

IRREDUCIBILITY ON GENERAL FUZZY AUTOMATA

M. HORRY

ABSTRACT. The aim of this paper is the study of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, δ -orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we obtain the relationships between them.

1. Introduction and Preliminaries

The concept of fuzzy automata was introduced by Wee in 1967 [12].

Let Σ be a set. A word in Σ is the product of a finite sequence of elements in Σ . Λ will denote the empty word and Σ^* the set of all words on Σ . The length $\ell(x)$ of the word $x \in \Sigma^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X , $\tilde{P}(X)$ will denote the set of all fuzzy sets on X and $P(X)$ will denote the set of all subsets on X .

A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the function f from $Q \times \Sigma$ into Q is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A word $x = x_1x_2 \dots x_n \in \Sigma^*$ is said to be accepted by A if there exist states q_0, q_1, \dots, q_n satisfying

- (1) $q_0 = s$
- (2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \dots, n$,
- (3) $q_n \in T$.

The empty word is accepted by A if and only if $s \in T$.

A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the function f from $Q \times \Sigma$ into $P(Q)$ is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A fuzzy finite-state automaton (FFA) is a six-tuple $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the initial state of \tilde{F} , Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \rightarrow [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \rightarrow Z$ is the output function. Associated with each fuzzy transition, there is a membership value

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in $[0, 1]$ called the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use $\delta(q_i, a_k, q_j)$ to refer both to a transition and its weight in the sense that whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, and otherwise, specifies the transition itself. The set of all transitions of \tilde{F} will be denoted by Δ . The above definition is generally accepted as a formal definition of a fuzzy finite-state automaton [4, 5, 6, 7, 8, 9].

In 2004, M. Doostfateme and S.C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [1].

In this paper, by using [2], [3], [10], [11] and [13], we define the concepts of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, $\tilde{\delta}$ -orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we obtain the relationships between them.

Definition 1.1. [1] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- (i) Q is a finite set of states, $Q = \{q_1, q_2, \dots, q_n\}$,
- (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \dots, a_m\}$,
- (iii) \tilde{R} is the set of fuzzy start states, $\tilde{R} \in \tilde{P}(Q)$,
- (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \dots, b_k\}$,
- (v) $\omega : Q \rightarrow Z$ is the output function,
- (vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \rightarrow [0, 1]$ is the augmented transition function,
- (vii) $F_1 : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is the membership assignment function,
- (viii) $F_2 : [0, 1]^* \rightarrow [0, 1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters μ and δ , where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the fuzzy set of all active states at time t_i , $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta, \forall i \geq 1\}.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in \text{Domain}(Q_{act}(t_i))$ and $T \subset \text{Domain}(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1.2. [1] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t + 1$, the following algorithm will assign a unified membership value to it:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1 , and will produce a membership value. Call this v_i .

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2 .

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j ,

$$\mu^{t+1}(q_j) = \tilde{F}_2[v_i] = \tilde{F}_2[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t + 1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k .
- $\mu^t(q_i)$ is the membership value of q_i at time t .
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t + 1$.

Definition 1.3. [13] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \rightarrow [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & \text{otherwise} \end{cases}$$

Also, if the input at time t_i be u_i , where $u_i \in \Sigma, \forall 1 \leq i \leq n$, then

$$\begin{aligned} \tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) &= \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p), \\ \tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) &= \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p)), \end{aligned}$$

and recursively

$$\begin{aligned} \tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) &= \bigvee \{ \tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots \\ &\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) \mid p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1}) \}. \end{aligned}$$

If $q \in Q_{act}(t_i)$, we should write q belongs to an element of Q_{act} . Hereafter, we simply denote it as: $q \in Q_{act}$.

2. Irreducibility on General Fuzzy Automata

Definition 2.1. Let $\tilde{F}_1^* = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1^*, F_1, F_2)$ and $\tilde{F}_2^* = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2^*, F_1, F_2)$ be max-min general fuzzy automata. Let η be a function of Q_2 onto Q_1

and let ξ be function of Σ_1 into Σ_2 . Extend ξ to a function ξ^* of Σ_1^* into Σ_2^* by $\xi^*(\Lambda) = \Lambda$ and for all $x \in \Sigma_1^*$, $\xi^*(x) = \xi(x_1)\xi(x_2)\dots\xi(x_n)$ where $x = x_1x_2\dots x_n$ and $x_i \in \Sigma_1$, $i = 1, 2, \dots, n$. Then (η, ξ) is called a covering of \tilde{F}_1^* by \tilde{F}_2^* , written $\tilde{F}_1^* \leq \tilde{F}_2^*$, if and only if for all $q_2 \in Q_2$, $q_1 \in Q_1$, and $x \in \Sigma_1^*$,

$$\tilde{\delta}_1^*((\eta(q_2), \mu^t(\eta(q_2))), x, q_1) = \vee \{\tilde{\delta}_2^*((q_2, \mu^t(q_2)), \xi^*(x), s) : \eta(s) = q_1, s \in Q_2\}$$

