IRREDUCIBILITY ON GENERAL FUZZY AUTOMATA

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Abstract. The aim of this paper is the study of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, $\tilde{\delta}$-orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we obtain the relationships between them.

1. Introduction and Preliminaries

The concept of fuzzy automata was introduced by Wee in 1967 [12].
Let Σ be a set. A word in Σ is the product of a finite sequence of elements in Σ. Λ will denote the empty word and Σ^* the set of all words on Σ. The length $\ell(x)$ of the word $x \in \Sigma^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ will denote the set of all fuzzy sets on X and $P(X)$ will denote the set of all subsets on X.
A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the function f from $Q \times \Sigma$ into Q is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.
A word $x = x_1x_2 \dots x_n \in \Sigma^*$ is said to be accepted by A if there exist states $q_0, q_1, \dots, q_n$ satisfying
(1) $q_0 = s$
(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \dots, n$,
(3)$q_n \in T$.
The empty word is accepted by A if and only if $s \in T$.
A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the function f from $Q \times \Sigma$ into $P(Q)$ is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.
A fuzzy finite-state automaton (FFA) is a six-tuple $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the initial state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition, there is a membership value

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in $[0,1]$ called the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use $\delta(q_i, a_k, q_j)$ to refer both to a transition and its weight in the sense that whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, and otherwise, specifies the transition itself. The set of all transitions of $\tilde{F}$ will be denoted by Δ. The above definition is generally accepted as a formal definition of a fuzzy finite-state automaton [4, 5, 6, 7, 8, 9].

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [1].

In this paper, by using [2], [3], [10], [11] and [13], we define the concepts of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, $\tilde{\delta}$-orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we obtain the relationships between them.

Definition 1.1. [1] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \in \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function,
(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,
(viii) $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the fuzzy set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1.2. [1] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i.

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 1.3. [13] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

Also, if the input at time t_i be u_i, where $u_i \in \Sigma, \forall 1 \leq i \leq n$, then

$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,

$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$,

and recursively

$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$
$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\}$.

If $q \in Q_{act}(t_i)$, we should write q belongs to an element of Q_{act}. Hereafter, we simply denote it as: $q \in Q_{act}$.

2. Irreducibility on General Fuzzy Automata

Definition 2.1. Let $\tilde{F}_1^* = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1^*, F_1, F_2)$ and $\tilde{F}_2^* = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2^*, F_1, F_2)$ be max-min general fuzzy automata. Let η be a function of Q_2 onto Q_1

and let ξ be function of Σ_1 into Σ_2. Extend ξ to a function ξ^* of Σ_1^* into Σ_2^* by $\xi^*(\Lambda) = \Lambda$ and for all $x \in \Sigma_1^*$, $\xi^*(x) = \xi(x_1)\xi(x_2)...\xi(x_n)$ where $x = x_1x_2...x_n$ and $x_i \in \Sigma_1$, $i = 1, 2, ..., n$. Then (η, ξ) is called a covering of $\tilde{F}_1^*$ by $\tilde{F}_2^*$, written $\tilde{F}_1^* \leq \tilde{F}_2^*$, if and only if for all $q_2 \in Q_2$, $q_1 \in Q_1$, and $x \in \Sigma_1^*$,

$$\tilde{\delta}_1^*((\eta(q_2), \mu^t(\eta(q_2))), x, q_1) = \vee\{\tilde{\delta}_2^*((q_2, \mu^t(q_2)), \xi^*(x), s) : \eta(s) = q_1, s \in Q_2\}$$

Example 2.2. Let $\tilde{F}_1^* = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1^*, F_1, F_2)$ and $\tilde{F}_2 = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2^*, F_1, F_2)$ be general fuzzy automata such that $Q_1 = \{q_0, q_1\}$, $Q_2 = \{q_0', q_1'\}$, $\Sigma_1 = \Sigma_2 = \{a\}$, $\tilde{R}_1 = \{(q_0, \mu_1^{t_0}(q_0))\} = \{(q_0, 1)\}$, $\tilde{R}_2 = \{(q_1', \mu_2^{t_0}(q_1'))\} = \{(q_1', 1)\}$ $F_1(\mu, \delta) = Min(\mu, \delta)$, $Z = \emptyset$, ω and F_2 are not applicable, $\delta_1(q_0, a, q_1) = 0.3$, $\delta_1(q_1, a, q_1) = 0.6$, $\delta_2(q_0', a, q_0') = 0.7$, $\delta_2(q_1', a, q_0') = 0.3$. Define $\eta : Q_2 \longrightarrow Q_1$ by $\eta(q_0') = q_1$ and $\eta(q_1') = q_0$. Let ξ be identity map on $\Sigma_1 = \Sigma_2$.
Since η is one to one, if we show that

$$\tilde{\delta}_1^*((\eta(q_1'), \mu_1^{t_0}(\eta(q_1')), x, \eta(q_0')) = \tilde{\delta}_2^*((q_1', \mu_2^{t_0}(q_1')), x, q_0')$$

for all $x \in \Sigma_1^* = \Sigma_2^*$, then (η, ξ) is a covering of $\tilde{F}_1^*$ by $\tilde{F}_2^*$.

