ON L-DOUBLE FUZZY ROUGH SETS

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Abstract. Our aim of this paper is to introduce the concept of L-double fuzzy rough sets in which both constructive and axiomatic approaches are used. In constructive approach, a pair of L-double fuzzy lower (resp. upper) approximation operators is defined and the basic properties of them are studied. From the viewpoint of the axiomatic approach, a set of axioms is constructed to characterize the L-double fuzzy upper (resp. lower) approximation of L-double fuzzy rough sets. Finally, from L-double fuzzy approximation operators, we generated Alexandrov L-double fuzzy topology.

1. Introduction and preliminaries

The theory of rough sets, proposed by Pawlak [21, 22], is an extension of the set theory for the study of intelligent systems characterized by insufficient and incomplete information. Recent years have witnessed its wide applications in various fields such as: machine learning, knowledge discovery, data mining, expert systems, pattern recognition, granular computing, graph theory, algebraic systems, partially ordered sets [4, 9, 16, 26, 36].

It is well known that the most important concepts in the rough set theory are the upper and the lower approximations derived from a binary relation R on the universe of discourse. There is at least two different basic approaches that have been formed for developing rough set theories, i.e., the constructive approach and the axiomatic approach. The constructive approach is suitable for practical applications of rough sets. The axiomatic approach, which is appropriate for studying the structures of rough set algebras, takes the lower and upper approximation operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators that are the same as the ones produced by using constructive approach.

The initiations and majority of studies on rough sets have been concentrated on constructive approaches. In Pawlak's rough set model [23], an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. In order to solve this problem, several authors have generalized the notion of approximation operators by using nonequivalence binary relations [17, 42]. This has led to various other approximation operators [28, 40, 48].

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Fuzzy generalizations of rough sets were first introduced by Dubois and Prade [8]. Radzikowska and Kerre [30] considered L-fuzzy rough sets as a further generalization of the notion of rough sets. That is, Radzikowska and Kerre extended fuzzy rough sets from unit interval $[0, 1]$ to residuated lattices. Liu [20], introduced the concept of generalized rough sets over fuzzy lattices. Through this concept, the crisp [24, 25] and fuzzy generalizations [8] of rough sets can also be put into one framework. Kubiak [18] and Šostak [37] introduced the notion of L-fuzzy topological space as a generalization of L-topological spaces which introduced by Chang [3] and Goguen [12]. Many works in L-fuzzy topology have been launched [14, 31, 32, 33]. Also, there are many works investigate the relation between fuzzy rough set and fuzzy topology [13, 27, 38, 43].

On the other hand, the concept of an intuitionistic fuzzy set, originally proposed by Atanassov [1, 2], is an important tool for dealing with imperfect and imprecise information. Compared with Zadeh's fuzzy sets [44], an intuitionistic fuzzy set gives the membership and nonmembership degree to which an element belongs to a set. Hence, coping with imperfect and imprecise information is more flexible and effective for intuitionistic fuzzy sets. In recent years, intuitionistic fuzzy set theory has been successfully applied in many practical fields, such as decision analysis and pattern recognition [19, 39, 41]. Combining intuitionistic fuzzy set theory and rough set theory may be a promising topic that deserves further investigation. Some research has already been carried out on this topic [15, 29, 35]. For example, Çoker [7] first revealed the relationship between intuitionistic fuzzy set theory and rough set theory and showed that a fuzzy rough set was in fact an intuitionistic L-fuzzy set. Çoker and his colleagues [5, 6] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [34], introduced the notion of intuitionistic gradation of openness (which we call it L-double fuzzy topology) as a generalization of intuitionistic fuzzy topology [6] and L-fuzzy topology [37]. Working under the name "intuitionistic" has doubts that were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were ended in 2005 by Gutierrez Garcia and Rodabaugh [10]. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

In this paper, we first introduce the concept of L-double fuzzy relation and use it as a tool to define L-double fuzzy lower approximation and L-double fuzzy upper approximation. Also, we study L-double fuzzy rough sets through both the constructive and axiomatic approaches. From the viewpoint of the constructive approach, we study the relations among L-double fuzzy lower approximation and L-double fuzzy upper approximation under arbitrary L-double fuzzy relation. The connections between special L-double fuzzy relations and L-double fuzzy lower and L-double fuzzy upper approximation operators are also examined. In the axiomatic approach, a set of axioms is constructed to characterize the L-double fuzzy upper (resp. lower) approximation of L-double fuzzy rough sets. Finally, from L-double fuzzy approximation operators, we generated Alexandrov L-double fuzzy topology.

It should be noted that there are previous works in intuitionistic fuzzy rough sets as [45, 46, 47]. In these works the authors define the notion of intuitionistic fuzzy rough sets by using intuitionistic fuzzy sets due to Atanassove [1, 2], but in our work we define L-double fuzzy rough sets by using L-fuzzy sets due to Goguen [11]. In [47] an intuitionistic fuzzy topology which is generated from intuitionistic fuzzy approximation operators is due to Çoker [5], but in our work the L-double fuzzy topology which is generated from L-double fuzzy approximation operators is due to Samanta and Mondal [34] and it is Alexandrov.

Throughout this paper, let $(L, \wedge, \vee, ', 0_L, 1_L)$ be a fuzzy lattice, i.e., a completely distributive lattice with an order reversing involution $' : L \to L$ where, 0_L and 1_M denotes the smallest and largest elements of lattice L. Let U be a non-empty set of objects called the universe. An L-fuzzy set is a map from U to L [11], let L^U, the set of all L-fuzzy sets in U. 0_U and 1_U are the smallest and largest elements of L^U. A map $R : U \times U \to L$ is called an L- fuzzy relation on U. $R(x, y)$ is referred to as the degree of relation between x and y, where $(x, y) \in U \times U$. R is referred to as a reflexive relation if $R(x, x) = 1_L$, $\forall x \in U$; R is referred to as a symmetric relation if $R(x, y) = R(y, x)$ $\forall x, y \in U$; R is referred to as a transitive relation if $R(x, z) \geq \bigvee_{y\in U}(R(x, y) \wedge R(y, z))$ for each $x, y, z \in U$ and R is referred to as a serial relation if for each $x \in U$, there exists $y \in U$ such that $R(x, y) = 1_L$. R is called an equivalence relation if it is reflexive, symmetric and transitive. For $\alpha \in L$ and $\lambda \in L^U$, the L-fuzzy sets $\alpha\lambda, \alpha \vee \lambda : U \to L$ are defined as follows:

$$(\alpha\lambda)(x) = \alpha \wedge \lambda(x), \text{ and } (\alpha \vee \lambda)(x) = \alpha \vee \lambda(x), \forall x \in U.$$

In addition, for each $x \in U$, we define the L-fuzzy set $\delta_x : U \to L$ as:

$$\delta_x(y) = \begin{cases} 1_L, & \text{if } \ y = x \\ 0_L, & \text{if } \ y \neq x. \end{cases}$$

Lemma 1.1. [20] *Any L-fuzzy set $\lambda \in L^U$ can be written as:*

$$\lambda = \bigvee_{x\in U} \lambda(x)\delta_x.$$

Definition 1.2. [20] Let U be an arbitrary universal set, L be a fuzzy lattice and R be an L-fuzzy relation on U. With each L-fuzzy set λ on U, we associate two L-fuzzy sets $\underline{R}\lambda, \overline{R}\lambda : U \to L$:

$$(\underline{R}\lambda)(x) = \bigwedge_{y\in U} ((R(x, y))' \vee \lambda(y)) \text{ and } (\overline{R}\lambda)(x) = \bigvee_{y\in U} (R(x, y) \wedge \lambda(y)), \forall x \in U.$$

$\underline{R}\lambda, \overline{R}\lambda$ are called the lower and upper approximations of the L-fuzzy set λ, respectively. The pair $(\underline{R}\lambda, \overline{R}\lambda)$ is referred to as a generalized rough set of λ relative to L.

Definition 1.3. [34] The pair $(\mathcal{T}, \mathcal{T}^*)$ of maps $\mathcal{T}, \mathcal{T}^* : L^U \to L$ is called an L-double fuzzy topology on U if it satisfies the following conditions:

(LDFT1) $\mathcal{T}(\lambda) \leq (\mathcal{T}^*(\lambda))'$, for each $\lambda \in L^U$,

(LDFT2) $\mathcal{T}(0_U) = \mathcal{T}(1_U) = 1_L$, $\mathcal{T}^*(0_U) = \mathcal{T}^*(1_U) = 0_L$,

(LDFT3) $\mathcal{T}(\lambda_1 \wedge \lambda_2) \geq \mathcal{T}(\lambda_1) \wedge \mathcal{T}(\lambda_2)$ and $\mathcal{T}^*(\lambda_1 \wedge \lambda_2) \leq \mathcal{T}^*(\lambda_1) \vee \mathcal{T}^*(\lambda_2)$, for any $\lambda_1, \lambda_2 \in L^U$.

