

MODIFIED K-STEP METHOD FOR SOLVING FUZZY INITIAL VALUE PROBLEMS

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ABSTRACT. We are concerned with the development of a k -step method for the numerical solution of fuzzy initial value problems. Convergence and stability of the method are also proved in detail. Moreover, a specific method of order 4 is found. The numerical results show that the proposed fourth order method is efficient for solving fuzzy differential equations.

1. Introduction

The still-standing problem in the theory of fuzzy differential equations (FDEs) is to find implementable numerical methods. Much effort has been made in this direction as well [1-9, 16, 26-28, 30-37, 41, 42]. For example, under Hukuhara or Seikkala differentiability concept, Abbasbandy and Allahviranloo [3] established a numerical method based on Taylor expansion, Allahviranloo et al. [4] proposed the predictor-corrector method (k -step method) and later they improved this method [5]. The k -step methods based on numerical differentiation have order k and stable methods are obtained for $k \leq 6$.

Here, to increase the order of stable k -step methods, we modify using one non-step point. This method is a powerful mathematical tool for solving linear and non-linear initial value problems (in the crisp case) and it can be implemented easily in practice.

The rest of the paper is organized as follows. In Section 2, some basic definitions and results are presented. A brief description of the modified k -step method is given in Section 3. Section 4 is devoted to the proposed method and its convergence. Finally, Sections 5 and 6, we focus on a fourth order method and present two numerical examples to illustrate our method. Conclusions are given in Section 7.

2. Preliminaries

2.1. Basic Concepts. Let $\mathbb{R}_{\mathcal{F}}$ be the set of all real fuzzy numbers which are normal, upper semicontinuous, convex and compactly supported fuzzy sets.

The parametric form of a fuzzy number is shown by $v = (\underline{v}(r), \overline{v}(r))$, where functions $\underline{v}(r)$ and $\overline{v}(r)$, $0 \leq r \leq 1$, satisfy the following conditions [32]:

- (1) $\underline{v}(r)$ is a monotonically increasing left continuous function.
- (2) $\overline{v}(r)$ is a monotonically decreasing left continuous function.
- (3) $\underline{v}(r) \leq \overline{v}(r)$, $0 \leq r \leq 1$.

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For $0 \leq r \leq 1$, denote $[v]^r = \{x \in \mathbb{R}; v(x) \geq r\}$ and $[v]^0 = \overline{\{x \in \mathbb{R}; v(x) > 0\}}$. Then, it is well-known that for any $r \in [0, 1]$, $[v]^r = [\underline{v}^r, \overline{v}^r]$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $[u + v]^r = [u]^r + [v]^r$ and $[\lambda.v]^r = \lambda[v]^r$, respectively, $\forall r \in [0, 1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals of \mathbb{R} and $\lambda[u]^r$ means the usual product of a scalar and a subset of \mathbb{R} .

Note that a crisp number α is simply represented by $\underline{v}(r) = \overline{v}(r) = \alpha$, $0 \leq r \leq 1$.

Let A and B two nonempty bounded subsets of \mathbb{R} . The Hausdorff distance between A and B is:

$$d_H(A, B) = \max \left[\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b| \right].$$

The metric d_H on $\mathbb{R}_{\mathcal{F}}$ is as follows,

$$d_{\infty}(u, v) = \sup \{d_H([u]^r, [v]^r), r \in [0, 1]\}, \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

Definition 2.1. Let I be a real interval. The mapping $f : I \rightarrow \mathbb{R}_{\mathcal{F}}$ is called a fuzzy function and its r -level set is denoted by

$$[f(t)]^r = [\underline{f}^r(t), \overline{f}^r(t)], \quad t \in I, \quad r \in [0, 1].$$

Definition 2.2. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and is denoted by $x \ominus y$.

Note that $x \ominus y \neq x + (-1)y$.

Definition 2.3. [6] Let I be an open interval in \mathbb{R} . A fuzzy function $f : I \rightarrow \mathbb{R}_{\mathcal{F}}$ is said to be Hukuhara differentiable at $t_0 \in I$, if for some $h_0 > 0$ the Hukuhara differences $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ exist in $\mathbb{R}_{\mathcal{F}}$, for all $0 < h < h_0$, and if there exists an element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\lim_{h \rightarrow 0^+} d_{\infty} \left(\frac{f(t_0 + h) \ominus f(t_0)}{h}, f'(t_0) \right) = 0$$

and

$$\lim_{h \rightarrow 0^+} d_{\infty} \left(\frac{f(t_0) \ominus f(t_0 - h)}{h}, f'(t_0) \right) = 0.$$

The fuzzy set $f'(t_0)$ is called the Hukuhara derivative of f at t_0 .

Theorem 2.4. [28] Let $f : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ be Hukuhara differentiable and denote $[f(t)]^r = [\underline{f}^r(t), \overline{f}^r(t)]$. Then, the boundary functions $\underline{f}^r(t)$ and $\overline{f}^r(t)$ are differentiable and

$$[f'(t)]^r = [(\underline{f}^r)'(t), (\overline{f}^r)'(t)], \quad r \in [0, 1]. \quad (1)$$

Let $(\underline{f}^r)'(t)$ and $(\overline{f}^r)'(t)$ also be continuous functions with respect to both t and r . This property is called *continuity condition*.

Theorem 2.5. [9] *Assuming the continuity condition holds, if one of the derivatives defined in [18, 20, 31, 35] or [37] exists and it is a fuzzy number, then so do the others and they are all equal. Their value is provided by (1).*

The continuity condition is assumed to hold for all fuzzy functions in the rest of the paper.