Now, for $\tilde{F}_1^*$, we have
$Q_{act}(t_0) = \tilde{R}_1 = \{(q_0, \mu_1^{t_0}(q_0))\} = \{(q_0, 1)\}, Q_{act}(t_i) = \{(q_1, \mu_1^{t_i}(q_1))\}, \forall i \geq 1$

$$\begin{aligned}
&\mu_1^{t_1}(q_1) = \tilde{\delta}_1((q_0, \mu_1^{t_0}(q_0)), a, q_1) = F_1(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_1)) = F_1(1, 0.3) = 0.3,\\
&\mu_1^{t_2}(q_1) = \tilde{\delta}_1((q_1, \mu_1^{t_1}(q_1)), a, q_1) = F_1(\mu_1^{t_1}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.6) = 0.3,\\
&\mu_1^{t_3}(q_1) = \tilde{\delta}_1((q_1, \mu_1^{t_2}(q_1)), a, q_1) = F_1(\mu_1^{t_2}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.3) = 0.3,\\
&\mu_1^{t_4}(q_1) = \tilde{\delta}_1((q_0, \mu_1^{t_3}(q_1)), a, q_1) = F_1(\mu_1^{t_3}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.3) = 0.3,\\
&\mu_1^{t_i}(q_1) = 0.3, \forall i \geq 5,\\
&\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), a, q_1) = 0.3,\\
&\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), aa, q_1) = 0.3 \wedge 0.3 = 0.3,\\
&\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), aaa, q_1) = 0.3 \wedge 0.3 \wedge 0.3 = 0.3,\\
&\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), a^n, q_1) = 0.3, \forall n \geq 4\ .
\end{aligned}$$

Also, for $\tilde{F}_2^*$, we have
$Q_{act}(t_0) = \tilde{R}_2 = \{(q_1', \mu_2^{t_0}(q_1'))\} = \{(q_1', 1)\}, Q_{act}(t_i) = \{(q_0', \mu_2^{t_i}(q_0'))\}, \forall i \geq 1$

$$\begin{aligned}
&\mu_2^{t_1}(q_0') = \tilde{\delta}_2((q_1', \mu_2^{t_0}(q_1')), a, q_0') = F_1(\mu_2^{t_0}(q_1'), \delta_2(q_1', a, q_0')) = F_1(1, 0.3) = 0.3,\\
&\mu_2^{t_2}(q_0') = \tilde{\delta}_2((q_0', \mu_2^{t_1}(q_0')), a, q_0') = F_1(\mu_2^{t_1}(q_0'), \delta_2(q_0', a, q_0')) = F_1(0.3, 0.7) = 0.3,\\
&\mu_2^{t_3}(q_0') = \tilde{\delta}_2((q_0', \mu_2^{t_2}(q_0')), a, q_0') = F_1(\mu_2^{t_2}(q_0'), \delta_2(q_0', a, q_0')) = F_1(0.3, 0.7) = 0.3,
\end{aligned}$$

$$\begin{aligned}
&\mu_2^{t_4}(q_0') = \tilde{\delta}_2((q_0', \mu_2^{t_3}(q_0')), a, q_0') = F_1(\mu_2^{t_3}(q_0'), \delta_2(q_0', a, q_0')) = F_1(0.3, 0.7) = 0.3,\\
&\mu_2^{t_i}(q_0') = 0.3, \forall i \geq 5,\\
&\tilde{\delta}_2^*((q_1', \mu_2^{t_0}(q_1')), a, q_0') = F_1(\mu_2^{t_0}(q_1'), \delta_2(q_1', a, q_0')) = F_1(1, 0.3) = 0.3,\\
&\tilde{\delta}_2^*((q_1', \mu_2^{t_0}(q_1')), aa, q_0')\\
&= \tilde{\delta}_2((q_1', \mu_2^{t_0}(q_1')), a, q_0') \wedge \tilde{\delta}_2((q_0', \mu_2^{t_1}(q_0')), a, q_0')\\
&= F_1(\mu_2^{t_0}(q_1'), \delta_2(q_1', a, q_0')) \wedge F_1(\mu_2^{t_1}(q_0'), \delta_2(q_0', a, q_0'))\\
&= F_1(1, 0.3) \wedge F_1(0.3, 0.7) = 0.3 \wedge 0.3 = 0.3,\\
&\tilde{\delta}_2^*((q_1', \mu_2^{t_0}(q_1')), a^n, q_0') = 0.3, \forall n \geq 3\ .
\end{aligned}$$

So for all $x \in \Sigma_1^* = \Sigma_2^*$, we have

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), x, q_1) = \tilde{\delta}_2^*((q_1', \mu_2^{t_0}(q_1')), x, q_0')$$

Thus (η, ξ) is a covering of $\tilde{F}_1^*$ by $\tilde{F}_2^*$.

Definition 2.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and let $\sim$ be an equivalence relation on Q. Then $\sim$ is called admissible relation on Q if and only if for all $p, q, r \in Q$, for all $a \in \Sigma$, if $p \sim q$ and $\tilde{\delta}((p, \mu^t(p)), a, r) > 0$, then there exists $s \in Q$ such that $\tilde{\delta}((q, \mu^t(q)), a, s) \geq \tilde{\delta}((p, \mu^t(p)), a, r)$ and $s \sim r$.