(LDFT4) $\mathcal{T}(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \mathcal{T}^*(\lambda_i)$, for any $\{\lambda_i : i \in \Gamma\} \subseteq L^U$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called an L-double fuzzy topological space.
An L-double fuzzy topology $(\mathcal{T}, \mathcal{T}^*)$ is called Alexandrov if it satisfies:

(LDFT5) $\mathcal{T}(\bigwedge_{i\in\Gamma}\lambda_i) \geq \bigwedge_{i\in\Gamma}\mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigwedge_{i\in\Gamma}\lambda_i) \leq \bigvee_{i\in\Gamma}\mathcal{T}^*(\lambda_i)$, for any $\{\lambda_i : i \in \Gamma\} \subseteq L^U$.

Example 1.4. Let $U = \{a, b, c\}$ and $L = I$. Define $\mu \in L^U$ as follows:

$$\mu(a) = 0.2, \qquad \mu(b) = 0.4, \qquad \mu(c) = 0.7.$$

Define $\mathcal{T}, \mathcal{T}^* : L^U \to L$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1_L, & \text{if} \quad \lambda \in \{0_U, 1_U\} \\ 0.5, & \text{if} \quad \lambda = \mu \\ 0_L, & \text{otherwise,} \end{cases} \qquad \mathcal{T}^*(\lambda) = \begin{cases} 0_L, & \text{if} \quad \lambda \in \{0_U, 1_U\} \\ 0.3, & \text{if} \quad \lambda = \mu \\ 1_L, & \text{otherwise.} \end{cases}$$

Then, $(\mathcal{T}, \mathcal{T}^*)$ is an L-double fuzzy topology on U.

Definition 1.5. [34] Let $f : (U, \mathcal{T}_1, \mathcal{T}_1^*) \to (V, \mathcal{T}_2, \mathcal{T}_2^*)$ be a map between L-double fuzzy topological spaces $(U, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(V, \mathcal{T}_2, \mathcal{T}_2^*)$. Then f is said to be continuous if $\forall \lambda \in L^V$, $\mathcal{T}_1(f^{\leftarrow}(\lambda)) \geq \mathcal{T}_2(\lambda)$ and $\mathcal{T}_1^*(f^{\leftarrow}(\lambda)) \leq \mathcal{T}_2^*(\lambda)$.

2. L-double Fuzzy Rough Set

Definition 2.1. Let U and V be two arbitrary sets. The pair (R, R^*) of maps $R, R^* : U \times V \to L$ is called an L-double fuzzy relation on $U \times V$ if $R(x, y) \leq (R^*(x, y))'$, $\forall\ (x, y) \in U \times V$. If $R, R^* : U \times U \to L$, (R, R^*) is called L-double fuzzy relation on U. $R(x, y)$ (resp. $R^*(x, y)$), referred to as the degree of relation (resp. non-relation) between x and y.

Definition 2.2. An L-double fuzzy relation (R, R^*) on U is called:

(i) L-double fuzzy reflexive if $R(x, x) = 1_L$ and $R^*(x, x) = 0_L$, $\forall x \in U$,

(ii) L-double fuzzy symmetric if $R(x, y) = R(y, x)$ and $R^*(x, y) = R^*(y, x)$, $\forall x, y \in U$,

(iii) L-double fuzzy transitive if for each $x, y, z \in U$, $R(x, z) \geq \bigvee_{y\in U}(R(x, y) \wedge R(y, z))$ and $R^*(x, z) \leq \bigwedge_{y\in U}(R^*(x, y) \vee R^*(y, z))$.

(iv) L-double fuzzy serial if for each $x \in U$, there exists $y \in U$ such that $R(x, y) = 1_L$ and $R^*(x, y) = 0_L$.

An L-double fuzzy relation (R, R^*) on U is called an L-double fuzzy equivalent relation if it is L-double fuzzy reflexive, L-double fuzzy symmetric and L-double fuzzy transitive. The triplet (U, R, R^*) is called an L-double fuzzy approximation space.

Example 2.3. Let $U = \{a, b, c\}$ and $L = I$. Define $R, R^* : U \times U \to L$ as follows:

$$R = \begin{pmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}, \quad R^* = \begin{pmatrix} 0 & 0 & 0.4 \\ 0 & 0 & 0.4 \\ 0.4 & 0.4 & 0 \end{pmatrix}$$

Then, (R, R^*) is an L-double fuzzy reflexive (symmetric, transitive, serial) relation.

Definition 2.4. Let U be an arbitrary universal set, L be a fuzzy lattice and (R, R^*) be an L-double fuzzy relation on U. With each L-fuzzy set λ on U, the pairs $(\underline{R}\lambda, \underline{R}^*\lambda)$, $(\overline{R}\lambda, \overline{R}^*\lambda)$ of maps $\underline{R}\lambda, \underline{R}^*\lambda, \overline{R}\lambda, \overline{R}^*\lambda : U \to L$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation of the L-fuzzy set λ, respectively, where

$$(\underline{R}\lambda)(x) = \bigwedge_{y \in U} ((R(x,y))' \vee \lambda(y)), \quad (\underline{R}^*\lambda)(x) = \bigvee_{y \in U} ((R^*(x,y))' \wedge \lambda'(y)), \forall x \in U$$

and

$$(\overline{R}\lambda)(x) = \bigvee_{y \in U} (R(x,y) \wedge \lambda(y)), \quad (\overline{R}^*\lambda)(x) = \bigwedge_{y \in U} (R^*(x,y) \vee \lambda'(y)), \forall x \in U.$$

The quaternary $(\underline{R}\lambda, \underline{R}^*\lambda, \overline{R}\lambda, \overline{R}^*\lambda)$ is called L-double fuzzy rough set of λ. The pairs $(\underline{R}, \underline{R}^*)$, $(\overline{R}, \overline{R}^*)$ of operators $\underline{R}, \underline{R}^*, \overline{R}, \overline{R}^* : L^U \to L^U$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation operators, respectively, and the triplets $(U, \underline{R}, \underline{R}^*)$, $(U, \overline{R}, \overline{R}^*)$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation spaces, respectively.

Remark 2.5. Let U be an arbitrary universal set, $R : U \times U \to L$ be an L-fuzzy relation on U and $(\underline{R}\lambda, \overline{R}\lambda)$ be a generalized rough set of $\lambda \in L^U$. Define a map $R^* : U \times U \to L$ by, $R^*(x,y) = (R(x,y))'$, $\forall (x,y) \in U \times U$. Define L-fuzzy sets $\underline{R}^*\lambda, \overline{R}^*\lambda : U \to L$ as:

$$(\underline{R}^*\lambda)(x) = (\underline{R}\lambda)'(x) \text{ and } (\overline{R}^*\lambda)(x) = (\overline{R}\lambda)'(x), \forall x \in U.$$

Then (R, R^*) is an L-double fuzzy relation on U and $(\underline{R}\lambda, \underline{R}^*\lambda, \overline{R}\lambda, \overline{R}^*\lambda)$ is an L-double fuzzy rough set of λ. Therefore, an L-double fuzzy rough set is a generalization of generalized rough set.

In the following theorems, we obtain the basic properties of L-double fuzzy lower approximation and L-double fuzzy upper approximation.