2.2. Fuzzy Initial Value Problem. Consider the fuzzy initial value problem (FIVP)

$$y' = f(t, y), \quad y(t_0) = y_0 \in \mathbb{R}_{\mathcal{F}}, \quad (2)$$

where $f : [t_0, T] \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ and $T \in \mathbb{R}$ is a given number. Using Theorem 2.4, we can translate FIVP given as (2) into a system of ODEs. If $y(t)$ is Hukuhara differentiable, then $[y'(t)]^r = [(y^r)'(t), (\overline{y}^r)'(t)]$. So, (2) translates into the following system of ODEs:

$$\begin{cases} (\underline{y}^r)'(t) = \underline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t)), \\ (\overline{y}^r)'(t) = \overline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t)), \\ \underline{y}^r(t_0) = \underline{y}_0^r, \\ \overline{y}^r(t_0) = \overline{y}_0^r, \end{cases} \quad (3)$$

where $[f(t, y)]^r = [\underline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t)), \overline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t))]$.

If the solution $(\underline{y}^r(t), \overline{y}^r(t))$ of the system (3) is expressed as valid level sets of a fuzzy number valued function and if derivatives $((\underline{y}^r)'(t), (\overline{y}^r)'(t))$ are valid level sets of a fuzzy-valued function, then we can construct the solution of FIVP (2) (for more details, see [6, 28]).

The following theorem shows that FIVP (2) is equivalent to the system (3).

Theorem 2.6. [6] *Let us consider FIVP (2), where $f : [t_0, T] \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is such that*

- (i) $[f(t, y)]^r = [\underline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t)), \overline{f}^r(t, \underline{y}^r(t), \overline{y}^r(t))]$;
- (ii) \underline{f}^r and \overline{f}^r are equicontinuous (i.e., for any $\varepsilon > 0$ and any $(t, x, y) \in [t_0, T] \times \mathbb{R}^2$, we have $|\underline{f}^r(t, x, y) - \underline{f}^r(t_1, x_1, y_1)| < \varepsilon$, $\forall r \in [0, 1]$, whenever $\|(t_1, x_1, y_1) - (t, x, y)\| < \delta$) and uniformly bounded on any bounded set;
- (iii) there exists $L > 0$ such that $d_{\infty}(f(t, x), f(t, y)) \leq Ld_{\infty}(x, y)$, for all $t \in [t_0, T]$, $x, y \in \mathbb{R}_{\mathcal{F}}$.

Then the FIVP (2) and the system of ODEs (3) are equivalent.

3. A Brief Description of the Modified k -step Method

We start this section by defining a k -step method.

Definition 3.1. A k -step method (hereinafter predictor) for solving the initial value problem is one whose difference equation for finding the approximation $y_0(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation:

$$y_0(t_{i+1}) = \sum_{j=1}^k a_{0j} y_0(t_{i-j+1}) + h \sum_{j=0}^k b_{0j} f(t_{i-j+1}, y_0(t_{i-j+1})) \quad (4)$$

for $i = k - 1, k, \dots, N - 1$, such that $t_0 \leq t_1 \leq \dots \leq t_N = T$, $h = \frac{(T-t_0)}{N} = t_{i+1} - t_i$ and $a_{01}, a_{02}, \dots, a_{0k}, b_{00}, b_{01}, \dots, b_{0k}$ are constant with the starting values $y_0 = \alpha_0, y_1 = \alpha_1, \dots, y_{k-1} = \alpha_{k-1}$.

If $b_{00} = 0$, then formula (4) is called an explicit formula and if $b_{00} \neq 0$, then the equation is referred to as an implicit formula.

The k -step methods based on numerical differentiation have order k and stable methods are obtained for $k \leq 6$. To increase the order of the stable k -step methods, we modify (4) by including a linear combination of one non-step point between t_i and t_{i+1} .

Definition 3.2. The modified k -step method with one slope (hereinafter corrector) is given by

$$y_{1,i+1} = \sum_{j=1}^k a_{1j} y_{1,i-j+1} + h \sum_{j=0}^k b_{1j} f(t_{i-j+1}, y_{1,i-j+1}) + hc_1 f(t_{i-\theta_1+1}, y_{1,i-\theta_1+1}) \quad (5)$$

where the a_{1j} , the b_{1j} , c_1 and $0 < \theta_1 < 1$ are $2k + 3$ arbitrary parameters.

We use the definition of convergence from Henrici [24], given next.

Definition 3.3. The modified k -step method is convergent if, for every problem $y' = f(t, y), t \in [t_0, T], y(t_0) = y_0$, satisfying Lipschitz continuity conditions for $t \in [t_0, T]$ and $y \in (-\infty, +\infty)$, $\varepsilon_n \rightarrow 0$, as $h \rightarrow 0$. Here, $\varepsilon_n = y_{1n} - y(t_n)$, where $y(x)$ is the solution of the differential equation, and y_{1n} is the value at $t = t_n$ calculated by (5) from a suitable set of starting values y_0, y_1, \dots, y_k .