Theorem 2.4. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and let $\sim$ be an equivalence relation on Q. Then $\sim$ is an admissible relation on Q if and only if for all $p, q, r \in Q$, for all $x \in \Sigma^*$, if $p \sim q$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$, then there exists $s \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), x, s) \geq \tilde{\delta}^*((p, \mu^t(p)), x, r)$ and $s \sim r$.*

Proof. Let $\sim$ be an admissible relation on Q, $x \in \Sigma^*$, $|x| = n$, $p \sim q$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$. If $n = 0$, then $x = \Lambda$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$ implies that $p = r$. Thus $\tilde{\delta}^*((q, \mu^t(q)), x, q) = \tilde{\delta}^*((p, \mu^t(p)), x, p) = 1$ and $p \sim q$. Hence the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, where $n > 0$. Let $x = ya$, where $a \in \Sigma$. Now
$\tilde{\delta}^*((p, \mu^t(p)), x, r) = \tilde{\delta}^*((p, \mu^t(p)), ya, r)$

$$= \vee\{\tilde{\delta}^*((p, \mu^t(p)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, r) : s \in Q\} > 0$$

Thus there exists $u \in Q$ such that

$$\tilde{\delta}^*((p, \mu^t(p)), ya, r) = \tilde{\delta}^*((p, \mu^t(p)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r)$$

So $\tilde{\delta}^*((p, \mu^t(p)), y, u) > 0$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$. By the induction hypothesis, there exists $s \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), y, s) \geq \tilde{\delta}^*((p, \mu^t(p)), y, u)$ and $s \sim u$. Now, since $\sim$ is an admissible relation on Q, $s \sim u$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$, then there exists $s' \in Q$ such that $\tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((u, \mu^t(u)), a, r)$ and $s' \sim r$. So we have $\tilde{\delta}^*((q, \mu^t(q)), x, s') = \tilde{\delta}^*((q, \mu^t(q)), ya, s') \geq \tilde{\delta}^*((q, \mu^t(q)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((p, \mu^t(p)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r) = \tilde{\delta}^*((p, \mu^t(p)), ya, r) = \tilde{\delta}^*((p, \mu^t(p)), x, r)$ and $s' \sim r$.
The converse is trivial. $\square$

Definition 2.5. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $\wp = \{Q_1, Q_2, ..., Q_k\}$ be a partition of Q. Then $\wp$ is called an admissible partition of Q if let $a \in X$, $p_1, p_2 \in Q_i, \forall i, 1 \leq i \leq k$, $\tilde{\delta}((p_1, \mu^t(p_1)), a, r) > 0$, for some $r \in Q$, then there exists $s \in Q$ and there exists $j, 1 \leq j \leq k$ such that

$$\tilde{\delta}((p_2, \mu^t(p_2)), a, s) \geq \tilde{\delta}((p_1, \mu^t(p_1)), a, r)$$

and $s, r \in Q_j$.

Example 2.6. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton.
(i) Let $1_Q = \{\{q\} : q \in Q\}$. Then 1_Q is an admissible partition of Q.
(ii) $\{Q\}$ is an admissible partition of Q.

Theorem 2.7. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $\wp = \{Q_1, Q_2, ..., Q_k\}$ be a partition of Q. The following are equivalent:*
(i) $\wp$ is an admissible partition of Q
(ii) let $x \in \Sigma^*$, $p_1, p_2 \in Q_i, \forall i, 1 \leq i \leq k$, $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$, for some $r \in Q$, then there exists $s \in Q$ and there exists $j, 1 \leq j \leq k$ such that

$$\tilde{\delta}^*((p_2, \mu^t(p_2)), x, s) \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), x, r)$$

and $s, r \in Q_j$.

Proof. $(i) \Rightarrow (ii)$ Let $\wp$ be an admissible partition of Q, $x \in \Sigma^*$, $|x| = n$, $p_1, p_2 \in Q_i$ and $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$. If $n = 0$, then $x = \Lambda$ and $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$ implies that $p_1 = r$. Thus $\tilde{\delta}^*((p_2, \mu^t(p_2)), x, p_2) = \tilde{\delta}^*((p_1, \mu^t(p_1)), x, p_1) = 1$ and $p_1, p_2 \in Q_i$. Hence the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, where $n > 0$. Let $x = ya$, where $a \in \Sigma$. Now
$\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r)$

$$= \vee\{\tilde{\delta}^*((p_1, \mu^t(p_1)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, r) : s \in Q\} > 0$$

Thus there exists $u \in Q$ such that

$$\tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r)$$

So $\tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) > 0$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$. By the induction hypothesis, there exists $s \in Q$ and there exists j such that $\tilde{\delta}^*((p_2, \mu^t(p_2)), y, s) \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u)$ and $s, u \in Q_j$. Now, since $\wp$ is an admissible partition of Q, $s, u \in Q_j$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$, then there exists $s' \in Q$ and there exists l such that $\tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((u, \mu^t(u)), a, r)$ and $s', r \in Q_l$. So we have $\tilde{\delta}^*((p_2, \mu^t(p_2)), x, s') = \tilde{\delta}^*((p_2, \mu^t(p_2)), ya, s') \geq$
$\tilde{\delta}^*((p_2, \mu^t(p_2)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r)$
$= \tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), x, r)$ and $s', r \in Q_l$.
The proof of $(ii) \Rightarrow (i)$ is trivial. □

Theorem 2.8. *Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. Let $\Pi = \{H_1, H_2, ..., H_n\}$ be an admissible partition of Q. Define*

$$\tilde{\delta}^\Pi : (\Pi \times [0,1]) \times \Sigma \times \Pi \longrightarrow [0,1]$$

by

$$\tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_j) = \vee\{\tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j\}$$

where $H_i \in \Pi$, $q \in H_i$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), ..., \mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i, j = 1, 2, ..., m$. Then $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\Pi, F_1, F_2)$ is a general fuzzy automaton.