Example 2.2. Let $\tilde{F}_1^* = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1^*, F_1, F_2)$ and $\tilde{F}_2 = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2^*, F_1, F_2)$ be general fuzzy automata such that $Q_1 = \{q_0, q_1\}$, $Q_2 = \{q'_0, q'_1\}$, $\Sigma_1 = \Sigma_2 = \{a\}$, $\tilde{R}_1 = \{(q_0, \mu_1^{t_0}(q_0))\} = \{(q_0, 1)\}$, $\tilde{R}_2 = \{(q'_1, \mu_2^{t_0}(q'_1))\} = \{(q'_1, 1)\}$, $F_1(\mu, \delta) = \text{Min}(\mu, \delta)$, $Z = \emptyset$, ω and F_2 are not applicable, $\delta_1(q_0, a, q_1) = 0.3$, $\delta_1(q_1, a, q_1) = 0.6$, $\delta_2(q'_0, a, q'_0) = 0.7$, $\delta_2(q'_1, a, q'_0) = 0.3$. Define $\eta : Q_2 \rightarrow Q_1$ by $\eta(q'_0) = q_1$ and $\eta(q'_1) = q_0$. Let ξ be identity map on $\Sigma_1 = \Sigma_2$. Since η is one to one, if we show that

$$\tilde{\delta}_1^*((\eta(q'_1), \mu_1^{t_0}(\eta(q'_1))), x, \eta(q'_0)) = \tilde{\delta}_2^*((q'_1, \mu_2^{t_0}(q'_1))), x, q'_0)$$

for all $x \in \Sigma_1^* = \Sigma_2^*$, then (η, ξ) is a covering of \tilde{F}_1^* by \tilde{F}_2^* .

Now, for \tilde{F}_1^* , we have

$$Q_{act}(t_0) = \tilde{R}_1 = \{(q_0, \mu_1^{t_0}(q_0))\} = \{(q_0, 1)\}, Q_{act}(t_i) = \{(q_1, \mu_1^{t_i}(q_1))\}, \forall i \geq 1$$

$$\mu_1^{t_1}(q_1) = \tilde{\delta}_1((q_0, \mu_1^{t_0}(q_0)), a, q_1) = F_1(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_1)) = F_1(1, 0.3) = 0.3,$$

$$\mu_1^{t_2}(q_1) = \tilde{\delta}_1((q_1, \mu_1^{t_1}(q_1)), a, q_1) = F_1(\mu_1^{t_1}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.6) = 0.3,$$

$$\mu_1^{t_3}(q_1) = \tilde{\delta}_1((q_1, \mu_1^{t_2}(q_1)), a, q_1) = F_1(\mu_1^{t_2}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.3) = 0.3,$$

$$\mu_1^{t_4}(q_1) = \tilde{\delta}_1((q_0, \mu_1^{t_3}(q_1)), a, q_1) = F_1(\mu_1^{t_3}(q_1), \delta_1(q_1, a, q_1)) = F_1(0.3, 0.3) = 0.3,$$

$$\mu_1^{t_i}(q_1) = 0.3, \forall i \geq 5,$$

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), a, q_1) = 0.3,$$

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), aa, q_1) = 0.3 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), aaa, q_1) = 0.3 \wedge 0.3 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), a^n, q_1) = 0.3, \forall n \geq 4.$$

Also, for \tilde{F}_2^* , we have

$$Q_{act}(t_0) = \tilde{R}_2 = \{(q'_1, \mu_2^{t_0}(q'_1))\} = \{(q'_1, 1)\}, Q_{act}(t_i) = \{(q'_0, \mu_2^{t_i}(q'_0))\}, \forall i \geq 1$$

$$\mu_2^{t_1}(q'_0) = \tilde{\delta}_2((q'_1, \mu_2^{t_0}(q'_1)), a, q'_0) = F_1(\mu_2^{t_0}(q'_1), \delta_2(q'_1, a, q'_0)) = F_1(1, 0.3) = 0.3,$$

$$\mu_2^{t_2}(q'_0) = \tilde{\delta}_2((q'_0, \mu_2^{t_1}(q'_0)), a, q'_0) = F_1(\mu_2^{t_1}(q'_0), \delta_2(q'_0, a, q'_0)) = F_1(0.3, 0.7) = 0.3,$$

$$\mu_2^{t_3}(q'_0) = \tilde{\delta}_2((q'_0, \mu_2^{t_2}(q'_0)), a, q'_0) = F_1(\mu_2^{t_2}(q'_0), \delta_2(q'_0, a, q'_0)) = F_1(0.3, 0.7) = 0.3,$$

$$\mu_2^{t_4}(q'_0) = \tilde{\delta}_2((q'_0, \mu_2^{t_3}(q'_0)), a, q'_0) = F_1(\mu_2^{t_3}(q'_0), \delta_2(q'_0, a, q'_0)) = F_1(0.3, 0.7) = 0.3,$$

$$\mu_2^{t_i}(q'_0) = 0.3, \forall i \geq 5,$$

$$\tilde{\delta}_2^*((q'_1, \mu_2^{t_0}(q'_1)), a, q'_0) = F_1(\mu_2^{t_0}(q'_1), \delta_2(q'_1, a, q'_0)) = F_1(1, 0.3) = 0.3,$$

$$\tilde{\delta}_2^*((q'_1, \mu_2^{t_0}(q'_1)), aa, q'_0)$$

$$= \tilde{\delta}_2((q'_1, \mu_2^{t_0}(q'_1)), a, q'_0) \wedge \tilde{\delta}_2((q'_0, \mu_2^{t_1}(q'_0)), a, q'_0)$$

$$= F_1(\mu_2^{t_0}(q'_1), \delta_2(q'_1, a, q'_0)) \wedge F_1(\mu_2^{t_1}(q'_0), \delta_2(q'_0, a, q'_0))$$

$$= F_1(1, 0.3) \wedge F_1(0.3, 0.7) = 0.3 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}_2^*((q'_1, \mu_2^{t_0}(q'_1)), a^n, q'_0) = 0.3, \forall n \geq 3.$$

So for all $x \in \Sigma_1^* = \Sigma_2^*$, we have

$$\tilde{\delta}_1^*((q_0, \mu_1^{t_0}(q_0)), x, q_1) = \tilde{\delta}_2^*((q'_1, \mu_2^{t_0}(q'_1)), x, q'_1)$$

Thus (η, ξ) is a covering of \tilde{F}_1^* by \tilde{F}_2^* .