Definition 3.4. A method is said to be stable if each root of the polynomial equation

$$\rho(\xi) = \xi^k - \sum_{j=1}^k \xi^{k-j} a_{1j} = 0,$$

is either inside or on the unit circle and is simple. And the method is consistent if the corrector is of order 1 or greater and if the predictor is of order zero or greater. Formally, this means that

$$1 - \sum_{j=1}^k a_{ij} = 0, \quad i = 0, 1, \quad (6)$$

and

$$k - \sum_{j=1}^k (k-j)a_{1j} = \sum_{j=0}^k b_{1j} + c_1. \quad (7)$$

4. Main Result

In this section, we use the modified k -step method (5) to solve FIVP (2). Let the fuzzy initial value be $y_0 = \alpha_0, y_1 = \alpha_1, \dots, y_{k-1} = \alpha_{k-1}$. According to Theorem 2.6 and using (4) and (5), for any $r \in [0, 1]$, and $i = k-1, \dots, N-1$, we obtain the predictor formula:

$$\left\{ \begin{array}{l} \underline{y}_{0,i+1}^r = \sum_{j=1}^k \underline{a}_{0j} \underline{y}_{0,i-j+1}^r + h \sum_{j=0}^k \underline{b}_{0j} \underline{f}_{0,i-j+1}^r, \\ \underline{y}_{00} = \underline{\alpha}_0, \underline{y}_{01} = \underline{\alpha}_1, \dots, \underline{y}_{0,k-1} = \underline{\alpha}_{k-1}, \\ \overline{y}_{0,i+1}^r = \sum_{j=1}^k \overline{a}_{0j} \overline{y}_{0,i-j+1}^r + h \sum_{j=0}^k \overline{b}_{0j} \overline{f}_{0,i-j+1}^r, \\ \overline{y}_{00} = \overline{\alpha}_0, \overline{y}_{01} = \overline{\alpha}_1, \dots, \overline{y}_{0,k-1} = \overline{\alpha}_{k-1}, \end{array} \right. \quad (8)$$

and the corrector formula:

$$\left\{ \begin{array}{l} \underline{y}_{1,i+1}^r = \sum_{j=1}^k \underline{a}_{1j} \underline{y}_{1,i-j+1}^r + h \sum_{j=0}^k \underline{b}_{1j} \underline{f}_{1,i-j+1}^r + h \underline{c}_1 \underline{f}(t_{i-\theta_1+1}, \underline{y}_{1,i-\theta_1+1}^r, \overline{y}_{1,i-\theta_1+1}^r) \\ \underline{y}_{10} = \underline{\alpha}_0, \underline{y}_{11} = \underline{\alpha}_1, \dots, \underline{y}_{1,k-1} = \underline{\alpha}_{k-1}, \\ \overline{y}_{1,i+1}^r = \sum_{j=1}^k \overline{a}_{1j} \overline{y}_{1,i-j+1}^r + h \sum_{j=0}^k \overline{b}_{1j} \overline{f}_{1,i-j+1}^r + h \overline{c}_1 \overline{f}(t_{i-\theta_1+1}, \underline{y}_{1,i-\theta_1+1}^r, \overline{y}_{1,i-\theta_1+1}^r), \\ \overline{y}_{10} = \overline{\alpha}_0, \overline{y}_{11} = \overline{\alpha}_1, \dots, \overline{y}_{1,k-1} = \overline{\alpha}_{k-1}. \end{array} \right. \quad (9)$$

We need the following lemmas to explore the convergence of the method.

Lemma 4.1. *Let*

$$\underline{E}_{0,h}^r(t_n, \underline{y}^r) = \underline{y}_{0,n+1}^r - \underline{y}^r(t_{n+1}), \quad \overline{E}_{0,h}^r(t_n, \overline{y}^r) = \overline{y}_{0,n+1}^r - \overline{y}^r(t_{n+1}), \quad (10)$$

where $0 \leq r \leq 1$. Here, $\underline{E}_{0,h}^r$ and $\overline{E}_{0,h}^r$ are the truncation error in the final predictor (8). Then, consistency implies that $\underline{E}_{0,h}^r \rightarrow 0, \overline{E}_{0,h}^r \rightarrow 0$, as $h \rightarrow 0$, uniformly for $t \in [t_0, T], r \in [0, 1]$.

Proof. Define

$$\underline{\chi}^r(\delta) = \max_{|t-t^*| \leq \delta} |\underline{y}^{r'}(t^*) - \underline{y}^{r'}(t)|, \quad t, t^* \in [t_0, T].$$

$\underline{\chi}^r(\delta)$ exists and approaches 0 as $\delta \rightarrow 0$, since $y' = f(t, y)$ exists and is continuous in the closed interval $[t_0, T]$. Hence,

$$\underline{y}^{r'}(t_{n-i}) = \underline{y}^{r'}(t_n) + \underline{\nu}_{-i}^r \underline{\chi}^r(ih),$$

and

$$\underline{y}^r(t_{n-i}) = \underline{y}^r(t_n) - ih[\underline{y}^{r'}(t_n) + \underline{\nu}_{-i}^r \underline{\chi}^r(ih)],$$

where,

$$|\underline{\nu}_{-i}^r| \leq 1, \quad |\underline{\nu}_{-i}^{r'}| \leq 1, \quad r \in [0, 1] \quad \text{and} \quad t_n, t_{n-i} \in [t_0, T].$$

Now,

$$\begin{aligned}\underline{E}_{0,h}^r(t_n, \underline{y}^r) &= \underline{y}_{0,n+1}^r - \underline{y}^r(t_{n+1}) \\ &= \sum_{i=1}^k \underline{a}_{0,i} [\underline{y}^r(t_n) - ih[\underline{y}^{r'}(t_n) + \underline{\nu}_{-i}^{r'} \underline{\chi}^r(ih)]] \\ &\quad + h \sum_{i=0}^k \underline{b}_{0,i} [\underline{y}^{r'}(t_n) + \underline{\nu}_{-i}^{r'} \underline{\chi}^r(ih)] - \underline{y}^r(t_n) - h[\underline{y}^r(t_n) + \underline{\nu}_{-1}^{r'} \underline{\chi}^r(h)].\end{aligned}$$