Proof. Let $q, q' \in H_i$, $a \in \Sigma$, $A = \{\tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j\}$ and
$B = \{\tilde{\delta}((q', \mu^t(q')), a, r) : r \in H_j\}$.
Suppose $\tilde{\delta}((q, \mu^t(q)), a, r) > 0$ for some $r \in H_j$. Since Π is an admissible partition of Q, there exists $r' \in H_j$ such that $\tilde{\delta}((q', \mu^t(q')), a, r') \geq \tilde{\delta}((q, \mu^t(q)), a, r)$. Similarly if $\tilde{\delta}((q', \mu^t(q)), a, r) > 0$ for some $r \in H_j$. Since Π is an admissible partition of Q,

there exists $r' \in H_j$ such that $\tilde{\delta}((q,\mu^t(q)),a,r') \geq \tilde{\delta}((q',\mu^t(q')),a,r)$. We show that $\tilde{\delta}((q,\mu^t(q)),a,r)=0$, for all $r \in H_j$ if and only if $\tilde{\delta}((q',\mu^t(q')),a,r)=0$, for all $r \in H_j$. Let $\tilde{\delta}((q,\mu^t(q)),a,r)=0$, for all $r \in H_j$ and let $\tilde{\delta}((q',\mu^t(q')),a,r)>0$ for some $r \in H_j$. By admissibility of Π, there exists $r' \in H_j$ such that $\tilde{\delta}((q,\mu^t(q)),a,r') \geq \tilde{\delta}((q',\mu^t(q')),a,r)$. Then $\tilde{\delta}((q',\mu^t(q')),a,r)=0$, which is a contradiction. Hence

$$\vee\{\tilde{\delta}((q,\mu^t(q)),a,r) : r \in H_j\} = \vee\{\tilde{\delta}((q',\mu^t(q')),a,r) : r \in H_j\}$$

So $\tilde{\delta}^{\Pi}$ is well defined. □

Corollary 2.9. *Let $\tilde{F}=(Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta},F_1,F_2)$ be a general fuzzy automaton and Π and τ be admissible partitions of Q. Consider the general fuzzy automata $\tilde{F}/\Pi=(\Pi,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^{\Pi},F_1,F_2)$ and $\tilde{F}/\tau=(\tau,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^{\tau},F_1,F_2)$. Define*

$$\tilde{\delta}^{\wedge} : ((\Pi\times\tau)\times[0,1])\times\Sigma\times(\Pi\times\tau) \longrightarrow [0,1]$$

by

$$\tilde{\delta}^{\wedge}(((H_i,K_j),\mu^t((H_i,K_j))),a,(H_u,K_v)) =$$
$$\tilde{\delta}^{\Pi}((H_i,\mu^t(H_i)),a,H_u) \wedge \tilde{\delta}^{\tau}((K_j,\mu^t(K_j)),a,K_v)$$

where $H_i,H_u \in \Pi$, $K_j,K_v \in \tau$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}),\mu^t(q_{i_2}),...,\mu^t(q_{i_m})\}$, $q_{i_s} \in H_i, s=1,2,...,m$, $\mu^t(k_j) \in \{\mu^t(q_{i'_1}),\mu^t(q_{i'_2}),...,\mu^t(q_{i'_m})\}$, $q_{i'_u} \in K_j, u=1,2,...,m'$ and

$$\mu^t((H_i,K_j)) \in \{(\mu^t(q_{i_1}),\mu^t(q_{i'_1})),(\mu^t(q_{i_2}),\mu^t(q_{i'_2})),...,(\mu^t(q_{i_m}),\mu^t(q_{i'_m}))\}.$$

Then $\tilde{F}/\Pi \wedge \tilde{F}/\tau = (\Pi\times\tau,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^{\wedge},F_1,F_2)$ is a general fuzzy automaton.

Theorem 2.10. *Let $\tilde{F}^*=(Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2)$ be a max-min general fuzzy automaton. Let $\Pi=\{H_i : i \in I\}$ be an admissible partition of Q. Then for $q \in H_i$*

$$\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j) = \vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_j\}$$

for all $H_i,H_j \in \Pi$, $x \in \Sigma^*$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}),\mu^t(q_{i_2}),...,\mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i, j=1,2,...,m$.

Proof. We show that
(i)$\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j) \leq \vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_j\}$,
for all $H_i,H_j \in \Pi$ and $x \in \Sigma^*$
(ii)$\tilde{\delta}^*((q,\mu^t(q)),x,r) \leq \tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)$, where $q \in H_i$, $r \in H_j$.