Definition 2.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and let \sim be an equivalence relation on Q . Then \sim is called admissible relation on Q if and only if for all $p, q, r \in Q$, for all $a \in \Sigma$, if $p \sim q$ and $\tilde{\delta}((p, \mu^t(p)), a, r) > 0$, then there exists $s \in Q$ such that $\tilde{\delta}((q, \mu^t(q)), a, s) \geq \tilde{\delta}((p, \mu^t(p)), a, r)$ and $s \sim r$.

Theorem 2.4. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and let \sim be an equivalence relation on Q . Then \sim is an admissible relation on Q if and only if for all $p, q, r \in Q$, for all $x \in \Sigma^*$, if $p \sim q$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$, then there exists $s \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), x, s) \geq \tilde{\delta}^*((p, \mu^t(p)), x, r)$ and $s \sim r$.

Proof. Let \sim be an admissible relation on Q , $x \in \Sigma^*$, $|x| = n$, $p \sim q$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$. If $n = 0$, then $x = \Lambda$ and $\tilde{\delta}^*((p, \mu^t(p)), x, r) > 0$ implies that $p = r$. Thus $\tilde{\delta}^*((q, \mu^t(q)), x, q) = \tilde{\delta}^*((p, \mu^t(p)), x, p) = 1$ and $p \sim q$. Hence the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, where $n > 0$. Let $x = ya$, where $a \in \Sigma$. Now

$$\begin{aligned} \tilde{\delta}^*((p, \mu^t(p)), x, r) &= \tilde{\delta}^*((p, \mu^t(p)), ya, r) \\ &= \vee \{ \tilde{\delta}^*((p, \mu^t(p)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, r) : s \in Q \} > 0 \end{aligned}$$

Thus there exists $u \in Q$ such that

$$\tilde{\delta}^*((p, \mu^t(p)), ya, r) = \tilde{\delta}^*((p, \mu^t(p)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r)$$

So $\tilde{\delta}^*((p, \mu^t(p)), y, u) > 0$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$. By the induction hypothesis, there exists $s \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), y, s) \geq \tilde{\delta}^*((p, \mu^t(p)), y, u)$ and $s \sim u$. Now, since \sim is an admissible relation on Q , $s \sim u$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$, then there exists $s' \in Q$ such that $\tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((u, \mu^t(u)), a, r)$ and $s' \sim r$. So we have $\tilde{\delta}^*((q, \mu^t(q)), x, s') = \tilde{\delta}^*((q, \mu^t(q)), ya, s') \geq \tilde{\delta}^*((q, \mu^t(q)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((p, \mu^t(p)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r) = \tilde{\delta}^*((p, \mu^t(p)), ya, r) = \tilde{\delta}^*((p, \mu^t(p)), x, r)$ and $s' \sim r$.

The converse is trivial. \square

Definition 2.5. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $\wp = \{Q_1, Q_2, \dots, Q_k\}$ be a partition of Q . Then \wp is called an admissible partition of Q if let $a \in X$, $p_1, p_2 \in Q_i, \forall i, 1 \leq i \leq k$, $\tilde{\delta}((p_1, \mu^t(p_1)), a, r) > 0$, for some $r \in Q$, then there exists $s \in Q$ and there exists $j, 1 \leq j \leq k$ such that

$$\tilde{\delta}((p_2, \mu^t(p_2)), a, s) \geq \tilde{\delta}((p_1, \mu^t(p_1)), a, r)$$

and $s, r \in Q_j$.

Example 2.6. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton.

- (i) Let $1_Q = \{\{q\} : q \in Q\}$. Then 1_Q is an admissible partition of Q .
- (ii) $\{Q\}$ is an admissible partition of Q .

Theorem 2.7. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $\wp = \{Q_1, Q_2, \dots, Q_k\}$ be a partition of Q . The following are equivalent:

- (i) \wp is an admissible partition of Q
(ii) let $x \in \Sigma^*$, $p_1, p_2 \in Q_i, \forall i, 1 \leq i \leq k$, $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$, for some $r \in Q$, then there exists $s \in Q$ and there exists $j, 1 \leq j \leq k$ such that

$$\tilde{\delta}^*((p_2, \mu^t(p_2)), x, s) \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), x, r)$$

and $s, r \in Q_j$.

Proof. (i) \Rightarrow (ii) Let \wp be an admissible partition of Q , $x \in \Sigma^*$, $|x| = n$, $p_1, p_2 \in Q_i$ and $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$. If $n = 0$, then $x = \Lambda$ and $\tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) > 0$ implies that $p_1 = r$. Thus $\tilde{\delta}^*((p_2, \mu^t(p_2)), x, p_2) = \tilde{\delta}^*((p_1, \mu^t(p_1)), x, p_1) = 1$ and $p_1, p_2 \in Q_i$. Hence the result is true for $n = 0$. Suppose the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, where $n > 0$. Let $x = ya$, where $a \in \Sigma$. Now

$$\begin{aligned} \tilde{\delta}^*((p_1, \mu^t(p_1)), x, r) &= \tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r) \\ &= \vee \{ \tilde{\delta}^*((p_1, \mu^t(p_1)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, r) : s \in Q \} > 0 \end{aligned}$$

Thus there exists $u \in Q$ such that

$$\tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r)$$

So $\tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) > 0$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$. By the induction hypothesis, there exists $s \in Q$ and there exists j such that $\tilde{\delta}^*((p_2, \mu^t(p_2)), y, s) \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u)$ and $s, u \in Q_j$. Now, since \wp is an admissible partition of Q , $s, u \in Q_j$ and $\tilde{\delta}^*((u, \mu^t(u)), a, r) > 0$, then there exists $s' \in Q$ and there exists l such that $\tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((u, \mu^t(u)), a, r)$ and $s', r \in Q_l$. So we have $\tilde{\delta}^*((p_2, \mu^t(p_2)), x, s') = \tilde{\delta}^*((p_2, \mu^t(p_2)), ya, s') \geq \tilde{\delta}^*((p_2, \mu^t(p_2)), y, s) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, s') \geq \tilde{\delta}^*((p_1, \mu^t(p_1)), y, u) \wedge \tilde{\delta}^*((u, \mu^t(u)), a, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), ya, r) = \tilde{\delta}^*((p_1, \mu^t(p_1)), x, r)$ and $s', r \in Q_l$.