Theorem 2.6. *Let U be an arbitrary universal set, L be a fuzzy lattice, and (R, R^*) be an L-double fuzzy relation on U, $\lambda \in L^U$. Then,*

(i) $\overline{R}\lambda \leq (\overline{R}^*\lambda)'$ *and* $\underline{R}\lambda \geq (\underline{R}^*\lambda)'$, $\forall \lambda \in L^U$;

(ii) $\underline{R}1_U = 1_U$ *and* $\underline{R}^*1_U = 0_U$;

(iii) $\overline{R}0_U = 0_U$ *and* $\overline{R}^*0_U = 1_U$;

(iv) $\underline{R}(\bigwedge_{i \in \Gamma} \lambda_i) = \bigwedge_{i \in \Gamma} \underline{R}\lambda_i$ *and* $\underline{R}^*(\bigwedge_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} \underline{R}^*\lambda_i$, *for each family* $\{\lambda_i : i \in \Gamma\} \subseteq L^U$;

(v) $\overline{R}(\bigvee_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} \overline{R}\lambda_i$ *and* $\overline{R}^*(\bigvee_{i \in \Gamma} \lambda_i) = \bigwedge_{i \in \Gamma} \overline{R}^*\lambda_i$, *for each family* $\{\lambda_i : i \in \Gamma\} \subseteq L^U$;

(vi) If $\lambda \leq \mu$, *then* $\underline{R}\lambda \leq \underline{R}\mu$ *and* $\underline{R}^*\lambda \geq \underline{R}^*\mu$;

(vii) If $\lambda \leq \mu$, *then* $\overline{R}\lambda \leq \overline{R}\mu$ *and* $\overline{R}^*\lambda \geq \overline{R}^*\mu$;

(viii) $\underline{R}\lambda \vee \underline{R}\mu \leq \underline{R}(\lambda \vee \mu)$ *and* $\underline{R}^*\lambda \wedge \underline{R}^*\mu \geq \underline{R}^*(\lambda \vee \mu)$, $\forall \lambda, \mu \in L^U$;

(ix) $\overline{R}(\lambda \wedge \mu) \leq \overline{R}\lambda \wedge \overline{R}\mu$ *and* $\overline{R}^*(\lambda \wedge \mu) \geq \overline{R}^*\lambda \vee \overline{R}^*\mu$, $\forall \lambda, \mu \in L^U$;

(x) $\overline{R}\lambda' = (\underline{R}\lambda)'$ *and* $\overline{R}^*\lambda' = (\underline{R}^*\lambda)'$;

(xi) $\underline{R}\lambda' = (\overline{R}\lambda)'$ *and* $\underline{R}^*\lambda' = (\overline{R}^*\lambda)'$.

Proof. (i) For each $x \in U$, $\lambda \in L^U$, we have

$$\begin{aligned}(\overline{R}^*\lambda)'(x) &= (\bigwedge_{y\in U}(R^*(x,y) \vee \lambda'(y)))' \\ &= \bigvee_{y\in U}((R^*(x,y))' \wedge \lambda(y)) \\ &\geq \bigvee_{y\in U}(R(x,y) \wedge \lambda(y)) \\ &= (\overline{R}\lambda)(x).\end{aligned}$$

Similarly, $(\underline{R}^*\lambda)'(x) = (\underline{R}\lambda)(x)$.

(ii) Since, for each $x \in U$,

$$(\underline{R}1_U)(x) = \bigwedge_{y\in U}((R(x,y))' \vee 1_U(y)) = 1_L$$

and

$$(\underline{R}^*1_U)(x) = \bigvee_{y\in U}((R^*(x,y))' \wedge (1_U(y))') = 0_L,$$

we have, $\underline{R}1_U = 1_U$ and $\underline{R}^*1_U = 0_U$.

(iii) Similar to (ii).

(iv) For each $x \in U$, $\{\lambda_i : i \in \Gamma\} \subseteq L^U$, we have

$$\begin{aligned}(\underline{R}^*(\bigwedge_{i\in\Gamma}\lambda_i))(x) &= \bigvee_{y\in U}((R^*(x,y))' \wedge (\bigwedge_{i\in\Gamma}\lambda_i)'(y)) \\ &= \bigvee_{y\in U}((R^*(x,y))' \wedge \bigvee_{i\in\Gamma}\lambda_i'(y)) \\ &= \bigvee_{y\in U}\bigvee_{i\in\Gamma}((R^*(x,y))' \wedge \lambda_i'(y)) \\ &= \bigvee_{i\in\Gamma}\bigvee_{y\in U}((R^*(x,y))' \wedge \lambda_i'(y)) \\ &= (\bigvee_{i\in\Gamma}\underline{R}^*\lambda_i)(x).\end{aligned}$$

Similarly, $(\underline{R}(\bigwedge_{i\in\Gamma}\lambda_i))(x) = (\bigwedge_{i\in\Gamma}\underline{R}\lambda_i)(x)$.

(v) Similar to (iv)

(vi) If $\lambda \leq \mu$, then $\forall x \in U$,

$$(\underline{R}\lambda)(x) = \bigwedge_{y\in U}((R(x,y))' \vee \lambda(y)) \leq \bigwedge_{y\in U}((R(x,y))' \vee \mu(y)) = (\underline{R}\mu)(x).$$

Thus $\underline{R}\lambda \leq \underline{R}\mu$. Also,

$$(\underline{R}^*\lambda)(x) = \bigvee_{y\in U}((R^*(x,y))' \wedge \lambda'(y)) \geq \bigvee_{y\in U}((R^*(x,y))' \wedge \mu'(y)) = (\underline{R}^*\mu)(x).$$

Thus $\underline{R}^*\lambda \geq \underline{R}^*\mu$.

(vii) Similar to (vi).

(viii) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, by (vi) we have $\underline{R}\lambda \leq \underline{R}(\lambda \vee \mu)$ and $\underline{R}\mu \leq \underline{R}(\lambda\vee\mu)$, this implies that $\underline{R}\lambda \vee \underline{R}\mu \leq \underline{R}(\lambda\vee\mu)$. Also, we have $\underline{R}^*\lambda \geq \underline{R}^*(\lambda\vee\mu)$ and $\underline{R}^*\mu \geq \underline{R}^*(\lambda \vee \mu)$, this implies that $\underline{R}^*\lambda \wedge \underline{R}^*\mu \geq \underline{R}^*(\lambda \vee \mu)$.

(ix) similar to (viii).

(x) For each $x \in U$, $\lambda \in L^U$, we have

$$\begin{aligned}(\overline{R}^*\lambda')(x) &= \bigwedge_{y\in U}(R^*(x,y) \vee (\lambda')'(y)) \\ &= (\bigvee_{y\in U}((R^*(x,y))' \wedge \lambda'(y)))' \\ &= ((\underline{R}^*\lambda)(x))' \\ &= (\underline{R}^*\lambda)'(x).\end{aligned}$$

Thus, $\overline{R}^*\lambda' = (\underline{R}^*\lambda)'$. Similarly, $\overline{R}\lambda' = (\underline{R}\lambda)'$.

(xi) Using the duality of $\underline{R}$ and $\overline{R}$ (resp. $\underline{R}^*$ and $\overline{R}^*$), we obtain $\underline{R}\lambda' = (\overline{R}\lambda)'$ and $\underline{R}^*\lambda' = (\overline{R}^*\lambda)'$. □

Theorem 2.7. *Let (R, R^*) be an L-double fuzzy relation on a universal set U. Then the following statements are equivalent:*

(i) (R, R^*) is an L-double fuzzy reflexive;

(ii) $\lambda \leq \overline{R}\lambda$ and $\lambda' \geq \overline{R}^*\lambda$;

(iii) $\underline{R}\lambda \leq \lambda$ and $\underline{R}^*\lambda \geq \lambda'$.

Proof. (i) $\Rightarrow$ (ii) Suppose that (R, R^*) is an L-double fuzzy reflexive. Then, $\forall x \in U$, $R(x,x) = 1_L$ and $R^*(x,x) = 0_L$. This implies that

$$\begin{aligned} \lambda(x) = 1_L \wedge \lambda(x) &= R(x,x) \wedge \lambda(x) \\ &\leq \textstyle\bigvee_{y \in U}(R(x,y) \wedge \lambda(y)) \\ &= (\overline{R}\lambda)(x), \end{aligned}$$

and

$$\begin{aligned} \lambda'(x) = 0_L \vee \lambda'(x) &= R^*(x,x) \vee \lambda'(x) \\ &\geq \textstyle\bigwedge_{y \in U}(R^*(x,y) \vee \lambda'(y)) \\ &= (\overline{R}^*\lambda)(x). \end{aligned}$$

(ii) $\Rightarrow$ (i) We assume that $\lambda \leq \overline{R}\lambda$ and $\lambda' \geq \overline{R}^*\lambda$, $\forall \lambda \in L^U$. If there exists some $x \in U$ such that $R(x,x) = a \neq 1_L$ and $R^*(x,x) = b \neq 0_L$, then we can define L-fuzzy set $\delta_x(y) : U \to L$ as:

$$\delta_x(y) = \begin{cases} 1_L, & \text{if} \quad y = x \\ 0_L, & \text{if} \quad y \neq x. \end{cases}$$

Then

$$\begin{aligned} (\overline{R}\delta_x)(x) &= \textstyle\bigvee_{y \in U}(R(x,y) \wedge \delta_x(y)) \\ &= R(x,x) \\ &= a \neq 1_L = \delta_x(x) \end{aligned}$$

and

$$\begin{aligned} (\overline{R}^*\delta_x)(x) &= \textstyle\bigwedge_{y \in U}(R^*(x,y) \vee \delta'_x(y)) \\ &= R^*(x,x) \\ &= b \neq 0_L = \delta'_x(x). \end{aligned}$$

Thus, $\delta_x \not\leq \overline{R}\delta_x$ and $\delta'_x \not\geq \overline{R}^*\delta_x$. This is a contradiction. Then $R(x,x) = 1_L$ and $R^*(x,x) = 0_L$, $\forall x \in U$.