For the proof of (i), let $H_i,H_j \in \Pi$ and $x \in \Sigma^*$. Let $|x|=n$. If $n=0$, then $x=\Lambda$. If $H_i=H_j$, then $\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)=1$ and $\vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_j\} = \vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_i\} = \tilde{\delta}^*((q,\mu^t(q)),x,q)=1$, where $q \in H_i$. If $H_i \neq H_j$, then $\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)=0$ and $H_i \bigcap H_j=\phi$. Since $H_i \bigcap H_j=\phi$ and $q \in H_i$, $\vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_j\}=0$. So

$$\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j) = \vee\{\tilde{\delta}^*((q,\mu^t(q)),x,r) : r \in H_j\}$$

Hence the result is true for $n=0$. Suppose that the result is true for all $y \in \Sigma^*$ and $|y|=n-1$, $n>0$. Let $x=ya$, where $a \in \Sigma$. Now for $q \in H_i$ and $s \in H_k$, we have

$$\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)=\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),ya,H_j)$$

$$=\vee\{\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),y,H_k)\wedge\tilde{\delta}^{\Pi^*}((H_k,\mu^t(H_k)),a,H_j):H_k\in\Pi\}$$

$$=\vee\{(\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,r):r\in H_k\})\wedge(\vee\{\tilde{\delta}^*((s,\mu^t(s)),a,p):p\in H_j\}):H_k\in\Pi\}$$

$$=\vee\{\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,r)\wedge\tilde{\delta}^*((s,\mu^t(s)),a,p):r\in H_k,p\in H_j\}:H_k\in\Pi\}$$

$$\le\vee\{\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,r)\wedge\tilde{\delta}^*((r,\mu^t(r)),a,p):r\in H_k,p\in H_j\}:H_k\in\Pi\}$$

$$=\{\vee\{\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,r)\wedge\tilde{\delta}^*((r,\mu^t(r)),a,p):r\in H_k,H_k\in\Pi\}:p\in H_j\}$$

$$=\vee\{\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,r)\wedge\tilde{\delta}^*((r,\mu^t(r)),a,p):r\in Q\}:p\in H_j\}$$

$$=\vee\{\tilde{\delta}^*((q,\mu^t(q)),ya,p):p\in H_j\}$$

$$=\vee\{\tilde{\delta}^*((q,\mu^t(q)),x,p):p\in H_j\}$$

For the proof of (ii), let $q\in H_i$, $r\in H_j$, $x\in\Sigma^*$ and $|x|=n$. If $n=0$, then $x=\Lambda$. If $q=r$, then $H_i=H_j$ and $\tilde{\delta}^*((q,\mu^t(q)),x,r)=1=\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)$. Suppose $q\neq r$. Then $\tilde{\delta}^*((q,\mu^t(q)),x,r)=0\le\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)$. If $n=1$, then $x=a\in\Sigma$ and $\tilde{\delta}^*((q,\mu^t(q)),a,r)=\tilde{\delta}((q,\mu^t(q)),a,r)\le\vee\{\tilde{\delta}((q,\mu^t(q)),a,r):r\in H_j\}=\tilde{\delta}^{\Pi}((H_i,\mu^t(H_i)),a,H_j)=\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),a,H_j)$.
Hence the result is true for $n=0$ and $n=1$. Suppose that the result is true for all $y\in\Sigma^*$ and $|y|=n-1$, $n>0$. Let $x=ya$, where $a\in\Sigma$. Then

$$\tilde{\delta}^*((q,\mu^t(q)),x,r)=\tilde{\delta}^*((q,\mu^t(q)),ya,r)$$

$$=\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,p)\wedge\tilde{\delta}((p,\mu^t(p)),a,r):p\in Q\}$$

$$=\vee\{\vee\{\tilde{\delta}^*((q,\mu^t(q)),y,p)\wedge\tilde{\delta}((p,\mu^t(p)),a,r):p\in H_k\}:H_k\in\Pi\}$$

$$\le\vee\{\vee\{\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),y,H_k)\wedge\tilde{\delta}^{\Pi}((H_k,\mu^t(H_k)),a,H_j):p\in H_k\}:H_k\in\Pi\}$$

$$=\vee\{\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),y,H_k)\wedge\tilde{\delta}^{\Pi}((H_k,\mu^t(H_k)),a,H_j):H_k\in\Pi\}$$

$$=\tilde{\delta}^{\Pi^*}((H_i,\mu^t(H_i)),x,H_j)$$

□

Definition 2.11. Let $\tilde{F}=(Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta},F_1,F_2)$ be a general fuzzy automaton and Π and τ be admissible partitions of Q. Π and τ are called $\tilde{\delta}$- orthogonal if
(i) $\Pi\cap\tau=1_Q$,
(ii) For all $H_i,H_u\in\Pi$, $K_j,K_v\in\tau$, $a\in\Sigma$, if $H_i\cap K_j=\{q_0\}$ and $H_u\cap K_v=\{p_0\}$, then
$\tilde{\delta}((q_0,\mu^t(q_0)),a,p_0)$
$$=\vee\{\tilde{\delta}((q_0,\mu^t(q_0)),a,p)\wedge\tilde{\delta}((q_0,\mu^t(q_0)),a,p'):p\in H_u,p'\in K_v\}$$

Theorem 2.12. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π and τ be admissible partitions of Q. Then Π and τ are $\tilde{\delta}$-orthogonal if and only if*
(i) $\Pi \cap \tau = 1_Q$,
(ii) For all $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $x \in \Sigma^*$, *if* $H_i \cap K_j = \{ q_0\}$ *and* $H_u \cap K_v = \{ p_0\}$, *then*
$\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0)$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\}$$