The proof of (ii) \Rightarrow (i) is trivial. \square

Theorem 2.8. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. Let $\Pi = \{H_1, H_2, \dots, H_n\}$ be an admissible partition of Q . Define

$$\tilde{\delta}^\Pi : (\Pi \times [0, 1]) \times \Sigma \times \Pi \longrightarrow [0, 1]$$

by

$$\tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_j) = \vee \{ \tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j \}$$

where $H_i \in \Pi$, $q \in H_i$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), \dots, \mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i, j = 1, 2, \dots, m$. Then $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\Pi, F_1, F_2)$ is a general fuzzy automaton.

Proof. Let $q, q' \in H_i, a \in \Sigma, A = \{ \tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j \}$ and $B = \{ \tilde{\delta}((q', \mu^t(q')), a, r) : r \in H_j \}$.

Suppose $\tilde{\delta}((q, \mu^t(q)), a, r) > 0$ for some $r \in H_j$. Since Π is an admissible partition of Q , there exists $r' \in H_j$ such that $\tilde{\delta}((q', \mu^t(q')), a, r') \geq \tilde{\delta}((q, \mu^t(q)), a, r)$. Similarly if $\tilde{\delta}((q', \mu^t(q')), a, r) > 0$ for some $r \in H_j$. Since Π is an admissible partition of Q ,

there exists $r' \in H_j$ such that $\tilde{\delta}((q, \mu^t(q)), a, r') \geq \tilde{\delta}((q', \mu^t(q')), a, r)$. We show that $\tilde{\delta}((q, \mu^t(q)), a, r) = 0$, for all $r \in H_j$ if and only if $\tilde{\delta}((q', \mu^t(q')), a, r) = 0$, for all $r \in H_j$. Let $\tilde{\delta}((q, \mu^t(q)), a, r) = 0$, for all $r \in H_j$ and let $\tilde{\delta}((q', \mu^t(q')), a, r) > 0$ for some $r \in H_j$. By admissibility of Π , there exists $r' \in H_j$ such that $\tilde{\delta}((q, \mu^t(q)), a, r') \geq \tilde{\delta}((q', \mu^t(q')), a, r)$. Then $\tilde{\delta}((q', \mu^t(q')), a, r) = 0$, which is a contradiction. Hence

$$\vee\{\tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j\} = \vee\{\tilde{\delta}((q', \mu^t(q')), a, r) : r \in H_j\}$$

So $\tilde{\delta}^\Pi$ is well defined. \square

Corollary 2.9. *Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and Π and τ be admissible partitions of Q . Consider the general fuzzy automata $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\Pi, F_1, F_2)$ and $\tilde{F}/\tau = (\tau, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\tau, F_1, F_2)$. Define*

$$\tilde{\delta}^\wedge : ((\Pi \times \tau) \times [0, 1]) \times \Sigma \times (\Pi \times \tau) \longrightarrow [0, 1]$$

by

$$\begin{aligned} \tilde{\delta}^\wedge(((H_i, K_j), \mu^t((H_i, K_j))), a, (H_u, K_v)) = \\ \tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_u) \wedge \tilde{\delta}^\tau((K_j, \mu^t(K_j)), a, K_v) \end{aligned}$$

where $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), \dots, \mu^t(q_{i_m})\}$, $q_{i_s} \in H_i$, $s = 1, 2, \dots, m$, $\mu^t(K_j) \in \{\mu^t(q_{j'_1}), \mu^t(q_{j'_2}), \dots, \mu^t(q_{j'_m})\}$, $q_{j'_u} \in K_j$, $u = 1, 2, \dots, m'$ and

$$\mu^t((H_i, K_j)) \in \{(\mu^t(q_{i_1}), \mu^t(q_{j'_1})), (\mu^t(q_{i_2}), \mu^t(q_{j'_2})), \dots, (\mu^t(q_{i_m}), \mu^t(q_{j'_m}))\}.$$

Then $\tilde{F}/\Pi \wedge \tilde{F}/\tau = (\Pi \times \tau, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\wedge, F_1, F_2)$ is a general fuzzy automaton.

Theorem 2.10. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton. Let $\Pi = \{H_i : i \in I\}$ be an admissible partition of Q . Then for $q \in H_i$*

$$\tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) = \vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_j\}$$

for all $H_i, H_j \in \Pi$, $x \in \Sigma^*$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), \dots, \mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i$, $j = 1, 2, \dots, m$.

Proof. We show that

$$(i) \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) \leq \vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_j\},$$

for all $H_i, H_j \in \Pi$ and $x \in \Sigma^*$

$$(ii) \tilde{\delta}^*((q, \mu^t(q)), x, r) \leq \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j), \text{ where } q \in H_i, r \in H_j.$$

For the proof of (i), let $H_i, H_j \in \Pi$ and $x \in \Sigma^*$. Let $|x| = n$. If $n = 0$, then $x = \Lambda$. If $H_i = H_j$, then $\tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) = 1$ and $\vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_j\} = \vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_i\} = \tilde{\delta}^*((q, \mu^t(q)), x, q) = 1$, where $q \in H_i$.