(ii) $\Leftrightarrow$ (iii) It is easy from Theorem 2.6(x),(xi) . □

Theorem 2.8. *Let (R, R^*) be an L-double fuzzy relation on a universal set U. Then the following statements are equivalent:*

(i) (R, R^*) is an L-double fuzzy transitive;

(ii) $\overline{R}(\overline{R}\lambda) \leq \overline{R}\lambda$ and $\overline{R}^*(\overline{R}^*\lambda)' \geq \overline{R}^*\lambda$

(iii) $\underline{R}(\underline{R}\lambda) \geq \underline{R}\lambda$ and $\underline{R}^*(\underline{R}^*\lambda)' \leq \underline{R}^*\lambda$.

Proof. (i) $\Leftrightarrow$ (ii) For each $\lambda \in L^U$, since

$$\begin{aligned}(\overline{R}(\overline{R}\lambda))(x) &= \bigvee_{y\in U}(R(x,y)\wedge(\overline{R}\lambda)(y))\\ &= \bigvee_{y\in U}(R(x,y)\wedge(\bigvee_{z\in U}(R(y,z)\wedge\lambda(z))))\\ &= \bigvee_{z\in U}(\bigvee_{y\in U}(R(x,y)\wedge R(y,z))\wedge\lambda(z)),\end{aligned}$$

and

$$\begin{aligned}(\overline{R}^*(\overline{R}^*\lambda)')(x) &= \bigwedge_{y\in U}(R^*(x,y)\vee(\overline{R}^*\lambda)(y))\\ &= \bigwedge_{y\in U}(R^*(x,y)\vee(\bigwedge_{z\in U}(R^*(y,z)\vee\lambda'(z))))\\ &= \bigwedge_{z\in U}(\bigwedge_{y\in U}(R^*(x,y)\vee R^*(y,z))\vee\lambda'(z)),\end{aligned}$$

we have $\overline{R}(\overline{R}\lambda)\leq\overline{R}\lambda$ and $\overline{R}^*(\overline{R}^*\lambda)'\geq\overline{R}^*\lambda \Leftrightarrow (R,R^*)$ is an L-double fuzzy transitive.

(i) $\Leftrightarrow$ (iii) is similar to (i) $\Leftrightarrow$ (ii). □

From Theorem 2.7 and Theorem 2.8, we have the following theorem:

Theorem 2.9. *If (R,R^*) is an L-double fuzzy reflexive and L-double fuzzy transitive relation on a universal set U, then the following properties hold:*

(i) $\overline{R}(\overline{R}\lambda)=\overline{R}\lambda$ and $\overline{R}^*(\overline{R}^*\lambda)'=\overline{R}^*\lambda$.
(ii) $\underline{R}(\underline{R}\lambda)=\underline{R}\lambda$ and $\underline{R}^*(\underline{R}^*\lambda)'=\underline{R}^*\lambda$.

Theorem 2.10. *Let (R,R^*) be an L-double fuzzy relation on a universal set U. Then the following statements are equivalent:*

(i) (R,R^*) is an L-double fuzzy serial;
(ii) $\underline{R}\lambda\leq\overline{R}\lambda$ and $\overline{R}^*\lambda\leq\underline{R}^*\lambda$, $\forall\lambda\in L^U$;
(iii) $\underline{R}0_U=0_U$ and $\underline{R}^*0_U=1_U$;
(iv) $\overline{R}1_U=1_U$ and $\underline{R}^*1_U=0_U$.

Proof. (i)$\Rightarrow$ (ii) For each $x\in U$,

$$\begin{aligned}(\overline{R}^*\lambda)(x) &= \bigwedge_{y\in U}(R^*(x,y)\vee\lambda'(y))\\ &= \lambda'(y_0)\quad(\text{since, from (i), }\exists y_0\in U,\text{ s.t, }R^*(x,y_0)=0_L)\\ &= (R^*(x,y_0))'\wedge\lambda'(y_0)\\ &\leq \bigvee_{y\in U}((R^*(x,y))'\wedge\lambda'(y))\\ &= (\underline{R}^*\lambda)(x).\end{aligned}$$

Similarly, $(\overline{R}\lambda)(x)\geq(\underline{R}\lambda)(x)$.

(ii)$\Rightarrow$ (iii) It is clear from Theorem 2.6(iii).
(iii)$\Rightarrow$ (iv) It is clear from Theorem 2.6(xi).
(iv)$\Rightarrow$ (i) $\forall x\in U$, since

$$(\overline{R}1_U)(x)=1_U(x)=\bigvee_{y\in U}(R(x,y)\wedge 1_U(y)),$$

and

$$(\overline{R}^*1_U)(x)=0_U(x)=\bigwedge_{y\in U}(R^*(x,y)\vee 0_U(y)).$$

Then, $\bigvee_{y\in U}R(x,y)=1_L$ and $\bigwedge_{y\in U}R^*(x,y)=0_L$. So, there exists $y_0\in U$ such that $R(x,y_0)=1_L$ and $R^*(x,y_0)=0_L$. □

3. **Axiomatic System of L-double Fuzzy Rough Sets**

This section gives the axiomatic approach of an L-double fuzzy upper (resp. lower) approximation of L-double fuzzy rough sets. The axiomatic approach aims to investigate the algebraic characters of L-double fuzzy rough sets, which may help to develop methods for application. It also provides a more general framework for the study generalized rough sets.

Theorem 3.1. *Let U be an arbitrary universe set, L be a fuzzy lattice and $f, f^* : L^U \to L^U$ be operators with $f(\lambda) \leq (f^*(\lambda))', \forall \lambda \in L^U$. Then, f and f^* satisfy the following axioms:*

$$f(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigvee_{i\in\Gamma} \alpha_i f(\lambda_i), \quad f^*(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigwedge_{i\in\Gamma} (\alpha_i' \vee f^*(\lambda_i))$$

for any given index set Γ, $\lambda_i \in L^U$ and $\alpha_i \in L$, iff there exists an L-double fuzzy relation (R, R^*) on U such that $\forall \lambda \in L^U$, $f(\lambda) = \overline{R}\lambda$ and $f^*(\lambda) = \overline{R}^*\lambda$; that is $f = \overline{R}$ and $f^* = \overline{R}^*$.

Proof. We use the operators f and f^* to construct an L-double fuzzy relation (R, R^*) on U as below:

$$R(x, y) = f(\delta_y)(x) \text{ and } R^*(x, y) = f^*(\delta_y)(x).$$

Then, we have

$$\begin{aligned}
(f^*(\lambda))(x) &= (f^*(\textstyle\bigvee_{y\in U} \lambda(y)\delta_y))(x), \quad \text{(by Lemma 1.1)}\\
&= \textstyle\bigwedge_{y\in U}(\lambda'(y) \vee f^*(\delta_y))(x)\\
&= \textstyle\bigwedge_{y\in U}(\lambda'(y) \vee f^*(\delta_y)(x))\\
&= \textstyle\bigwedge_{y\in U}(\lambda'(y) \vee R^*(x, y))\\
&= (\overline{R}^*\lambda)(x).
\end{aligned}$$

Thus, $f^*(\lambda) = \overline{R}^*\lambda$ and $f^* = \overline{R}^*$. Similarly, $f = \overline{R}$.