But, from (6) we have

$$\sum_{i=1}^k \underline{a}_{0,i} = 1.$$

Therefore,

$$\underline{E}_{0,h}^r = O(h) \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

Similarly,

$$\overline{E}_{0,h}^r = O(h) \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

□

Lemma 4.2. *Consider*

$$\left\{ \begin{array}{l} \underline{y}_{2,n+1}^r = \sum_{j=1}^k \underline{a}_{1j} \underline{y}_{2,n-j+1}^r + h \sum_{j=0}^k \underline{b}_{1j} \underline{f}_{2,n-j+1}^r + h \underline{c}_1 \underline{f}(t_{n+1}, \underline{y}_{2,n+1}^r, \overline{y}_{2,n+1}^r) \\ \underline{y}_{20}^r = \underline{\alpha}_0, \underline{y}_{21}^r = \underline{\alpha}_1, \dots, \underline{y}_{2,k-1}^r = \underline{\alpha}_{k-1}, \\ \overline{y}_{2,n+1}^r = \sum_{j=1}^k \overline{a}_{1j} \overline{y}_{2,n-j+1}^r + h \sum_{j=0}^k \overline{b}_{1j} \overline{f}_{2,n-j+1}^r + h \overline{c}_1 \overline{f}(t_{n+1}, \underline{y}_{2,n+1}^r, \overline{y}_{2,n+1}^r), \\ \overline{y}_{20}^r = \overline{\alpha}_0, \overline{y}_{21}^r = \overline{\alpha}_1, \dots, \overline{y}_{2,k-1}^r = \overline{\alpha}_{k-1}. \end{array} \right. \quad (11)$$

where $r \in [0, 1]$ and $n = k-1, \dots, N-1$. Let

$$\underline{E}_{2,h}^r(t_n, \underline{y}^r) = \underline{y}_{2,n+1}^r - \underline{y}^r(t_{n+1}), \quad \overline{E}_{2,h}^r(t_n, \overline{y}^r) = \overline{y}_{2,n+1}^r - \overline{y}^r(t_{n+1}), \quad r \in [0, 1]. \quad (12)$$

Here, $\underline{E}_{2,h}^r$ and $\overline{E}_{2,h}^r$ are the truncation errors in the final corrector (11). Then, consistency implies that

$$\frac{\underline{E}_{2,h}^r}{h} \rightarrow 0, \quad \frac{\overline{E}_{2,h}^r}{h} \rightarrow 0$$

as $h \rightarrow 0$ uniformly for $t \in [t_0, T]$.

Proof. Similar to Lemma 4.1,

$$\begin{aligned}
\overline{E}_{2,h}^r(t_n, \overline{y}^r) &= \overline{y}_{2,n+1}^r - \overline{y}^r(t_{n+1}) \\
&= \overline{y}^r(t_n) \left[\sum_{i=1}^k \overline{a}_{1,i} - 1 \right] \\
&+ h \left[-(\overline{y}^r)'(t_n) - \sum_{i=1}^k i \overline{a}_{1,i} (\overline{y}^r)'(t_n) + h \sum_{i=0}^k \overline{b}_{1,i} (\overline{y}^r)'(t_n) \right. \\
&+ \left. \overline{c}_1 \overline{f}^r \left(t_n, \sum_{m=1}^k \underline{a}_{0,m} \underline{y}^r(t_n) + O(h), \sum_{m=1}^k \overline{a}_{0,m} \overline{y}^r(t_n) + O(h) \right) \right] \\
&+ h \overline{\chi}(kh) \overline{B},
\end{aligned}$$

where \overline{B} is bounded, as $h \rightarrow 0$.

Since $\sum_{m=1}^k \underline{a}_{0,m} = 1$ and $\sum_{m=1}^k \overline{a}_{0,m} = 1$ and f satisfies the Lipschitz condition, the last term can be replaced by

$$\overline{c}_1 (\overline{y}^r)'(t_n) + O(h).$$

Now, using

$$\sum_{i=1}^k \underline{a}_{1,i} = 1,$$

and

$$k - \sum_{j=1}^k (k-j) \overline{a}_{1,j} = \sum_{j=0}^k \overline{b}_{1,j} + \overline{c}_1,$$

we get

$$\overline{E}_{2,h}^r(t_n, \overline{y}^r) = h \overline{\chi}(kh) \overline{B} + O(h^2).$$

Therefore,

$$\frac{\overline{E}_{2,h}^r}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Similarly,

$$\frac{E_{1,h}^r}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

□

We are now ready to state and prove the main result.