If $H_i \neq H_j$, then $\tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) = 0$ and $H_i \cap H_j = \phi$. Since $H_i \cap H_j = \phi$ and $q \in H_i$, $\vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_j\} = 0$. So

$$\tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) = \vee\{\tilde{\delta}^*((q, \mu^t(q)), x, r) : r \in H_j\}$$

Hence the result is true for $n = 0$. Suppose that the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$. Now for $q \in H_i$ and $s \in H_k$, we have

$$\begin{aligned}
& \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j) = \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), ya, H_j) \\
& = \vee \{ \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), y, H_k) \wedge \tilde{\delta}^{\Pi^*}((H_k, \mu^t(H_k)), a, H_j) : H_k \in \Pi \} \\
& = \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, r) : r \in H_k \} \wedge \vee \{ \tilde{\delta}^*((s, \mu^t(s)), a, p) : p \in H_j \} : H_k \in \Pi \} \\
& = \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, r) \wedge \tilde{\delta}^*((s, \mu^t(s)), a, p) : r \in H_k, p \in H_j \} : H_k \in \Pi \} \\
& \leq \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) : r \in H_k, p \in H_j \} : H_k \in \Pi \} \\
& = \{ \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) : r \in H_k, H_k \in \Pi \} : p \in H_j \} \\
& = \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) : r \in Q \} : p \in H_j \} \\
& = \vee \{ \tilde{\delta}^*((q, \mu^t(q)), ya, p) : p \in H_j \} \\
& = \vee \{ \tilde{\delta}^*((q, \mu^t(q)), x, p) : p \in H_j \}
\end{aligned}$$

For the proof of (ii), let $q \in H_i$, $r \in H_j$, $x \in \Sigma^*$ and $|x| = n$. If $n = 0$, then $x = \Lambda$. If $q = r$, then $H_i = H_j$ and $\tilde{\delta}^*((q, \mu^t(q)), x, r) = 1 = \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j)$. Suppose $q \neq r$. Then $\tilde{\delta}^*((q, \mu^t(q)), x, r) = 0 \leq \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j)$. If $n = 1$, then $x = a \in \Sigma$ and $\tilde{\delta}^*((q, \mu^t(q)), a, r) = \tilde{\delta}((q, \mu^t(q)), a, r) \leq \vee \{ \tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j \} = \tilde{\delta}^{\Pi}((H_i, \mu^t(H_i)), a, H_j) = \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), a, H_j)$. Hence the result is true for $n = 0$ and $n = 1$. Suppose that the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$. Then

$$\begin{aligned}
& \tilde{\delta}^*((q, \mu^t(q)), x, r) = \tilde{\delta}^*((q, \mu^t(q)), ya, r) \\
& = \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, p) \wedge \tilde{\delta}((p, \mu^t(p)), a, r) : p \in Q \} \\
& = \vee \{ \vee \{ \tilde{\delta}^*((q, \mu^t(q)), y, p) \wedge \tilde{\delta}((p, \mu^t(p)), a, r) : p \in H_k \} : H_k \in \Pi \} \\
& \leq \vee \{ \vee \{ \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), y, H_k) \wedge \tilde{\delta}^{\Pi}((H_k, \mu^t(H_k)), a, H_j) : p \in H_k \} : H_k \in \Pi \} \\
& = \vee \{ \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), y, H_k) \wedge \tilde{\delta}^{\Pi}((H_k, \mu^t(H_k)), a, H_j) : H_k \in \Pi \} \\
& = \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_j)
\end{aligned}$$

□

Definition 2.11. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and Π and τ be admissible partitions of Q . Π and τ are called $\tilde{\delta}$ -orthogonal if

- (i) $\Pi \cap \tau = 1_Q$,
- (ii) For all $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $a \in \Sigma$, if $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$, then

$$\begin{aligned}
& \tilde{\delta}((q_0, \mu^t(q_0)), a, p_0) \\
& = \vee \{ \tilde{\delta}((q_0, \mu^t(q_0)), a, p) \wedge \tilde{\delta}((q_0, \mu^t(q_0)), a, p') : p \in H_u, p' \in K_v \}
\end{aligned}$$

Theorem 2.12. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π and τ be admissible partitions of Q . Then Π and τ are $\tilde{\delta}$ -orthogonal if and only if

- (i) $\Pi \cap \tau = 1_Q$,
(ii) For all $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $x \in \Sigma^*$, if $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$, then

$$\begin{aligned} & \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v \} \end{aligned}$$

Proof. Let Π and τ are $\tilde{\delta}$ -orthogonal. Then clearly (i) holds.

For the proof of (ii), let $H_i, H_u \in \Pi$, $K_j, K_v \in \tau$, $x \in \Sigma^*$, $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$. Suppose $|x| = n$. If $n = 0$, then $x = \Lambda$. If $q_0 \neq p_0$, then $\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 0$, $q_0 \notin H_u$ and $q_0 \notin K_v$. So $q_0 \neq p$, $\forall p \in H_u$ and $q_0 \neq p'$, $\forall p' \in K_v$. Hence

$$\vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v \} = 0.$$

Suppose $q_0 = p_0$. Then $\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 1$, $H_i = H_u$ and $K_j = K_v$. Thus

$$\begin{aligned} & \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v \} \\ &= \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = 1. \end{aligned}$$

Hence if $n = 0$, then

$$\begin{aligned} & \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v \} \end{aligned}$$

Suppose that the result is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$. Then

$$\begin{aligned} & \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) = \tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p_0) \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p_0) : r \in Q \} \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge (\vee \{ \tilde{\delta}^*((r, \mu^t(r)), a, p) \wedge \\ & \quad \tilde{\delta}^*((r, \mu^t(r)), a, p') : p \in H_u, p' \in K_v \}) : r \in Q \} \\ &= \vee \{ (\vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) \}) \wedge (\vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \\ & \quad \tilde{\delta}^*((r, \mu^t(r)), a, p') \}) : p \in H_u, p' \in K_v, r \in Q \} \\ &= \vee \{ (\vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p) : r \in Q \}) \wedge \\ & \quad (\vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), y, r) \wedge \tilde{\delta}^*((r, \mu^t(r)), a, p') : r \in Q \}) : p \in H_u, p' \in K_v \} \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), ya, p') : p \in H_u, p' \in K_v \} \\ &= \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v \}. \end{aligned}$$