Conversely, if there exists an L-double fuzzy relation (R, R^*) on U such that $f = \overline{R}$ and $f^* = \overline{R}^*$. From Theorem 2.6(i), clearly $f(\lambda) \leq (f^*(\lambda))', \forall \lambda \in L^U$. For any given index set Γ, $\lambda_i \in L^U$ and $\alpha_i \in L$, we have

$$\begin{aligned}
(f^*(\textstyle\bigvee_{i\in\Gamma} \alpha_i\lambda_i))(x) &= (\overline{R}^*(\textstyle\bigvee_{i\in\Gamma} \alpha_i\lambda_i))(x)\\
&= \textstyle\bigwedge_{y\in U}(R^*(x, y) \vee (\bigvee_{i\in\Gamma}(\alpha_i\lambda_i)(y))')\\
&= \textstyle\bigwedge_{y\in U}(R^*(x, y) \vee (\bigvee_{i\in\Gamma}(\alpha_i \wedge \lambda_i(y)))')\\
&= \textstyle\bigwedge_{y\in U}(R^*(x, y) \vee (\bigwedge_{i\in\Gamma}(\alpha_i' \vee \lambda_i'(y))))\\
&= \textstyle\bigwedge_{y\in U} \bigwedge_{i\in\Gamma}(R^*(x, y) \vee \alpha_i' \vee \lambda_i'(y))\\
&= \textstyle\bigwedge_{i\in\Gamma}(\alpha_i' \vee \bigwedge_{y\in U}((R^*(x, y) \vee \lambda_i'(y))))\\
&= \textstyle\bigwedge_{i\in\Gamma}(\alpha_i' \vee (\overline{R}^*\lambda_i)(x))\\
&= \textstyle\bigwedge_{i\in\Gamma}(\alpha_i' \vee (f^*(\lambda_i))(x))\\
&= (\textstyle\bigwedge_{i\in\Gamma}(\alpha_i' \vee (f^*(\lambda_i))))(x).
\end{aligned}$$

Thus, $f^*(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigwedge_{i\in\Gamma}(\alpha_i' \vee (f^*(\lambda_i)))$.
Similarly, $f(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigvee_{i\in\Gamma} \alpha_i f(\lambda_i)$. □

Theorem 3.2. *Let U be an arbitrary universe set, L be a fuzzy lattice and $f, f^* : L^U \to L^U$ be operators with $f(\lambda) \le (f^*(\lambda))', \forall \lambda \in L^U$. Then, f and f^* satisfy the following axioms:*

$$f(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigvee_{i\in\Gamma} \alpha_i f(\lambda_i), \quad f^*(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigwedge_{i\in\Gamma}(\alpha_i' \vee f^*(\lambda_i))$$

for any given index set Γ, $\lambda_i \in L^U$ and $\alpha_i \in L$ iff there exists an L-double fuzzy relation (R, R^*) on U such that $\forall\lambda \in L^U$, $g(\lambda) = \underline{R}\lambda$ and $g^*(\lambda) = \underline{R}^*\lambda$; that is $g = \underline{R}$ and $g^* = \underline{R}^*$, where $g, g^* : L^U \to L^U$ defined as:

$$g(\lambda) = (f(\lambda'))', \text{ and } g^*(\lambda) = (f^*(\lambda'))'.$$

Proof. By using the L-double fuzzy relation (R, R^*) on U which defined by:

$$R(x, y) = (g(\delta_y')(x))' \text{ and } R^*(x, y) = g^*(\delta_y')(x),$$

the result can be obtain by the same manner of Theorem 3.1. □

Definition 3.3. Let U be an arbitrary universe set and $\lambda, \mu \in L^U$. The pair $(\lambda\mu, (\lambda\mu)^*)$ is called an L-double fuzzy inner product of λ, μ, where

$$\lambda\mu = \bigvee_{x\in U}(\lambda(x) \wedge \mu(x)), \text{ and } (\lambda\mu)^* = \bigwedge_{x\in U}(\lambda(x) \vee \mu(x)).$$

Lemma 3.4. *The L-double fuzzy inner product has the following basic properties:*

(i) $\lambda\mu = \mu\lambda$ and $(\lambda\mu)^* = (\mu\lambda)^*$, $\forall\lambda, \mu \in L^U$.

(ii) $\bigvee_{i\in\Gamma}(\alpha_i\lambda_i)\mu = \bigvee_{i\in\Gamma}(\alpha_i \wedge \lambda_i\mu)$ and $(\bigwedge_{i\in\Gamma}(\alpha_i \vee \lambda_i)\mu)^* = \bigwedge_{i\in\Gamma}(\alpha_i \vee (\lambda_i\mu)^*)$, $\forall\ \lambda_i, \mu \in L^U$ and $\alpha_i \in L$.

(iii) If $\lambda\mu = \lambda\nu$ (resp. $(\lambda\mu)^* = (\lambda\nu)^*$) $\forall\lambda \in L^U$, then $\mu = \nu$.

Proof. (i) This follows directly from the definition of L-double fuzzy inner product.

(ii) For any given index set Γ, $\forall\lambda_i, \mu \in L^U$ and $\alpha_i \in L$, we have

$$\begin{aligned}(\textstyle\bigwedge_{i\in\Gamma}(\alpha_i \vee \lambda_i)\mu)^* &= \textstyle\bigwedge_{x\in U}(\bigwedge_{i\in\Gamma}(\alpha_i \vee \lambda_i)(x) \vee \mu(x)) \\ &= \textstyle\bigwedge_{x\in U}(\bigwedge_{i\in\Gamma}(\alpha_i \vee \lambda_i(x)) \vee \mu(x)) \\ &= \textstyle\bigwedge_{i\in\Gamma}\bigwedge_{x\in U}(\alpha_i \vee \lambda_i(x) \vee \mu(x)) \\ &= \textstyle\bigwedge_{i\in\Gamma}(\alpha_i \vee \bigwedge_{x\in U}(\lambda_i(x) \vee \mu(x))) \\ &= \textstyle\bigwedge_{i\in\Gamma}(\alpha_i \vee (\lambda_i\mu)^*).\end{aligned}$$

Similarly, $\bigvee_{i\in\Gamma}(\alpha_i\lambda_i)\mu = \bigvee_{i\in\Gamma}(\alpha_i \wedge \lambda_i\mu)$.

(iii) Suppose that, $\lambda\mu = \lambda\nu$ (resp. $(\lambda\mu)^* = (\lambda\nu)^*$) , $\forall\lambda \in L^U$. If $\mu \ne \nu$, then there exists $x \in U$ such that $\mu(x) \ne \nu(x)$. Using L-fuzzy set δ_x, we have

$$\delta_x\mu = \bigvee_{y\in U}(\delta_x(y) \wedge \mu(y)) = \mu(x),$$

and

$$\delta_x\nu = \bigvee_{y\in U}(\delta_x(y) \wedge \nu(y)) = \nu(x).$$

This contradicts $\delta_x\mu = \delta_x\nu$. Thus $\mu = \nu$. On the other hand,

$$(\delta_x'\mu)^* = \bigwedge_{y\in U}(\delta_x'(y) \vee \mu(y)) = \mu(x),$$

and
$$(\delta'_x\nu)^* = \bigwedge_{y\in U}(\delta'_x(y)\vee\nu(y)) = \nu(x).$$
This contradicts $(\delta'_x\mu)^* = (\delta'_x\nu)^*$. Thus, $\mu=\nu$. $\square$

Theorem 3.5. *Let U be an arbitrary universe set and L be a fuzzy lattice. Suppose that $f, f^*: L^U \to L^U$ are operators with $f(\lambda)\le (f^*(\lambda))', \forall\lambda\in L^U$ such that f and f^* satisfy the following axioms:*
$$f(\bigvee_{i\in\Gamma}\alpha_i\lambda_i) = \bigvee_{i\in\Gamma}\alpha_i f(\lambda_i),\quad f^*(\bigvee_{i\in\Gamma}\alpha_i\lambda_i) = \bigwedge_{i\in\Gamma}(\alpha'_i\vee f^*(\lambda_i))$$
for any given index set Γ, $\lambda_i\in L^U$ and $\alpha_i\in L$. Then there exist:

(i) an L-double fuzzy reflexive relation (R,R^*) on U iff $\lambda\le f(\lambda)$ and $f^*(\lambda)\le\lambda'$, $\forall\lambda\in L^U$;

(ii) an L-double fuzzy symmetric relation (R,R^*) on U iff $\lambda f(\nu)=\nu f(\lambda)$ and $(\lambda f^*(\nu))^* = (\nu' f^*(\lambda'))^*$, $\forall\lambda\in L^U$;

(iii) an L-double fuzzy transitive relation (R,R^*) on U iff $f(f(\lambda))\le f(\lambda)$ and $f^*((f^*(\lambda))')\ge f^*(\lambda)$.

Proof. By Theorem 3.1, there exists an L-double fuzzy relation (R,R^*) on U such that $f(\lambda)=\overline{R}\lambda$ and $f^*(\lambda)=\overline{R}^*\lambda$, $\forall\lambda\in L^U$.

(i) It is immediately from Theorem 2.7.

(ii) If (R,R^*) is an L-double fuzzy symmetric relation on U, then $R(x,y)=R(y,x)$ and $R^*(x,y)=R^*(y,x)$, $\forall x,y\in U$. Thus, we have
$$\begin{aligned}(\lambda f^*(\nu))^* = (\lambda\overline{R}^*\nu)^* &= \textstyle\bigwedge_{x\in U}(\lambda(x)\vee(\overline{R}^*\nu)(x))\\ &= \textstyle\bigwedge_{x\in U}(\lambda(x)\vee(\bigwedge_{y\in U}(R^*(x,y)\vee\nu'(y))))\\ &= \textstyle\bigwedge_{x\in U}\bigwedge_{y\in U}(\lambda(x)\vee R^*(x,y)\vee\nu'(y))\\ &= \textstyle\bigwedge_{y\in U}\bigwedge_{x\in U}(\nu'(y)\vee R^*(y,x)\vee\lambda(x))\\ &= \textstyle\bigwedge_{y\in U}(\nu'(y)\vee(\bigwedge_{x\in U}(R^*(y,x)\vee\lambda(x))))\\ &= \textstyle\bigwedge_{y\in U}(\nu'(y)\vee(\overline{R}^*\lambda')(y))\\ &= (\nu'\overline{R}^*\lambda')^* = (\nu' f^*(\lambda'))^*.\end{aligned}$$
Similarly, $\lambda f(\nu)=\nu f(\lambda)$.