Theorem 4.3. *Assume that the conditions of Theorem 2.6, Definitions 3.1 and 3.2 hold. Let $(\underline{y}^r(t_n), \overline{y}^r(t_n))$ and $(\underline{y}_{1,n}^r, \overline{y}_{1,n}^r)$ be the solutions of (3) and (9), respectively, for all $r \in [0, 1]$ and $t_n \in [t_0, T]$. Then, the method (9) is convergent if it is stable and consistent. In other words, stability and consistency imply that $\underline{y}_{1,N}^r \rightarrow \underline{y}^r(t_N)$ and $\overline{y}_{1,N}^r \rightarrow \overline{y}^r(t_N)$, as $h \rightarrow 0$.*

Proof. Define the errors in the predictor (8) as $\underline{\varepsilon}_{0,n}^r, \bar{\varepsilon}_{0,n}^r$ and the errors in the corrector (9) as $\underline{\varepsilon}_{1,n}^r, \bar{\varepsilon}_{1,n}^r$, i.e.,

$$\underline{\varepsilon}_{0,n}^r = \underline{y}_{0,n}^r - \underline{y}^r(t_n), \quad \bar{\varepsilon}_{0,n}^r = \bar{y}_{0,n}^r - \bar{y}^r(t_n)$$

and

$$\underline{\varepsilon}_{1,n}^r = \underline{y}_{1,n}^r - \underline{y}^r(t_n), \quad \bar{\varepsilon}_{1,n}^r = \bar{y}_{1,n}^r - \bar{y}^r(t_n).$$

Then, from (10) and (12) we have

$$\underline{\varepsilon}_{1,n+1}^r = \underline{y}_{1,n+1}^r - \underline{y}_{2,n+1}^r + \underline{E}_{2,h}^r(t_n, \underline{y}^r), \quad \bar{\varepsilon}_{1,n+1}^r = \bar{y}_{1,n+1}^r - \bar{y}_{2,n+1}^r + \bar{E}_{2,h}^r(t_n, \bar{y}^r), \quad (13)$$

and

$$\underline{\varepsilon}_{0,n}^r = \underline{E}_{0,h}^r(t_n, \underline{y}^r), \quad \bar{\varepsilon}_{0,n}^r = \bar{E}_{0,h}^r(t_n, \bar{y}^r). \quad (14)$$

When we substitute for $\underline{y}_1 - \underline{y}_2$ and $\bar{y}_1 - \bar{y}_2$ in (13), terms of the form $h\underline{c}_1 (f(\underline{y}_{1,n}) - f(\underline{y}_{2,n}))$ and $h\bar{c}_1 (f(\bar{y}_{1,n}) - f(\bar{y}_{2,n}))$ will be included, which, by the Lipschitz condition, can be replaced by $h\underline{c}_1 \underline{L}(\underline{y}_{1,n} - \underline{y}_{2,n})$ and $h\bar{c}_1 \bar{L}(\bar{y}_{1,n} - \bar{y}_{2,n})$, respectively, where \underline{L} and \bar{L} are bounded. Since the formulas are explicit, the $\underline{y}_{1,n} - \underline{y}_{2,n}$ and $\bar{y}_{1,n} - \bar{y}_{2,n}$ terms can be substituted repeatedly. The process stops after a maximum of N steps, resulting in a polynomial in h with bounded coefficients. Thus, (13) can be rewritten as

$$\begin{aligned} \underline{\varepsilon}_{1,n+1}^r &= \underline{E}_{2,h}^r(t_n, \underline{y}^r) + \sum_{i=1}^k \underline{a}_{1,i} \underline{\varepsilon}_{1,n-i}^r \\ &+ h \sum_{i=0}^k (\underline{\varepsilon}_{1,n-i}^r P_{1i}^r(h, t_n) + \underline{\varepsilon}_{0,n-i}^r P_{2i}^r(h, t_n)), \end{aligned} \quad (15)$$

and

$$\begin{aligned} \bar{\varepsilon}_{1,n+1}^r &= \bar{E}_{2,h}^r(t_n, \bar{y}^r) + \sum_{i=1}^k \bar{a}_{1,i} \bar{\varepsilon}_{1,n-i}^r \\ &+ h \sum_{i=0}^k (\bar{\varepsilon}_{1,n-i}^r P_{3i}^r(h, t_n) + \bar{\varepsilon}_{0,n-i}^r P_{4i}^r(h, t_n)), \end{aligned} \quad (16)$$

where $r \in [0, 1]$, $P_{1i}^r, P_{2i}^r, P_{3i}^r$ and P_{4i}^r are polynomials in h with bounded coefficients.

Henrici [22, Lemma 5.5] shows that if

$$\xi^k - \sum_{i=0}^{k-1} a_{1,i} \xi^{k-i}$$

is the polynomial of a stable method, and if

$$\frac{1}{1 - \sum_{i=0}^{k-1} a_{1,i} \xi^{k-i}} = \sum_{p=0}^{\infty} \omega_p \xi^p, \quad (17)$$

then there exists a constant $\Gamma < \infty$ such that $|\omega_p| \leq \Gamma$, $p = 0, 1, \dots$. Multiply (15) and (16) by $\underline{\omega}_p$ and $\bar{\omega}_p$, respectively, and sum from $n = k$ to $N - 1$ to get

$$\begin{aligned} & \underline{\varepsilon}_{1,N}^r \underline{\omega}_0 + \underline{\varepsilon}_{1,N-1}^r (\underline{\omega}_1 - \underline{a}_{1,0} \underline{\omega}_0) + \underline{\varepsilon}_{1,N-2}^r (\underline{\omega}_2 - \underline{a}_{1,0} \underline{\omega}_1 - \underline{a}_{1,1} \underline{\omega}_0) + \dots \\ & \quad + \underline{\varepsilon}_{1,k+1}^r (\underline{\omega}_{N-k-1} - \underline{a}_{1,0} \underline{\omega}_{N-k-2} - \dots - \underline{a}_{1,k} \underline{\omega}_{N-2k-2}) \\ & = \sum_{n=k}^{N-1} \underline{\omega}_{N-n-1} \underline{E}_{2,h}^r(t_n, \underline{y}^r) + \sum_{i=0}^k (\text{bounded multiples of } \underline{\varepsilon}_{1,i}^r \text{ and } \underline{\varepsilon}_{0,i}^r) \\ & \quad + h \sum_{n=k}^{N+1} \underline{\omega}_{N-n-1} \sum_{i=0}^k [\underline{\varepsilon}_{1,n-i}^r P_{1i}^r(h, t_n) + \underline{\varepsilon}_{0,n-i}^r P_{2i}^r(h, t_n)], \end{aligned} \quad (18)$$