The converse is trivial. \square

Theorem 2.13. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π and τ be admissible partitions of Q that are $\tilde{\delta}$ -orthogonal. Then $\tilde{F}^* \leq \tilde{F}^*/\Pi \wedge \tilde{F}^*/\tau$.*

Proof. Define $\eta : \Pi \times \tau \rightarrow Q$ by $\eta((H_i, K_j)) = q_0$ where $H_i \cap K_j = \{q_0\}$. Since Π and τ are $\tilde{\delta}$ -orthogonal, η is one-to-one. Let ξ be identity map of Σ into Σ . Let $H_i, H_u \in \Pi, K_j, K_v \in \tau$ and $x \in \Sigma^*$. Suppose $H_i \cap K_j = \{q_0\}$ and $H_u \cap K_v = \{p_0\}$. Then

$$\begin{aligned} & \tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j))))), x, \eta((H_u, K_v))) \\ &= \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \end{aligned}$$

Now, since Π and τ are $\tilde{\delta}$ -orthogonal, we have

$$\begin{aligned} & \tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v)) \\ &= \tilde{\delta}^{\Pi^*}((H_i, \mu^t(H_i)), x, H_u) \wedge \tilde{\delta}^{\tau^*}((K_j, \mu^t(K_j)), x, K_v) \\ &= (\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) : p \in H_u\}) \wedge (\vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p' \in K_v\}) \\ &= \vee\{\tilde{\delta}^*((q_0, \mu^t(q_0)), x, p) \wedge \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p') : p \in H_u, p' \in K_v\} \\ &= \tilde{\delta}^*((q_0, \mu^t(q_0)), x, p_0) \end{aligned}$$

Thus

$$\begin{aligned} & \tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j))))), x, \eta((H_u, K_v))) = \\ & \tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v)) \end{aligned}$$

Now, since η is one-to-one, we have

$$\begin{aligned} & \tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_u, K_v)) \\ &= \vee\{\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_r, K_s)) : \\ & \eta((H_r, K_s)) = \eta((H_u, K_v)), (H_r, K_s) \in \Pi \times \tau\} \end{aligned}$$

Hence

$$\begin{aligned} & \tilde{\delta}^*((\eta((H_i, K_j)), \mu^t(\eta((H_i, K_j))))), x, \eta((H_u, K_v))) \\ &= \vee\{\tilde{\delta}^{\wedge^*}((H_i, K_j), \mu^t((H_i, K_j))), x, (H_r, K_s)) : \\ & \eta((H_r, K_s)) = \eta((H_u, K_v)), (H_r, K_s) \in \Pi \times \tau\} \end{aligned}$$

Consequently, $\tilde{F}^* \leq \tilde{F}^*/\Pi \wedge \tilde{F}^*/\tau$. \square

Definition 2.14. Let $\tilde{F}_1 = (Q_1, \Sigma_1, \tilde{R}_1, Z, \omega, \tilde{\delta}_1, F_1, F_2)$ and $\tilde{F}_2 = (Q_2, \Sigma_2, \tilde{R}_2, Z, \omega, \tilde{\delta}_2, F_1, F_2)$ be general fuzzy automata. Let Ω be a function of $Q_2 \times \Sigma_2$ into Σ_1 and $Q = Q_1 \times Q_2$. Define

$$\tilde{\delta}^\Omega : Q \times ([0, 1] \times [0, 1]) \times \Sigma_2 \times Q \rightarrow [0, 1]$$

by

$$\begin{aligned} & \tilde{\delta}^\Omega(((q_1, q_2), \mu^t((q_1, q_2))), b, (p_1, p_2)) \\ &= \tilde{\delta}_1((q_1, \mu^t(q_1)), \Omega(q_2, b), p_1) \wedge \tilde{\delta}_2((q_2, \mu^t(q_2)), b, p_2) \end{aligned}$$

where $\mu^t((q_1, q_2)) = (\mu^t(q_1), \mu^t(q_2))$. Then $\tilde{F} = (Q', \Sigma_2, \tilde{R}, Z, \omega, \tilde{\delta}^\Omega, F_1, F_2)$ is a general fuzzy automaton and write $\tilde{F} = \tilde{F}_1 \Omega \tilde{F}_2$.

Theorem 2.15. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and Π be an admissible partition of Q . If there exists an admissible partition τ of Q such that Π and τ are $\tilde{\delta}$ -orthogonal, then there exists a general fuzzy automaton \tilde{F}' such that $\tilde{F}^* \leq \tilde{F}' \Omega \tilde{F}^* / \tau$.*

Proof. Let $\Pi = \{H_1, H_2, \dots, H_n\}$ and $\tau = \{K_1, K_2, \dots, K_m\}$ be $\tilde{\delta}$ -orthogonal partitions of Q . Let $\tilde{F}' = (\tau, \Pi \times \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}', F_1, F_2)$, where

$$\tilde{\delta}'((K_j, \mu^t(K_j)), (H_i, a), K_v) = \vee \{ \tilde{\delta}^*((q_0, \mu^t(q_0)), a, p) : p \in K_v \}$$

and $H_i \cap K_j = \{q_0\}$. Since τ is admissible, $\tilde{\delta}'$ is well defined.