Conversely, if $\lambda f(\nu)=\nu f(\lambda)$ and $(\lambda f^*(\nu))^* = (\nu' f^*(\lambda'))^*$, $\forall\lambda\in L^U$, then for L-fuzzy sets $\lambda=\delta_x$ and $\nu=\delta_y$, we have
$$\begin{aligned}\lambda f(\nu)=\lambda\overline{R}\nu &= \delta_x\overline{R}\delta_y\\ &= \textstyle\bigvee_{z\in U}(\delta_x(z)\wedge\overline{R}\delta_y(z))\\ &= \overline{R}\delta_y(x)\\ &= \textstyle\bigvee_{z\in U}(R(x,z)\wedge\delta_y(z))\\ &= R(x,y).\end{aligned}$$
Similarly,
$$\begin{aligned}\nu f(\lambda)=\nu\overline{R}\lambda &= \delta_y\overline{R}\delta_x\\ &= \textstyle\bigvee_{z\in U}(\delta_y(z)\wedge\overline{R}\delta_x(z))\\ &= \overline{R}\delta_x(y)\\ &= \textstyle\bigvee_{z\in U}(R(y,z)\wedge\delta_x(z))\\ &= R(y,x).\end{aligned}$$
Thus $R(x,y)=R(y,x)$.

Also, if we put $\lambda = \delta'_x$ and $\nu = \delta_y$, then we have

$$\begin{aligned}(\lambda f^*(\nu))^* = (\lambda \overline{R}^* \nu)^* &= (\delta'_x \overline{R}^* \delta_y)^* \\ &= \textstyle\bigwedge_{z\in U}(\delta'_x(z) \vee \overline{R}^* \delta_y(z)) \\ &= \overline{R}^* \delta_y(x) \\ &= \textstyle\bigwedge_{z\in U}(R^*(x,z) \vee \delta'_y(z)) \\ &= R^*(x,y).\end{aligned}$$

Similarly,

$$\begin{aligned}(\nu' f^*(\lambda'))^* = (\nu' \overline{R}^* \lambda')^* &= (\delta'_y \overline{R}^* \delta_x)^* \\ &= \textstyle\bigwedge_{z\in U}(\delta'_y(z) \vee \overline{R}^* \delta_x(z)) \\ &= \overline{R}^* \delta_x(y) \\ &= \textstyle\bigwedge_{z\in U}(R^*(y,z) \vee \delta'_x(z)) \\ &= R^*(y,x).\end{aligned}$$

Thus $R^*(x,y) = R^*(y,x)$. Hence, (R, R^*) is an L-double fuzzy symmetric relation on U.

(iii) It follows immediately from Theorem 2.8. □

Theorem 3.6. *Let U be an arbitrary universe set and L be a fuzzy lattice. Suppose that $f, f^* : L^U \to L^U$ are operators with $f(\lambda) \le (f^*(\lambda))', \forall \lambda \in L^U$ such that f and f^* satisfy the following axioms:*

$$f(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigvee_{i\in\Gamma} \alpha_i f(\lambda_i), \quad f^*(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigwedge_{i\in\Gamma} (\alpha'_i \vee f^*(\lambda_i))$$

for any given index set Γ, $\lambda_i \in L^U$ and $\alpha_i \in L$. Let $g, g^* : L^U \to L^U$ defined as:

$$g(\lambda) = (f(\lambda'))', \text{ and } g^*(\lambda) = (f^*(\lambda'))'.$$

Then there exist:

(i) an L-double fuzzy reflexive relation (R, R^*) on U iff $g(\lambda) \le \lambda$ and $\lambda' \le g^*(\lambda)$, $\forall \lambda \in L^U$;

(ii) an L-double fuzzy symmetric relation (R, R^*) on U iff $\lambda g(\nu) = \nu g(\lambda)$ and $(\lambda g^*(\nu))^* = (\nu' g^*(\lambda'))^*$, $\forall \lambda \in L^U$;

(iii) an L-double fuzzy transitive relation (R, R^*) on U iff $g(\lambda) \le g(g(\lambda))$ and $g^*((g^*(\lambda))') \le g^*(\lambda)$.

Proof. By Theorem 3.2, there exists an L-double fuzzy relation (R, R^*) on U such that $g(\lambda) = \underline{R}\lambda$ and $g^*(\lambda) = \underline{R}^*\lambda$, $\forall \lambda \in L^U$.

(i) and (iii) are followed immediately from Theorem 2.7, and Theorem 2.8, respectively.

(ii) it can be proved by the same manner of Theorem 3.5(ii). □

Lemma 3.7. *Let U be an arbitrary universe set and L be a fuzzy lattice. Let $f, f^* : L^U \to L^U$ be operators with $f(\lambda) \le (f^*(\lambda))', \forall \lambda \in L^U$. If f and f^* satisfy the following axioms:*

$$\lambda f(\nu) = \nu f(\lambda) \text{ and } (\lambda f^*(\nu))^* = (\nu' f^*(\lambda'))^*, \forall \lambda, \mu \in L^U,$$

then,

$$f(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigvee_{i\in\Gamma} \alpha_i f(\lambda_i) \text{ and } f^*(\bigvee_{i\in\Gamma} \alpha_i\lambda_i) = \bigwedge_{i\in\Gamma} (\alpha'_i \vee f^*(\lambda_i)),$$

for any given index set Γ, $\lambda_i \in L^U$ *and* $\alpha_i \in L$.

Proof. For each $\nu \in L^U$, we have

$$\begin{aligned}
(\nu \bigwedge_{i\in\Gamma}(\alpha_i' \vee f^*(\lambda_i)))^* &= (\bigwedge_{i\in\Gamma}(\alpha_i' \vee f^*(\lambda_i))\nu)^*, \quad (\text{ by Lemma 3.4(i)})\\
&= \bigwedge_{i\in\Gamma}(\alpha_i' \vee ((f^*(\lambda_i))\nu)^*), \quad (\text{ by Lemma 3.4(ii)})\\
&= \bigwedge_{i\in\Gamma}(\alpha_i' \vee (\nu(f^*(\lambda_i)))^*), \quad (\text{ by Lemma 3.4(i)})\\
&= \bigwedge_{i\in\Gamma}(\alpha_i' \vee (\lambda_i'(f^*(\nu')))^*)\\
&= ((\bigwedge_{i\in\Gamma}(\alpha_i' \vee \lambda_i'))f^*(\nu'))^*, \quad (\text{ by Lemma 3.4(ii)})\\
&= (\nu f^*((\bigwedge_{i\in\Gamma}(\alpha_i' \vee \lambda_i'))'))^*\\
&= (\nu f^*(\bigvee_{i\in\Gamma}(\alpha_i\lambda_i)))^*.
\end{aligned}$$

Then, by Lemma 3.4(iii), we have $f^*(\bigvee_{i\in\Gamma}(\alpha_i\lambda_i)) = \bigwedge_{i\in\Gamma}(\alpha_i' \vee f^*(\lambda_i))$. Similarly, $f(\bigvee_{i\in\Gamma}\alpha_i\lambda_i) = \bigvee_{i\in\Gamma}(\alpha_i f(\lambda_i))$. □

From Theorem 3.5, and Lemma 3.7, we have the following result.

Theorem 3.8. *Let* U *be an arbitrary universe set and* L *be a fuzzy lattice. Suppose that* $f, f^* : L^U \to L^U$ *are operators with* $f(\lambda) \leq (f^*(\lambda))', \forall\lambda \in L^U$ *such that* f *and* f^* *satisfy the following axioms:*

(i) $f(f(\lambda)) \leq f(\lambda)$ *and* $f^*(\lambda) \leq f^*((f^*(\lambda))')$, $\forall\lambda \in L^U$;
(ii) $\lambda \leq f(\lambda)$ *and* $f^*(\lambda) \leq \lambda'$, $\forall\lambda \in L^U$, $\forall\lambda \in L^U$;
(iii) $\lambda f(\nu) = \nu f(\lambda)$ *and* $(\lambda f^*(\nu))^* = (\nu' f^*(\lambda'))^*$, $\forall\lambda, \nu \in L^U$.