and

$$\begin{aligned} & \bar{\varepsilon}_{1,N}^r \bar{\omega}_0 + \bar{\varepsilon}_{1,N-1}^r (\bar{\omega}_1 - \bar{a}_{1,0} \bar{\omega}_0) + \bar{\varepsilon}_{1,N-2}^r (\bar{\omega}_2 - \bar{a}_{1,0} \bar{\omega}_1 - \bar{a}_{1,1} \bar{\omega}_0) + \dots \\ & \quad + \bar{\varepsilon}_{1,k+1}^r (\bar{\omega}_{N-k-1} - \bar{a}_{1,0} \bar{\omega}_{N-k-2} - \dots - \bar{a}_{1,k} \bar{\omega}_{N-2k-2}) \\ & = \sum_{n=k}^{N-1} \bar{\omega}_{N-n-1} \bar{E}_{2,h}^r(t_n, \bar{y}^r) + \sum_{i=0}^k (\text{bounded multiples of } \bar{\varepsilon}_{1,i}^r \text{ and } \bar{\varepsilon}_{0,i}^r) \\ & \quad + h \sum_{n=k}^{N+1} \bar{\omega}_{N-n-1} \sum_{i=0}^k [\bar{\varepsilon}_{1,n-i}^r P_{3i}^r(h, t_n) + \bar{\varepsilon}_{0,n-i}^r P_{4i}^r(h, t_n)]. \end{aligned} \quad (19)$$

From (17), $\underline{\omega}_0 = \bar{\omega}_0 = 1$ and

$$\underline{\omega}_m - \underline{a}_{1,0} \underline{\omega}_{m-1} - \underline{a}_{1,0} \underline{\omega}_{m-2} - \dots - \underline{a}_{1,0} \underline{\omega}_{m-k-1} = 0,$$

and

$$\bar{\omega}_m - \bar{a}_{1,0} \bar{\omega}_{m-1} - \bar{a}_{1,0} \bar{\omega}_{m-2} - \dots - \bar{a}_{1,0} \bar{\omega}_{m-k-1} = 0.$$

Therefore, the left hand sides of (18) and (19) equal $\underline{\varepsilon}_{1,N}^r$ and $\bar{\varepsilon}_{1,N}^r$, respectively. The last terms of the right hand sides involve each $\underline{\varepsilon}_{1,i}^r, \bar{\varepsilon}_{1,i}^r, \underline{\varepsilon}_{0,i}^r$ and $\bar{\varepsilon}_{0,i}^r$ no more than $k + 1$ times. Thus, (18) and (19) give the bounds

$$\begin{aligned} |\underline{\varepsilon}_{1,N}^r| & \leq \underline{\Gamma}(N - k) \max |E_{2,h}^r(t_n, \underline{y}^r)| \\ & \quad + M_1 \sum_{i=0}^k (|\underline{\varepsilon}_{1,i}^r| + |\underline{\varepsilon}_{0,i}^r|) + h C_1 \sum_{n=0}^{N-1} (|\underline{\varepsilon}_{1,n}^r| + |\underline{\varepsilon}_{0,n}^r|), \end{aligned}$$

and

$$\begin{aligned} |\bar{\varepsilon}_{1,N}^r| & \leq \bar{\Gamma}(N - k) \max |\bar{E}_{2,h}^r(t_n, \bar{y}^r)| \\ & \quad + M_2 \sum_{i=0}^k (|\bar{\varepsilon}_{1,i}^r| + |\bar{\varepsilon}_{0,i}^r|) + h C_2 \sum_{n=0}^{N-1} (|\bar{\varepsilon}_{1,n}^r| + |\bar{\varepsilon}_{0,n}^r|), \end{aligned}$$

where M_1, M_2, C_1 and C_2 are bounded for all h less than or equal to some h_0 . Now, $N - k \leq \frac{(T-t_0)}{h}$,

$$\left| \frac{E_{2,h}^r(t_n, \underline{y}^r)}{h} \right| \leq \mu_1(h) \rightarrow 0 \text{ as } h \rightarrow 0,$$

and

$$\left| \frac{\overline{E}_{2,h}^r(t_n, \overline{y}^r)}{h} \right| \leq \mu_2(h) \rightarrow 0, \text{ as } h \rightarrow 0.$$

Therefore,

$$|\underline{\varepsilon}_{1,N}^r| \leq (T - t_0)\underline{\Gamma}\mu_1(h) + S_1(h) + hC_1 \sum_{n=0}^{N-1} (|\underline{\varepsilon}_{1,n}^r| + |\underline{\varepsilon}_{0,n}^r|), \quad (20)$$

and

$$|\overline{\varepsilon}_{1,N}^r| \leq (T - t_0)\overline{\Gamma}\mu_2(h) + S_2(h) + hC_2 \sum_{n=0}^{N-1} (|\overline{\varepsilon}_{1,n}^r| + |\overline{\varepsilon}_{0,n}^r|), \quad (21)$$

where $S_1(h), S_2(h)$ are functions of the errors in the initial conditions which approach 0 as $h \rightarrow 0$, and C_1, C_2 are bounded for $h \leq h_0$.