Define $\Omega : \Pi \times \Sigma \rightarrow \Pi \times \Sigma$ to be the identity map and define $\eta : \tau \times \Pi \rightarrow Q$ by $\eta((k_j, H_i)) = q_0$ where $H_i \cap K_j = \{q_0\}$. Then η is one-to-one and onto. Let ξ be identity map of Σ into Σ . Let $x = \Lambda$. Since η is one-to-one, so we have

$$\begin{aligned} \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), \Lambda, \eta((K_v, H_u))) &= 1 \iff \\ \eta((K_j, H_i)) = \eta((K_v, H_u)) &\iff (K_j, H_i) = (K_v, H_u) \iff \\ \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), \Lambda, (K_v, H_u)) &= 1 \end{aligned}$$

Let $H_u \cap K_v = \{p_0\}$ and $a \in \Sigma$. Since Π and τ are $\tilde{\delta}$ -orthogonal, so we have

$$\begin{aligned} \tilde{\delta}((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), a, \eta((K_v, H_u))) &= \tilde{\delta}((q_0, \mu^t(q_0)), a, p_0) \\ &= \vee \{ \tilde{\delta}((q_0, \mu^t(q_0)), a, p') \wedge \tilde{\delta}((q_0, \mu^t(q_0)), a, p) : p' \in K_v, p \in H_u \} \\ &= (\vee \{ \tilde{\delta}((q_0, \mu^t(q_0)), a, p') : p' \in K_v \}) \wedge (\vee \{ \tilde{\delta}((q_0, \mu^t(q_0)), a, p) : p \in H_u \}) \\ &= \tilde{\delta}'((K_j, \mu^t(K_j)), (H_i, a), K_v) \wedge \tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_u) \\ &= \tilde{\delta}'((K_j, \mu^t(K_j)), \Omega(H_i, a), K_v) \wedge \tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_u) \\ &= \tilde{\delta}^\Omega(((K_j, H_i), \mu^t((K_j, H_i))), a, (K_v, H_u)) \end{aligned}$$

Suppose that

$$\begin{aligned} \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), y, \eta((K_v, H_u))) &= \\ \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), y, (K_v, H_u)) & \end{aligned}$$

is true for all $y \in \Sigma^*$ and $|y| = n - 1$, $n > 0$. Let $x = ya$, where $a \in \Sigma$.

We show that

$$\begin{aligned} \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), x, \eta((K_v, H_u))) &= \\ \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), x, (K_v, H_u)) & \end{aligned}$$

Now,

$$\begin{aligned} \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), x, \eta((K_v, H_u))) &= \\ &= \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), ya, \eta((K_v, H_u))) \\ &= \vee \{ \tilde{\delta}^*((\eta((K_j, H_i)), \mu^t(\eta((K_j, H_i))))), y, \eta((K, H)) \} \wedge \\ &\tilde{\delta}((\eta((K, H)), \mu^t(\eta((K, H))))), a, \eta((K_v, H_u))) : \eta((K, H)) \in \eta(\tau \times \Pi) \\ &= \vee \{ \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), y, (K, H)) \} \wedge \\ &\tilde{\delta}^\Omega(((K, H), \mu^t((K, H))), a, (K_v, H_u)) : (K, H) \in \tau \times \Pi \\ &= \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), ya, (K_v, H_u)) \\ &= \tilde{\delta}^{\Omega^*}(((K_j, H_i), \mu^t((K_j, H_i))), x, (K_v, H_u)). \end{aligned}$$

□

Definition 2.16. Let Q be a nonempty set and Π, τ be partitions of Q . Then $\Pi \leq \tau$ if for all $A \in \Pi$, there exists $B \in \tau$ such that $A \subseteq B$.

Example 2.17. Let Q be a nonempty set and let $\Pi = \{H_1, H_2, \dots, H_n\}$ and $\tau = \{H_1 \cup H_2 \cup \dots \cup H_m, H_{m+1} \cup H_{m+2} \cup \dots \cup H_n\}$ be partitions of Q , where $m < n$. Then $\Pi \leq \tau$.

Definition 2.18. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and Π be an admissible partition of Q . Then Π is called maximal if $\Pi \neq \{Q\}$ and if τ is any admissible partition of Q such that $\Pi \leq \tau \leq \{Q\}$, then either $\tau = \Pi$ or $\tau = \{Q\}$.

Definition 2.19. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. Then \tilde{F} is called irreducible if $|Q| > 1$, 1_Q and $\{Q\}$ are the only admissible partitions of Q .

Theorem 2.20. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $\Pi = \{H_1, H_2, \dots, H_n\}$ be an admissible partition of Q . Then Π is maximal if and only if $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^\Pi, F_1, F_2)$ is irreducible, where

$$\tilde{\delta}^\Pi((H_i, \mu^t(H_i)), a, H_j) = \vee \{ \tilde{\delta}((q, \mu^t(q)), a, r) : r \in H_j \}$$

$H_i \in \Pi$, $q \in H_i$, $a \in \Sigma$, $\mu^t(H_i) \in \{\mu^t(q_{i_1}), \mu^t(q_{i_2}), \dots, \mu^t(q_{i_m})\}$ and $q_{i_j} \in H_i$, $j = 1, 2, \dots, m$.