Proof. Let Π and τ are $\tilde{\delta}$-orthogonal. Then clearly (i) holds.
For the proof of (ii), let $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $x \in \Sigma^*$, $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$. Suppose $|x| = n$. If $n = 0$, then $x = \Lambda$. If $q_0 \neq p_0$, then $\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 0$, $q_0 \notin H_u$ and $q_0 \notin K_v$. So $q_0 \neq p, \forall p \in H_u$ and $q_0 \neq p', \forall p' \in K_v$. Hence

$$\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\} = 0.$$

Suppose $q_0 = p_0$. Then $\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 1$, $H_i = H_u$ and $K_j = K_v$. Thus

$$\begin{aligned}&\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\}\\ = &\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 1.\end{aligned}$$

Hence if $n = 0$, then
$\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0)$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\}$$

Suppose that the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$. Then
$\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = \tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p_0)$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p_0) : r \in Q\}$$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge (\vee\{\tilde{\delta}^*((r, \mu^t(r)), a, p) \wedge$$

$$\tilde{\delta}^*((r, \mu^t(r)), a, p') : p \in H_u, p' \in K_v\}) : r \in Q\}$$

$$= \vee\{(\vee(\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p))) \wedge (\vee(\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p'))) : p \in H_u, p' \in K_v, r \in Q\}$$

$$= \vee\{(\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) : r \in Q\})\wedge$$

$$(\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p') : r \in Q\}) : p \in H_u, p' \in K_v\}$$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p') : p \in H_u, p' \in K_v\}$$

$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\}.$$

The converse is trivial. □

Theorem 2.13. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π and τ be admissible partitions of Q that are $\tilde{\delta}$-orthogonal. Then $\tilde{F}^* \leq \tilde{F}^*/\Pi \wedge \tilde{F}^*/\tau$.*

Proof. Define $\eta : \Pi \times \tau \longrightarrow Q$ by $\eta((H_i, K_j)) = q_0$ where $H_i \cap K_j = \{q_0\}$. Since Π and τ are $\tilde{\delta}$-orthogonal, η is one-to-one. Let ξ be identity map of Σ into Σ. Let $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$ and $x \in \Sigma^*$. Suppose $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$. Then

$$\tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j)))), x, \eta((H_u, K_v)))$$
$$= \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0)$$

Now, since Π and τ are $\tilde{\delta}$-orthogonal, we have

$$\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v))$$
$$= \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_u) \wedge \tilde{\delta}^{\tau^*}((K_j, \mu^t(K_j)), x, K_v)$$
$$= (\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) : p \in H_u\}) \wedge (\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p' \in K_v\})$$
$$= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \{tilde\delta^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\}$$
$$= \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0)$$

Thus

$$\tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j)))), x, \eta((H_u, K_v))) =$$
$$\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v))$$

Now, since η is one-to-one, we have

$$\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v))$$
$$= \vee\{\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_r, K_s)) :$$
$$\eta((H_r, K_s)) = \eta((H_u, K_v)), (H_r, K_s) \in \Pi \times \tau\}$$

Hence

$$\tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j)))), x, \eta((H_u, K_v)))$$
$$= \vee\{\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_r, K_s)) :$$
$$\eta((H_r, K_s)) = \eta((H_u, K_v)), (H_r, K_s) \in \Pi \times \tau\}$$

Consequently, $\tilde{F}^* \leq \tilde{F}^*/\Pi \wedge \tilde{F}^*/\tau$. □

Definition 2.14. Let $\tilde{F}_1 = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1, F_1, F_2)$ and $\tilde{F}_2 = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2, F_1, F_2)$ be general fuzzy automata. Let Ω be a function of $Q_2 \times \Sigma_2$ into Σ_1 and $Q = Q_1 \times Q_2$. Define

$$\tilde{\delta}^{\Omega} : Q \times ([0,1] \times [0,1]) \times \Sigma_2 \times Q \longrightarrow [0,1]$$

by

$$\tilde{\delta}^{\Omega}(((q_1, q_2), \mu^t((q_1, q_2))), b, (p_1, p_2)))$$
$$= \tilde{\delta}_1((q_1, \mu^t(q_1)), \Omega(q_2, b), p_1) \wedge \tilde{\delta}_2((q_2, \mu^t(q_2)), b, p_2)$$

where $\mu^t((q_1, q_2)) = (\mu^t(q_1), \mu^t(q_2))$. Then $\tilde{F} = (Q', \Sigma_2, \tilde{R}, Z, \omega, \tilde{\delta}^{\Omega}, F_1, F_2)$ is a general fuzzy automaton and write $\tilde{F} = \tilde{F}_1 \Omega \tilde{F}_2$.