Let $\underline{E}_N^r = \max\{|\underline{\varepsilon}_{1,N}^r|, |\underline{\varepsilon}_{0,N}^r|\}$ and $\overline{E}_N^r = \max\{|\overline{\varepsilon}_{1,N}^r|, |\overline{\varepsilon}_{0,N}^r|\}$. So, from lemma 4.2, we get

$$\underline{E}_N^r < h \sum_{n=0}^{N-1} B_1 |\underline{E}_n^r| + \lambda_1(h),$$

and

$$\overline{E}_N^r < h \sum_{n=0}^{N-1} B_2 |\overline{E}_n^r| + \lambda_2(h),$$

where B_1, B_2 are bounded and $\lambda_1(h) \rightarrow 0, \lambda_2(h) \rightarrow 0$, as $h \rightarrow 0$.

Initially, $\underline{\varepsilon}_{1,i}^r = \underline{\varepsilon}_{0,i}^r = \underline{E}_i^r$ and $\overline{\varepsilon}_{1,i}^r = \overline{\varepsilon}_{0,i}^r = \overline{E}_i^r$, for $i = 0, 1, \dots, k$, and $r \in [0, 1]$. Let these be bounded by $K \rightarrow 0$, as $h \rightarrow 0$. Then,

$$\underline{E}_N^r < (\lambda_1(h) + K)(1 + hB_1)^N \leq (\lambda_1(h) + K)e^{(T-t_0)B_1} \rightarrow 0 \text{ as } h \rightarrow 0,$$

and

$$\overline{E}_N^r < (\lambda_2(h) + K)(1 + hB_2)^N \leq (\lambda_2(h) + K)e^{(T-t_0)B_2} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Thus the proof is complete. \square

5. A Fourth Order Method

If $k = 1$ is used in (9), then we have

$$\begin{cases} \underline{y}_{1,n+1}^r = \underline{a}_{11}\underline{y}_{1,n}^r + h(\underline{b}_{10}\underline{f}_{1,n+1}^r + \underline{b}_{11}\underline{f}_{1,n}^r) + h\underline{c}_1\underline{f}_{1,n-\theta_1+1}^r, \\ \underline{y}_{10} = \underline{\alpha}_0, \\ \overline{y}_{1,n+1}^r = \overline{a}_{11}\overline{y}_{1,n}^r + h(\overline{b}_{10}\overline{f}_{1,n+1}^r + \overline{b}_{11}\overline{f}_{1,n}^r) + h\overline{c}_1\overline{f}_{1,n-\theta_1+1}^r, \\ \overline{y}_{10} = \overline{\alpha}_0, \end{cases} \quad (22)$$

where, $n = 0, 1, \dots, N$, $h = \frac{(T-t_0)}{N} = t_{n+1} - t_n$, $\underline{a}_{11}, \bar{a}_{11}, \underline{b}_{10}, \bar{b}_{10}, \underline{b}_{11}, \bar{b}_{11}, \underline{c}_1, \bar{c}_1$ and θ_1 are arbitrary and $\theta_1 \neq 0$ or 1 .

Now, we try to choose the coefficients to yield the maximum stable order possible. From (6), (7), and expansion of each term in (22) by Taylor's series about t_n and equating the coefficients of like powers of h , we obtain:

$$\begin{aligned} \underline{a}_{11} &= 1 = \bar{a}_{11} \\ \underline{b}_{10} + \underline{b}_{11} + \underline{c}_1 &= 1 = \bar{b}_{10} + \bar{b}_{11} + \bar{c}_1 \\ \underline{b}_{10} + (1 - \theta_1)\underline{c}_1 &= \frac{1}{2} = \bar{b}_{10} + (1 - \theta_1)\bar{c}_1 \\ \frac{1}{2}\underline{b}_{10} + \frac{1}{2}(1 - \theta_1)^2\underline{c}_1 &= \frac{1}{6} = \frac{1}{2}\bar{b}_{10} + \frac{1}{2}(1 - \theta_1)^2\bar{c}_1. \end{aligned} \quad (23)$$

The principal terms of the truncation errors are given by

$$\frac{1}{4}\underline{C}_4 h^4 (\underline{y}^r)^{(4)}(t_n) + O(h^5) \quad \text{and} \quad \frac{1}{4}\bar{C}_4 h^4 (\bar{y}^r)^{(4)}(t_n) + O(h^5), \quad (24)$$

where $\underline{C}_4 = 1 - 4\underline{b}_{10} - 4\underline{c}_1(1 - \theta_1)^3$ and $\bar{C}_4 = 1 - 4\bar{b}_{10} - 4\bar{c}_1(1 - \theta_1)^3$.

For $\underline{b}_{11} = \bar{b}_{11} = \frac{1}{6}$, we have

$$\underline{a}_{11} = \bar{a}_{11} = 1, \quad \underline{b}_{10} = \bar{b}_{10} = \frac{1}{6}, \quad \underline{c}_1 = \bar{c}_1 = \frac{2}{3}, \quad \theta_1 = \frac{1}{2}, \quad \underline{C}_4 = \bar{C}_4 = 0.$$

Therefore, the equations (22) become

$$\left\{ \begin{array}{l} \underline{y}_{1,n+1}^r = \underline{y}_{1,n}^r + \frac{h}{6}(\underline{f}_{1,n+1}^r + \underline{f}_{1,n}^r) + \frac{2h}{3}\underline{f}_{n+\frac{1}{2}}^r, \\ \underline{y}_{10} = \underline{\alpha}_0, \\ \bar{y}_{1,n+1}^r = \bar{y}_{1,n}^r + \frac{h}{6}(\bar{f}_{1,n+1}^r + \bar{f}_{1,n}^r) + \frac{2h}{3}\bar{f}_{n+\frac{1}{2}}^r, \\ \bar{y}_{10} = \bar{\alpha}_0. \end{array} \right. \quad (25)$$

This is a fourth order implicit one step method with one slope.