Then, there exists an L*-double fuzzy equivalent relation* (R, R^*) *on* U *such that* $f(\lambda) = \overline{R}\lambda$ *and* $f^*(\lambda) = \overline{R}^*\lambda$, $\forall\lambda \in L^U$.

When we define the L-double fuzzy lower approximation operation $(\underline{R}, \underline{R}^*)$ using the formulas $\underline{R}\lambda = (\overline{R}\lambda')'$ and $\underline{R}^*\lambda = (\overline{R}^*\lambda')'$, then, $(\underline{R}\lambda, \underline{R}^*\lambda, \overline{R}\lambda, \overline{R}^*\lambda)$ is an L-double fuzzy rough set of λ. Observe that Theorem 3.8, gives an axiomatic system of L-double fuzzy rough sets.

Lemma 3.9. *Let* U *be an arbitrary universe set and* L *be a fuzzy lattice. Suppose that* $f, f^* : L^U \to L^U$ *are operators with* $f(\lambda) \leq (f^*(\lambda))', \forall\lambda \in L^U$. *Then, the axioms* $f(f(\lambda)) \leq f(\lambda)$ *and* $\lambda \leq f(\lambda)$ *(resp.* $f^*(\lambda) \leq f^*((f^*(\lambda))')$ *and* $f^*(\lambda) \leq \lambda'$*) are equivalent to* $\lambda \vee f(f(\lambda)) = f(\lambda)$ *(resp.* $\lambda' \wedge f^*((f^*(\lambda))') = f^*(\lambda)$*).*

Proof. If $\lambda \vee f(f(\lambda)) = f(\lambda)$, then $f(f(\lambda)) \leq f(\lambda)$ and $\lambda \leq f(\lambda)$.
Similarly, If $\lambda' \wedge f^*((f^*(\lambda))') = f^*(\lambda)$, then $f^*(\lambda) \leq f^*((f^*(\lambda))')$ and $f^*(\lambda) \leq \lambda'$

Conversely, if $f(f(\lambda)) \leq f(\lambda)$ and $\lambda \leq f(\lambda)$, then $f(f(\lambda)) \leq f(\lambda) \leq f(f(\lambda))$ which implies that $f(\lambda) = f(f(\lambda))$. Thus $f(\lambda) = \lambda \vee f(\lambda) = \lambda \vee f(f(\lambda))$.

Similarly, if $f^*(\lambda) \leq f^*((f^*(\lambda))')$ and $f^*(\lambda) \leq \lambda'$, then

$$\begin{aligned}
\lambda' \wedge f^*((f^*(\lambda))') &\leq f^*((f^*(\lambda))')\\
&\leq ((f^*(\lambda))')'\\
&= f^*(\lambda)\\
&\leq \lambda' \wedge f^*((f^*(\lambda))').
\end{aligned}$$

Thus, $\lambda' \wedge f^*((f^*(\lambda))') = f^*(\lambda)$. □

Theorem 3.8 and Lemma 3.9, imply the following theorem:

Theorem 3.10. *Let U be an arbitrary universe set and L be a fuzzy lattice. Suppose that $f, f^* : L^U \to L^U$ are operators with $f(\lambda) \leq (f^*(\lambda))', \forall \lambda \in L^U$ such that f and f^* satisfy the following axioms:*

(i) $\lambda \vee f(f(\lambda)) = f(\lambda)$ and $\lambda' \vee f^*((f^*(\lambda))') = f^*(\lambda)$, $\forall \lambda \in L^U$;

(ii) $\lambda f(\nu) = \nu f(\lambda)$ and $(\lambda f^*(\nu))^* = (\nu' f^*(\lambda'))^*$, $\forall \lambda, \nu \in L^U$.

Then, there exists an L-double fuzzy equivalent relation (R, R^*) on U such that $f(\lambda) = \overline{R}\lambda$ and $f^*(\lambda) = \overline{R}^*\lambda$, $\forall \lambda \in L^U$.

4. Induced L-double Fuzzy Topology from L-double Fuzzy Approximation Operators

In this section we will show that an L-double fuzzy upper (resp. lower) approximation operator on U induces Alexandrov L-double fuzzy topology on U.

Theorem 4.1. *Let $(U, \overline{R}, \overline{R}^*)$ be an L-double fuzzy upper approximation space with $\overline{R}\lambda \geq (\overline{R}^*\lambda)'$, $\forall \lambda \in L^U$. Define $\mathcal{T}_{\overline{R}}, \mathcal{T}^*_{\overline{R}^*} : L^U \to L$ as follows: $\forall \lambda \in L^U$*

$$\mathcal{T}_{\overline{R}}(\lambda) = \bigwedge_{x \in U} ((\overline{R}\lambda)'(x) \vee \lambda(x)),$$

$$\mathcal{T}^*_{\overline{R}^*}(\lambda) = \bigvee_{x \in U} ((\overline{R}^*\lambda)'(x) \wedge \lambda'(x)).$$

Then, $(\mathcal{T}_{\overline{R}}, \mathcal{T}^*_{\overline{R}^*})$ is Alexandrov L-double fuzzy topology on U.

Proof. (LDFT1) $\forall \lambda \in L^U$, we have:

$$\begin{aligned} \mathcal{T}_{\overline{R}}(\lambda) &= \textstyle\bigwedge_{x \in U} ((\overline{R}\lambda)'(x) \vee \lambda(x)) \\ &\leq \textstyle\bigwedge_{x \in U} ((\overline{R}^*\lambda)(x) \vee \lambda(x)) \\ &= (\textstyle\bigvee_{x \in U} ((\overline{R}^*\lambda)'(x) \wedge \lambda'(x)))' \\ &= (\mathcal{T}^*_{\overline{R}^*}(\lambda))' \end{aligned}$$

(LDFT2) It is clear.

(LDFT3) For each $\lambda_1, \lambda_2 \in L^U$ we have:

$$\begin{aligned} \mathcal{T}_{\overline{R}}(\lambda_1 \wedge \lambda_2) &= \textstyle\bigwedge_{x \in U} ((\overline{R}(\lambda_1 \wedge \lambda_2))'(x) \vee (\lambda_1 \wedge \lambda_2)(x)) \\ &= \textstyle\bigwedge_{x \in U} (((\overline{R}(\lambda_1 \wedge \lambda_2))'(x) \vee \lambda_1(x)) \wedge ((\overline{R}(\lambda_1 \wedge \lambda_2))'(x) \vee \lambda_2(x))) \\ &\geq \textstyle\bigwedge_{x \in U} (((\overline{R}(\lambda_1))'(x) \vee \lambda_1(x))) \wedge \textstyle\bigwedge_{x \in U} (((\overline{R}(\lambda_2))'(x) \vee \lambda_2(x))) \\ &= \mathcal{T}_{\overline{R}}(\lambda_1) \wedge \mathcal{T}_{\overline{R}}(\lambda_2). \end{aligned}$$

Similarly, $\mathcal{T}^*_{\overline{R}^*}(\lambda_1 \wedge \lambda_2) \leq \mathcal{T}^*_{\overline{R}^*}(\lambda_1) \vee \mathcal{T}^*_{\overline{R}^*}(\lambda_2)$.

(LDFT4) For each family $\{\lambda_i \in L^U : i \in \Gamma\}$ we have:

$$\begin{aligned} \mathcal{T}_{\overline{R}}(\textstyle\bigvee_{i \in \Gamma} \lambda_i) &= \textstyle\bigwedge_{x \in U} ((\overline{R}(\bigvee_{i \in \Gamma} \lambda_i))'(x) \vee (\bigvee_{i \in \Gamma} \lambda_i)(x)) \\ &= \textstyle\bigwedge_{x \in U} ((\bigvee_{i \in \Gamma} \overline{R}(\lambda_i))'(x) \vee (\bigvee_{i \in \Gamma} \lambda_i)(x)) \qquad \text{(by Theorem 2.6(v))} \\ &= \textstyle\bigwedge_{x \in U} (\bigwedge_{i \in \Gamma} (\overline{R}(\lambda_i))'(x) \vee (\bigvee_{i \in \Gamma} \lambda_i)(x)) \\ &\geq \textstyle\bigwedge_{i \in \Gamma} \bigwedge_{x \in U} ((\overline{R}(\lambda_i))'(x) \vee \lambda_i(x)) \\ &= \textstyle\bigwedge_{i \in \Gamma} \mathcal{T}_{\overline{R}}(\lambda_i). \end{aligned}$$

Similarly, $\mathcal{T}^*_{\overline{R}^*}(\bigvee_{i \in \Gamma} \lambda_i) \leq \bigvee_{i \in \Gamma} \mathcal{T}^*_{\overline{R}^*}(\lambda_i)$.