Theorem 2.15. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π be an admissible partition of Q. If there exists an admissible partition τ of Q such that Π and τ are $\tilde{\delta}$-orthogonal, then there exists a general fuzzy automaton $\tilde{F}'$ such that $\tilde{F}^* \leq \tilde{F}' \Omega \tilde{F}^*/\tau$.*

Proof. Let $\Pi = \{H_1, H_2, ..., H_n\}$ and $\tau = \{K_1, K_2, ..., K_m\}$ be $\tilde{\delta}$-orthogonal partitions of Q. Let $\tilde{F}' = (\tau, \Pi \times \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}', F_1, F_2)$, where

$$\tilde{\delta}'((K_j, \mu^t(K_j)), (H_i, a), K_v) = \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), a, p) : p \in K_v\}$$

and $H_i \cap K_j = \{q_0\}$. Since τ is admissible, $\tilde{\delta}'$ is well defined.
Define $\Omega : \Pi \times \Sigma \longrightarrow \Pi \times \Sigma$ to be the identity map and define $\eta : \tau \times \Pi \longrightarrow Q$ by $\eta((k_j, H_i)) = q_0$ where $H_i \cap K_j = \{q_0\}$. Then η is one-to-one and onto. Let ξ be identity map of Σ into Σ. Let $x = \Lambda$. Since η is one-to-one, so we have

$$\tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), \Lambda, \eta((K_v, H_u))) = 1 \Longleftrightarrow$$
$$\eta((K_j, H_i)) = \eta((K_v, H_u)) \Longleftrightarrow (K_j, H_i) = (K_v, H_u) \Longleftrightarrow$$
$$\tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), \Lambda, (K_v, H_u)) = 1$$

Let $H_u \cap K_v = \{p_0\}$ and $a \in \Sigma$. Since Π and τ are $\tilde{\delta}$-orthogonal, so we have

$$\tilde{\delta}((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), a, \eta((K_v, H_u))) = \tilde{\delta}((q_0, \mu^t(q_0)), a, p_0)$$
$$= \vee\{\tilde{\delta}((q_0, \mu^t(q_0)), a, p') \wedge \tilde{\delta}((q_0, \mu^t(q_0)), a, p) : p' \in K_v, p \in H_u\}$$
$$= (\vee\{\tilde{\delta}((q_0, \mu^t(q_0)), a, p') : p' \in K_v\}) \wedge (\vee\{\tilde{\delta}((q_0, \mu^t(q_0)), a, p) : p \in H_u\})$$
$$= \tilde{\delta}'((K_j, \mu^t(K_j)), (H_i, a), K_v) \wedge \tilde{\delta}^{\Pi}((H_i, \mu^t(H_i)), a, H_u)$$
$$= \tilde{\delta}'((K_j, \mu^t(K_j)), \Omega(H_i, a), K_v) \wedge \tilde{\delta}^{\Pi}((H_i, \mu^t(H_i)), a, H_u)$$
$$= \tilde{\delta}^{\Omega}(((K_j, H_i), \mu^t((K_j, H_i))), a, (K_v, H_u))$$

Suppose that

$$\tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), y, \eta((K_v, H_u))) = \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), y, (K_v, H_u))$$

is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$.
We show that

$$\tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), x, \eta((K_v, H_u))) = \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), x, (K_v, H_u))$$

Now,

$$\tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), x, \eta((K_v, H_u)))$$
$$= \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), ya, \eta((K_v, H_u)))$$
$$= \vee\{\tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i)))), y, \eta((K, H))) \wedge$$
$$\tilde{\delta}((\eta((K, H)), \mu^t(\eta((K, H)))), a, \eta((K_v, H_u))) : \eta((K, H)) \in \eta(\tau \times \Pi)\}$$
$$= \vee\{\tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), y, (K, H)) \wedge$$
$$\tilde{\delta}^{\Omega}(((K, H), \mu^t((K, H))), a, (K_v, H_u)) : (K, H) \in \tau \times \Pi\}$$
$$= \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), ya, (K_v, H_u))$$
$$= \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), x, (K_v, H_u)).$$

□

Definition 2.16. Let Q be a nonempty set and Π, τ be partitions of Q. Then $\Pi \leq \tau$ if for all $A \in \Pi$, there exists $B \in \tau$ such that $A \subseteq B$.

Example 2.17. Let Q be a nonempty set and let $\Pi = \{H_1, H_2, ..., H_n\}$ and $\tau = \{H_1 \cup H_2 \cup ... \cup H_m, H_{m+1} \cup H_{m+2} \cup ... \cup H_n\}$ be partitions of Q, where $m < n$. Then $\Pi \leq \tau$.

Definition 2.18. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and Π be an admissible partition of Q. Then Π is called maximal if $\Pi \neq \{Q\}$ and if τ is any admissible partition of Q such that $\Pi \leq \tau \leq \{Q\}$, then either $\tau = \Pi$ or $\tau = \{Q\}$.

Definition 2.19. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. Then $\tilde{F}$ is called irreducible if $|Q| > 1$, 1_Q and $\{Q\}$ are the only admissible partitions of Q.

Theorem 2.20. *Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $\Pi = \{H_1, H_2, ..., H_n\}$ be an admissible partition of Q. Then Π is maximal if and only if $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^{\Pi}, F_1, F_2)$ is irreducible, where*

$$\tilde{\delta}^{\Pi}((H_i, \mu^t(H_i)), a, H_j) = \vee\{\tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j\}$$

$H_i \in \Pi$, $q \in H_i$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), ..., \mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i, j = 1, 2, ..., m$.