Proof. Suppose Π is maximal. Then $\Pi \neq \{Q\}$. So $|\Pi| > 1$. Let $\tilde{\Pi}$ be an admissible partition of Π , and $\tilde{\Pi} \neq 1_\Pi$. Then there exists $\tau \subseteq \Pi = \{H_1, H_2, \dots, H_n\}$ such that $\tau \in \tilde{\Pi}$ and $|\tau| \geq 1$. Let $\tau = \{H_1, H_2, \dots, H_m\}$, $1 < m \leq n$ and $\Pi' = \{H_1 \cup H_2 \cup \dots \cup H_m, H_{m+1} \cup H_{m+2} \cup \dots \cup H_n\}$. Then $\Pi < \Pi'$ and Π' is a partition of Q . Now, we show that Π' is admissible. Let either $q, p \in H_1 \cup H_2 \cup \dots \cup H_m$ or $q, p \in H_{m+1} \cup H_{m+2} \cup \dots \cup H_n$. Without loss of generality, we may assume that $q \in H_1$, $p \in H_2$ and $\tilde{\delta}((p, \mu^t(p)), a, r) > 0$, where $r \in H_i$. Then

$$\tilde{\delta}^\Pi((H_2, \mu^t(H_2)), a, H_i) = \vee \{ \tilde{\delta}((p, \mu^t(p)), a, r) : r \in H_i \} > 0$$

Since $\tilde{\Pi}$ is an admissible partition of Π , there exists $B \in \tilde{\Pi}$ such that

$$\tilde{\delta}^\Pi((H_1, \mu^t(H_1)), a, B) \geq \tilde{\delta}^\Pi((H_2, \mu^t(H_2)), a, H_i)$$

and B, H_i belong to the same element of $\tilde{\Pi}$. Hence

$$\vee \{ \tilde{\delta}((q, \mu^t(q)), a, u) : u \in B \} \geq \vee \{ \tilde{\delta}((p, \mu^t(p)), a, r) : r \in H_i \}$$

Then there exists $u \in B$ such that $\tilde{\delta}((q, \mu^t(q)), a, u) \geq \tilde{\delta}((p, \mu^t(p)), a, r) > 0$. Since B, H_i belong to the same element of $\tilde{\Pi}$, then u and r belong to the same element of Π' . Hence Π' is admissible. Since Π is maximal, then $\Pi' = \{H_1 \cup H_2 \cup \dots \cup H_n\} = \{Q\}$. So $\tau = \{H_1, H_2, \dots, H_n\} = \{\Pi\}$ and $\Pi \in \tilde{\Pi}$. This implies that $\tilde{\Pi} = \{\Pi\}$. Thus \tilde{F}/Π is irreducible.

Conversely, suppose \tilde{F}/Π is irreducible. Let τ be an admissible partition of Q such that $\Pi \leq \tau \leq \{Q\}$, and $\tau \neq \Pi$. Let $\tau = \{H_1 \cup H_2 \cup \dots \cup H_m, H_{m+1}, H_{m+2}, \dots, H_n\}$, $1 < m \leq n$ and $\tilde{\tau} = \{\{H_1, H_2, \dots, H_m\}, \{H_{m+1}\}, \{H_{m+2}\}, \dots, \{H_n\}\}$. Then $\tilde{\tau} \neq 1_\Pi$. Now, we show that $\tilde{\tau}$ is an admissible partition of Π .

Without loss of generality, we may consider H_1, H_2 . Suppose

$$\tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i) = \vee \{ \tilde{\delta}((p, \mu^t(p)), a, r') : r' \in H_i \} > 0$$

for some $H_i \in \Pi$ and $p \in H_1$. Let $r \in H_i$ such that

$$\tilde{\delta}((p, \mu^t(p)), a, r) = \vee \{ \tilde{\delta}((p, \mu^t(p)), a, r') : r' \in H_i \} > 0.$$

Since τ is an admissible partition of Q , for all $q \in H_2$, there exists $t_q \in Q$ such that $\tilde{\delta}((q, \mu^t(q)), a, t_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r)$ and r, t_q are the same element of τ , for all $q \in H_2$. If $H_i \notin \{H_1, H_2, \dots, H_m\}$, then $H_i = H_j$, $m+1 \leq j \leq n$ and $r, t_q \in H_i$, for all $q \in H_2$. So

$$\tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_i) \geq \tilde{\delta}((q, \mu^t(q)), a, t_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r) = \tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i)$$

Suppose $H_i \in \{H_1, H_2, \dots, H_m\}$, then $r, t_q \in H_1 \cup H_2 \cup \dots \cup H_m$, for all $q \in H_2$. Let $\tilde{\delta}((q, \mu^t(q)), a, t'_q) = \vee \{ \tilde{\delta}((q, \mu^t(q)), a, t_q) : q \in H_2 \}$. Then we have $t'_q \in H_k$, $1 \leq k \leq m$ and

$$\tilde{\delta}^{\Pi}((H_2, \mu^t(H_2)), a, H_k) = \vee \{ \tilde{\delta}((q, \mu^t(q)), a, r') : r' \in H_k \} \geq \tilde{\delta}((q, \mu^t(q)), a, t'_q) \geq \tilde{\delta}((p, \mu^t(p)), a, r) = \tilde{\delta}^{\Pi}((H_1, \mu^t(H_1)), a, H_i)$$

and $H_k, H_i \in \{H_1, H_2, \dots, H_m\}$. Consequently $\tilde{\tau}$ is an admissible partition of Π . Since \tilde{F}/Π is irreducible, $\tilde{\tau} = \{\Pi\}$ and so $\tau = \{H_1 \cup H_2 \cup \dots \cup H_n\} = \{Q\}$. Hence Π is maximal. \square

3. Conclusion

In this paper, we have introduced the concept of a covering of a max-min general fuzzy automaton by another, admissible relations, admissible partitions of a max-min general fuzzy automaton, $\tilde{\delta}$ -orthogonality of admissible partitions, irreducible max-min general fuzzy automata. Then we have obtained the relationships between them, for example, we have shown that Π is maximal if and only if $\tilde{F}/\Pi = (\Pi, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^{\Pi}, F_1, F_2)$ is irreducible.

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MOHAMMAD HORRY, SHAHID CHAMRAN UNIVERSITY OF KERMAN, KERMAN, IRAN
E-mail address: mohhorry@chamran-edu.ir, mohhorry@yahoo.com