(LDFT5) For each family $\{\lambda_i \in L^U : i \in \Gamma\}$ we have:

$$\begin{array}{ll} \mathcal{T}_{\overline{R}}(\bigwedge_{i\in\Gamma}\lambda_i) & = \bigwedge_{x\in U}((\overline{R}(\bigwedge_{i\in\Gamma}\lambda_i))'(x) \vee (\bigwedge_{i\in\Gamma}\lambda_i)(x)) \\ & \geq \bigwedge_{x\in U}((\overline{R}(\lambda_i))'(x) \vee (\bigwedge_{i\in\Gamma}\lambda_i)(x)) \\ & = \bigwedge_{i\in\Gamma}\bigwedge_{x\in U}((\overline{R}(\lambda_i))'(x) \vee \lambda_i(x)) \\ & = \bigwedge_{i\in\Gamma}\mathcal{T}_{\overline{R}}(\lambda_i). \end{array}$$

Similarly, $\mathcal{T}^*_{\overline{R}^*}(\bigwedge_{i\in\Gamma}\lambda_i) \leq \bigvee_{i\in\Gamma}\mathcal{T}^*_{\overline{R}^*}(\lambda_i)$. □

Theorem 4.2. *Let $(U, \underline{R}, \underline{R}^*)$ be an L-double fuzzy lower approximation space with $\underline{R}\lambda \leq (\underline{R}^*\lambda)'$, $\forall\lambda \in L^U$. Define $\mathcal{T}_{\underline{R}}, \mathcal{T}^*_{\underline{R}^*} : L^U \to L$ as follows: $\forall\lambda \in L^U$*

$$\mathcal{T}_{\underline{R}}(\lambda) = \bigwedge_{x\in U}((\underline{R}\lambda)(x) \vee \lambda'(x)),$$

$$\mathcal{T}^*_{\underline{R}^*}(\lambda) = \bigvee_{x\in U}((\underline{R}^*\lambda)(x) \wedge \lambda(x)).$$

Then, $(\mathcal{T}_{\underline{R}}, \mathcal{T}^*_{\underline{R}^*})$ is Alexandrov L-double fuzzy topology on U.

Proof. It can be proved by the same manner of Theorem 4.1. □

Definition 4.3. Let (U, R_1, R_1^*) and (V, R_2, R_2^*) be two L-double fuzzy approximation spaces. The map $f : (U, R_1, R_1^*) \to (V, R_2, R_2^*)$ is called R-map if $R_1(x, y) \leq R_2(f(x), f(y))$ and $R_1^*(x, y) \geq R_2^*(f(x), f(y))$, $\forall(x, y) \in U \times V$.

Definition 4.4. Let $(U, \overline{R}_1, \overline{R}_1^*)$ and $(V, \overline{R}_2, \overline{R}_2^*)$ be two L-double fuzzy upper approximation spaces. The map $f : (U, \overline{R}_1, \overline{R}_1^*) \to (V, \overline{R}_2, \overline{R}_2^*)$ is called $\overline{R}$-map if $\overline{R}_1(f^{\leftarrow}(\lambda)) \leq f^{\leftarrow}(\overline{R}_2\lambda)$ and $\overline{R}_1^*(f^{\leftarrow}(\lambda)) \geq f^{\leftarrow}(\overline{R}_2^*\lambda)$, $\forall\lambda \in L^V$.

Definition 4.5. Let $(U, \underline{R}_1, \underline{R}_1^*)$ and $(V, \underline{R}_2, \underline{R}_2^*)$ be two L-double fuzzy lower approximation spaces. The map $f : (U, \underline{R}_1, \underline{R}_1^*) \to (V, \underline{R}_2, \underline{R}_2^*)$ is called $\underline{R}$-map if $f^{\leftarrow}(\underline{R}_2\lambda) \leq \underline{R}_1(f^{\leftarrow}(\lambda))$ and $f^{\leftarrow}(\underline{R}_2^*\lambda) \geq \underline{R}_1^*(f^{\leftarrow}(\lambda))$, $\forall\lambda \in L^V$.

Theorem 4.6. *If the map $f : (U, R_1, R_1^*) \to (V, R_2, R_2^*)$ is an R-map, then $f : (U, \overline{R}_1, \overline{R}_1^*) \to (V, \overline{R}_2, \overline{R}_2^*)$ is an $\overline{R}$-map.*

Proof. $\forall\lambda \in L^V$, $x \in U$,

$$\begin{array}{ll} f^{\leftarrow}(\overline{R}_2\lambda)(x) & = (\overline{R}_2\lambda)(f(x)) \\ & = \bigvee_{z\in V}(R_2(f(x), z) \wedge \lambda(z)) \\ & \geq \bigvee_{f(y)=z\in V}(R_2(f(x), f(y)) \wedge \lambda(f(y))) \\ & \geq \bigvee_{y\in U}(R_1(x, y) \wedge f^{\leftarrow}(\lambda)(y)) \\ & = \overline{R}_1(f^{\leftarrow}(\lambda))(x). \end{array}$$

Similarly, $f^{\leftarrow}(\overline{R}_2^*\lambda)(x) \leq \overline{R}_1^*(f^{\leftarrow}(\lambda))(x)$. □

Theorem 4.7. *If the map $f : (U, R_1, R_1^*) \to (V, R_2, R_2^*)$ is an R-map, then $f : (U, \underline{R}_1, \underline{R}_1^*) \to (V, \underline{R}_2, \underline{R}_2^*)$ is an $\underline{R}$-map.*

Proof. Similar to Theorem 4.6. □

Theorem 4.8. *If the map $f : (U, \overline{R}_1, \overline{R}_1^*) \to (V, \overline{R}_2, \overline{R}_2^*)$ is an $\overline{R}$-map, then $f : (U, \mathcal{T}_{\overline{R}_1}, \mathcal{T}^*_{\overline{R}_1^*}) \to (V, \mathcal{T}_{\overline{R}_2}, \mathcal{T}^*_{\overline{R}_2^*})$ is continuous.*

Proof. $\forall \lambda \in L^V$,

$$\begin{aligned} \mathcal{T}_{\overline{R}_1}(f^{\leftarrow}(\lambda)) &= \bigwedge_{x\in U}((\overline{R}_1(f^{\leftarrow}(\lambda)))'(x) \vee f^{\leftarrow}(\lambda)(x)) \\ &\geq \bigwedge_{x\in U}((f^{\leftarrow}(\overline{R}_2\lambda))'(x) \vee f^{\leftarrow}(\lambda)(x)) \\ &= \bigwedge_{x\in U}(f^{\leftarrow}((\overline{R}_2\lambda)')(x) \vee f^{\leftarrow}(\lambda)(x)) \\ &= \bigwedge_{x\in U}((\overline{R}_2\lambda)'(f(x)) \vee \lambda(f(x))) \\ &\geq \bigwedge_{y\in V}((\overline{R}_2\lambda)'(y) \vee \lambda(y)) \\ &= \mathcal{T}_{\overline{R}_2}(\lambda). \end{aligned}$$

Similarly, $\mathcal{T}^*_{\overline{R}^*_1}(f^{\leftarrow}(\lambda)) \leq \mathcal{T}^*_{\overline{R}^*_2}(\lambda)$. □

Theorem 4.9. *If the map* $f : (U, \underline{R}_1, \underline{R}^*_1) \to (V, \underline{R}_2, \underline{R}^*_2)$ *is an* $\underline{R}$*-map, then* $f : (U, \mathcal{T}_{\underline{R}_1}, \mathcal{T}^*_{\underline{R}^*_1}) \to (V, \mathcal{T}_{\underline{R}_2}, \mathcal{T}^*_{\underline{R}^*_2})$ *is continuous.*

Proof. Similar to Theorem 4.8. □

5. Conclusions

As a suitable mathematical model to handle partial knowledge in data bases, the rough set theory is emerging as a powerful theory and has been found that it has successive applications in the fields of artificial intelligence such as pattern recognition, machine learning, and automated knowledge acquisition. As it is well known, there are at least two approaches to the study of the rough set theory, namely the constructive and axiomatic approaches. In this paper, we introduced the concept of L-double fuzzy rough sets in which both constructive and axiomatic approaches were considered. In constructive approach, a pair of L-double fuzzy lower and L-double fuzzy upper approximation operators were defined and the basic properties of them were studied. From the viewpoint of the axiomatic approach, a set of axioms was constructed to characterize the L-double fuzzy upper approximation of L-double fuzzy rough sets. L-double fuzzy rough sets were viewed as a generalization of generalized rough sets [20]. Finally, from L-double fuzzy approximation operators, we generated Alexandrov L-double fuzzy topology.

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