Proof. Suppose Π is maximal. Then $\Pi \neq \{Q\}$. So $|\Pi| > 1$. Let $\tilde{\Pi}$ be an admissible partition of Π, and $\tilde{\Pi} \neq 1_{\Pi}$. Then there exists $\tau \subseteq \Pi = \{H_1, H_2, ..., H_n\}$ such that $\tau \in \tilde{\Pi}$ and $|\tau| \geq 1$. Let $\tau = \{H_1, H_2, ..., H_m\}$, $1 < m \leq n$ and $\Pi' = \{H_1 \cup H_2 \cup ... \cup H_m, H_{m+1} \cup H_{m+2} \cup ... \cup H_n\}$. Then $\Pi < \Pi'$ and Π' is a partition of Q. Now, we show that Π' is admissible. Let either $q, p \in H_1 \cup H_2 \cup ... \cup H_m$ or $q, p \in H_{m+1} \cup H_{m+2} \cup ... \cup H_n$. Without loss of generality, we may assume that $q \in H_1$, $p \in H_2$ and $\tilde{\delta}((p, \mu^t(p)), a, r) > 0$, where $r \in H_i$. Then

$$\tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_i) = \vee\{\tilde{\delta}((p, \mu^t(p)), a, r) : r \in H_i\} > 0$$

Since $\tilde{\Pi}$ is an admissible partition of Π, there exists $B \in \tilde{\Pi}$ such that

$$\tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, B) \geq \tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_i)$$

and B, H_i belong to the same element of $\tilde{\Pi}$. Hence

$$\vee\{\tilde{\delta}((q, \mu^t(q)), a, u) : u \in B\} \geq \vee\{\tilde{\delta}((p, \mu^t(p)), a, r) : r \in H_i\}$$

Then there exists $u \in B$ such that $\tilde{\delta}((q, \mu^t(q)), a, u) \geq \tilde{\delta}((p, \mu^t(p)), a, r) > 0$. Since B, H_i belong to the same element of $\tilde{\Pi}$, then u and r belong to the same element of Π'. Hence Π' is admissible. Since Π is maximal, then $\Pi' = \{H_1 \cup H_2 \cup ... \cup H_n\} = \{Q\}$. So $\tau = \{H_1, H_2, ..., H_n\} = \{\Pi\}$ and $\Pi \in \tilde{\Pi}$. This implies that $\tilde{\Pi} = \{\Pi\}$. Thus $\tilde{F}/\Pi$ is irreducible.

Conversely, suppose $\tilde{F}/\Pi$ is irreducible. Let τ be an admissible partition of Q such that $\Pi \leq \tau \leq \{Q\}$, and $\tau \neq \Pi$. Let $\tau = \{H_1 \cup H_2 \cup ... \cup H_m, H_{m+1}, H_{m+2}, ..., H_n\}$, $1 < m \leq n$ and $\tilde{\tau} = \{\{H_1, H_2, ..., H_m\}, \{H_{m+1}\}, \{H_{m+2}\}, ..., \{H_n\}\}$. Then $\tilde{\tau} \neq 1_{\Pi}$. Now, we show that $\tilde{\tau}$ is an admissible partition of Π.

Without loss of generality, we may consider H_1, H_2. Suppose

$$\tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i) = \vee\{\tilde{\delta}((p, \mu^t(p)), a, r') : r' \in H_i\} > 0$$

for some $H_i \in \Pi$ and $p \in H_1$. Let $r \in H_i$ such that

$$\tilde{\delta}((p, \mu^t(p)), a, r) = \vee\{\tilde{\delta}((p, \mu^t(p)), a, r') : r' \in H_i\} > 0.$$

Since τ is an admissible partition of Q, for all $q \in H_2$, there exists $t_q \in Q$ such that $\tilde{\delta}((q, \mu^t(q)), a, t_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r)$ and r, t_q are the same element of τ, for all $q \in H_2$. If $H_i \notin \{H_1, H_2, ..., H_m\}$, then $H_i = H_j, m+1 \leq j \leq n$ and $r, t_q \in H_i$, for all $q \in H_2$. So

$$\tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_i) \geq \tilde{\delta}((q, \mu^t(q)), a, t_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r) = \tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i)$$

Suppose $H_i \in \{H_1, H_2, ..., H_m\}$, then $r, t_q \in H_1 \cup H_2 \cup ... \cup H_m$, for all $q \in H_2$. Let $\tilde{\delta}((q, \mu^t(q)), a, t'_q) = \vee\{\tilde{\delta}((q, \mu^t(q)), a, t_q) : q \in H_2\}$. Then we have $t'_q \in H_k, 1 \leq k \leq m$ and

$$\tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_k) = \vee\{\tilde{\delta}((q, \mu^t(q)), a, r') : r' \in H_k\} \geq \tilde{\delta}((q, \mu^t(q)), a, t'_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r) = \tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i)$$

and $H_k, H_i \in \{H_1, H_2, ..., H_m\}$. Consequently $\tilde{\tau}$ is an admissible partition of Π. Since $\tilde{F}/\Pi$ is irreducible, $\tilde{\tau} = \{\Pi\}$ and so $\tau = \{H_1 \cup H_2 \cup ... \cup H_n\} = \{Q\}$. Hence Π is maximal. $\square$

3. **Conclusion**

In this paper, we have introduced the concept of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, $\tilde{\delta}$-orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we have obtained the relationships between them, for example, we have shown that Π is maximal if and only if $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^{\Pi}, F_1, F_2)$ is irreducible.

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