6. Numerical Experiments

In this section, we solve numerically two problems and we compare the approximate solutions with the exact solutions.

Example 6.1. Consider the fuzzy initial value problem [5],

$$\left\{ \begin{array}{l} y'(t) = -y(t), \\ y(0) = (0.96 + 0.04r, 1.01 - 1.01r), \quad 0 \leq r \leq 1. \end{array} \right.$$

The exact solution at $t = 0.1$ is:

$$\underline{y}^r(0.1) = (0.985 + 0.01r)e^{-0.1} - (1 - r)0.025e^{0.1},$$

$$\overline{y}^r(0.1) = (0.985 + 0.015r)e^{-0.1} + (1 - r)0.025e^{0.1},$$

for $r \in [0, 1]$.

Table 1 shows the comparison of the exact and approximate solutions at $t = 0.1$, for any $r \in [0, 1]$, using (25) with $N = 5$ ($h = 0.02$).

r	\underline{y}_1	\underline{y}	Error	\overline{y}_1	\overline{y}	Error
0	0.86363551	0.86363558	7.0×10^{-8}	0.91889405	0.91889412	7.0×10^{-8}
0.1	0.86775569	0.86775576	7.0×10^{-8}	0.91748838	0.91748845	7.0×10^{-8}
0.2	0.87187589	0.87187595	5.0×10^{-8}	0.91608271	0.91608278	7.0×10^{-8}
0.3	0.87599608	0.87599613	5.0×10^{-8}	0.91467705	0.91467711	6.0×10^{-8}
0.4	0.88011627	0.88011631	4.0×10^{-8}	0.91327140	0.91327144	4.0×10^{-8}
0.5	0.88423647	0.88423650	3.0×10^{-8}	0.91186573	0.91186577	4.0×10^{-8}
0.6	0.88835665	0.88835668	3.0×10^{-8}	0.91046007	0.91046010	3.0×10^{-8}
0.7	0.89247684	0.89247686	2.0×10^{-8}	0.90905441	0.90905443	2.0×10^{-8}
0.8	0.89659704	0.89659705	1.0×10^{-8}	0.90764875	0.90764876	1.0×10^{-8}
0.9	0.90071722	0.90071723	1.0×10^{-8}	0.90624307	0.90624308	1.0×10^{-8}
1	0.90483741	0.90483741	0.0×10^{-8}	0.90483741	0.90483741	0.0×10^{-8}

TABLE 1. Comparison of the Exact and Approximate Solutions at $t = 0.1$

Example 6.2. Consider the fuzzy initial value problem [5],

$$\begin{cases} y'(t) = -y(t) + t + 1, \\ y(0) = (0.96 + 0.04r, 1.01 - 0.01r), \quad 0 \leq r \leq 1. \end{cases}$$

The exact solution at $t = 0.1$ is given by

$$\underline{y}^r(0.1) = 0.1 + (0.985 + 0.015r)e^{-0.1} + (-0.025 + 0.025r)e^{0.1},$$

$$\overline{y}^r(0.1) = 0.1 + (0.985 + 0.015r)e^{-0.1} + (0.025 - 0.025r)e^{0.1},$$

for $r \in [0, 1]$.

Table 2 shows the comparison of the exact and approximate solutions at $t = 0.1$ for any $r \in [0, 1]$, using (25) with $N = 10$ ($h = 0.01$).

r	\underline{y}_1	\underline{y}	Error	\bar{y}_1	\bar{y}	Error
0	0.9636355824	0.9636355838	1.4×10^{-10}	1.0188941283	1.0188941297	1.4×10^{-10}
0.1	0.9677557658	0.9677557672	1.4×10^{-10}	1.0174884571	1.0174884585	1.4×10^{-10}
0.2	0.9718759492	0.9718759506	1.4×10^{-10}	1.0160827860	1.0160827873	1.3×10^{-10}
0.3	0.9759961327	0.9759961340	1.3×10^{-10}	1.0146771149	1.0146771162	1.3×10^{-10}
0.4	0.9801163162	0.9801163175	1.3×10^{-10}	1.0132714438	1.0132714450	1.2×10^{-10}
0.5	0.9842364997	0.9842365009	1.2×10^{-10}	1.0118657726	1.0118657738	1.2×10^{-10}
0.6	0.9883566832	0.9883566843	1.1×10^{-10}	1.0104601015	1.0104601027	1.2×10^{-10}
0.7	0.9924768666	0.9924768677	1.1×10^{-10}	1.0090544305	1.0090544315	1.0×10^{-10}
0.8	0.9965970501	0.9965970511	1.0×10^{-10}	1.0076487593	1.0076487603	1.0×10^{-10}
0.9	1.0007172336	1.0007172346	1.0×10^{-10}	1.0062430882	1.0062430892	1.0×10^{-10}
1	1.0048374171	1.0048374180	0.9×10^{-11}	1.0048374171	1.0048374180	0.9×10^{-11}

TABLE 2. Comparison of the Exact and Approximate Solutions at $t = 0.1$

7. Conclusions

A modified k -step method with one non-step point to solve fuzzy initial value problem was presented. The convergence of this method was proved. An important property of this method is its rapid convergence to the solution.

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