apply the approach of Bailey and Gillett [2] on the program FP_θ. Roughly speaking, this approach first sets $\theta=1$ and solves $FP_{(\theta=1)}$. Then it determines the minimal value of θ, say θ^* such that the optimal solution remains feasible. Thus the optimal solution for $FP_{(\theta=1)}$ will be optimal for the family of problems FP_θ for $\theta^*<\theta\leq 1$. Now it solves the program $FP_{(\theta^*-\epsilon)}$. The new optimal solution is tested as before, to determine the value of θ, say θ^{**}, which allows it to stay feasible. This optimal solution will be optimal in the interval $(\theta^{**},\theta^*]$. The procedure is iterated until $\theta\leq\epsilon$. Thus this approach gives the optimal solution of FP_θ for every value of θ. The parameter ϵ is called the step size of the method.

We denote by x^l the optimal solution of FP_θ when θ belongs to an interval $I_l, l=1,\ldots,L$ where L is the number of interval of θ needed to solve FP_θ and determine the membership degree, $\lambda(x^l)$ for x^l by (2). To get a crisp approximate solution x to FP, we introduce a cost matrix $c=\sum_{l=1}^{l} x^l$, i.e, the sum of all the fuzzy optimal solutions. The matrix x should be as near as possible to the matrix c. Hence x is an optimal solution of the following deterministic integer program:

$$DP \quad \begin{cases} max \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \sum_{j=1}^{n} x_{ij} \leq h_i, \quad i = 1, \dots, m \\ \sum_{i=1}^{m} x_{ij} \leq v_j, \quad j = 1, \dots n \\ \sum_{m-i+j=k} x_{ij} \leq d_k, \quad k = 1, \dots, m+n-1 \\ \sum_{i+j=k+1} x_{ij} \leq a_k, \quad k = 1, \dots, m+n-1 \\ x_{ij} \in \{0, 1\} \end{cases}$$

The reconstruction algorithm is as follows:

Heuristics Fuzzy-Reconstruction
Set $\theta = 1$; $\epsilon = 0.15$;$l = 1$ $c_{ij} = 1 \forall i, j$
While $\theta > 0$ **do**
  Compute x^l solution to FP_θ.
  Compute the smallest value θ^* for with x^l remains feasible for FP_θ.
  Set $\theta = \theta^* - \epsilon$, $c_{ij} = c_{ij} + x_{ij}^l \forall i, j$.
  $l = l + 1$.
End while
Compute x solution to DP.

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5. Computational Results

To simulate the noise in the process of measurement of the projection, we independently added a random noise from a normal distribution with average $\mu = 0$ and variance $\sigma^2 = 1$ to each perfect projection. All the violations were fixed and equal to 2.

The experiment was conducted as follows: We first selected a subset of images, determined the projections of each image and randomly added noise. Then we applied our heuristics to find the approximate images. Finally, we compared the results of our heuristics and the results obtained by solving the deterministic program BIF. The images in Table 1 are an example.

The main criterion for evaluating the performance of our heuristics is the ability to reconstruct binary images and to eliminate noise. Several measures of the difference between two binary images M and M' of size $m \times n$ have been proposed. Here we use the relative mean error:

$$RME = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} |M_i j - M'_i j|}{\sum_{i=1}^{m} \sum_{j=1}^{n} M_i j} 100\%.$$

Since, we have a stochastic heuristic, we repeated each test 10 times and estimated it by the mean of the 10 RME values. In general, the choice of the step ϵ is application-dependent. In our heuristic, we set $\epsilon = 0.15$. Table 1 shows an example of reconstruction. For each pair of original and reconstruction images, we give the optimal objective value of obtained by FP.

The quality of reconstruction is not very good because the number of projections is small. A more accurate reconstruction can be obtained by increasing the number of projections, but the problem will be harder to solve.

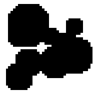
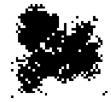
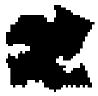
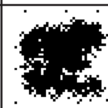
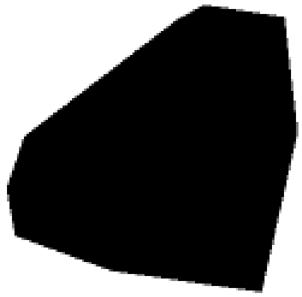
Size	Original image	Reconstruction
(40*40)		
(40*40)		
(128*128)		

Table 1. Original Images and their Reconstructions

Size	MRE (BIF)	MRE (FP)	CPU(FP)
40*40	21	20	127
40*40	26	25	134
128*128	6	5	200
256*256	4	3	480
256*256	4	3	625

Table 2. The Error Values Mre Measured on the Reconstructed Images

The results of our computational experiments are summarized in Table 2. The columns in order show the size of the original image, the MRE obtained for the program BIF, the MRE obtained by the fuzzy approach FP and the total CPU time in seconds for the program FP. All results were obtained using a PC with 3.8 Ghz processor and 512 Mbs of RAM.

Without any expert knowledge, the results obtained by both programs BIF and FP are quite similar. However, the program FP gives a smaller MRE in all cases in Table 2.

6. **Conclusion**

In this paper, we have introduced a new approach based on the use of fuzzy integer programming for reconstructing binary images using expert knowledge. The heuristics can be adapted to solve other cases of noisy projections when only some directions are noisy.

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