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SEMISIMPLE SEMIHYPERGROUPS IN TERMS OF HYPERIDEALS AND FUZZY HYPERIDEALS

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ABSTRACT. In this paper, we define prime (semiprime) hyperideals and prime (semiprime) fuzzy hyperideals of semihypergroups. We characterize semihypergroups in terms of their prime (semiprime) hyperideals and prime (semiprime) fuzzy hyperideals.

1. Introduction

The concept of a fuzzy set, introduced by Zadeh in his classic paper [19], provides a natural framework for generalizing some of the notions of classical algebraic structures. Fuzzy semigroups have been first considered by Kuroki [13]. The idea of hyperstructures first introduced by Marty in 1934 [15], as a generalization of ordinary algebraic structures. Fuzzy sets and hyperstructures are extensively studied by many mathematicians. A recent book [6] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of numerous applications of the algebraic hyperstructures. The relations between fuzzy sets and hyperstructures have been already considered by Corsini [3,4,5], Cristea [7,8], Stefanescu and Cristea [17], Davvaz [9,10], Leoreanu [16], Tofan [18], Kehagias [13], and others. In [11], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. We extend this idea to prime (semiprime) hyperideals of a semihypergroup in this paper.

In [1], Ahsan et al. have shown that a semigroup S is semisimple if and only if each fuzzy ideal of S is semiprime if and only if each fuzzy ideal of S is the intersection of prime fuzzy ideals of S which contain it.

Here we give parallel characterizations for semihypergroups in crisp as well as in fuzzy form and prove that a semihypergroup H is semisimple if and only if each hyperideal of H is semiprime if and only if each hyperideal of H is the intersection of prime hyperideals of H which contain it. We also extend this property of semihypergroups in fuzzy context and prove that a semihypergroup H is semisimple if and only if for all fuzzy hyperideals λ and μ of H, $\lambda \circ \mu = \lambda \wedge \mu$ if and only if every fuzzy hyperideal of H is semiprime if and only if each fuzzy hyperideal of H is the intersection of prime fuzzy hyperideals of H which contain it.

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2. Definitions and Basic Results

A hypergroupoid is a non-empty set H equipped with a *hyperoperation*, that is a map $\circ : H \times H \longrightarrow \wp^*(H)$, where $\wp^*(H)$ denotes the set of all non-empty subsets of H (see [6]). We denote by $x \circ y$, the hyperproduct of elements x, y of H.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$(x \circ y) \circ z = x \circ (y \circ z) \text{ for all } x, y, z \text{ in } H.$$

Throughout this paper H will denote a semihypergroup with hyperoperation "$\circ$".

Let A, B be subsets of H. Then the hyperproduct of A and B is defined as:

$$A \circ B = \underset{a \in A, b \in B}{\cup} a \circ b$$

We shall write $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$.

A non-empty subset S of a semihypergroup H is called a *subsemihypergroup* of H if for all $x, y \in S$, $x \circ y \subseteq S$.

If a semihypergroup H contains an element e with the property that, for all $x \in H$, $x \in x \circ e$ (resp. $x \in e \circ x$), we say that e is a right (resp. left) identity of H. If $x \circ e = \{x\}$ (resp. $e \circ x = \{x\}$), for all x in H, then e is called *scalar right* (resp. *left*) *identity* in H.

In [11], it is defined that if $A \in \wp^*(H)$, then A is called,

(i) a *right hyperideal* in H if

$$x \in A \Longrightarrow x \circ y \subseteq A, \forall y \in H.$$

(ii) a *left hyperideal* in H if

$$x \in A \Longrightarrow y \circ x \subseteq A, \forall y \in H.$$

(iii) a hyperideal in H if it is both a left and a right hyperideal in H.

If $a \in H$, then the smallest right (left) hyperideal of H containing a is called the *principal right (left) hyperideal* of H and is denoted by $< a >_r$ $(< a >_l)$, respectively. The hyper ideal of H generated by a is denoted by $< a >$. It is given in [9] that

$$< a >_r = (a \circ H) \cup \{a\} \quad (< a >_l = (H \circ a) \cup \{a\})$$

and

$$< a > = ((H \circ a) \circ H) \cup Ha \cup aH.$$

If H contains an identity element, then $< a >_r = a \circ H$, $< a >_l = H \circ a$ and $< a > = H \circ a \circ H$.

Definition 2.1. A semihypergroup H is called *semisimple*, if for each $h \in H$ there exist $x, y, z \in H$ such that $h \in x \circ h \circ y \circ h \circ z$.

Definition 2.2. A subset A of H is called *idempotent* if $A \circ A = A$.

Let H be a semihypergroup. By a fuzzy subset λ of H, we mean a mapping $\lambda : H \longrightarrow [0, 1]$.

For any fuzzy subset λ of H and for any $t \in [0, 1]$, the set

$$U(\lambda;t)=\{x\in H|\lambda(x)\geq t\}$$

is called a *level subset* of λ.

For any two fuzzy subsets λ and μ of H, $\lambda\leq\mu$ means that, for all $x\in H$, $\lambda(x)\leq\mu(x)$.

For $x\in H$, define

$$X_x=\{(y,z)\in H\times H : x\in y\circ z\}$$

For any two fuzzy subsets λ and μ of H, define

$$\lambda\circ\mu : H\longrightarrow[0,1], x\longmapsto\lambda\circ\mu(x):=\begin{cases}\bigvee\limits_{(y,z)\in X_x}\min\{\lambda(y),\mu(z)\} & \text{if } X_x\neq\emptyset\\ 0 & \text{if } X_x=\emptyset\end{cases}$$

For a non-empty family of fuzzy subsets $\{\lambda_i\}_{i\in I}$, of a semihypergroup H, the fuzzy subsets $\bigvee\limits_{i\in I}\lambda_i$ and $\bigwedge\limits_{i\in I}\lambda_i$ of H are defined as follows:

$$\bigvee_{i\in I}\lambda_i : H\longrightarrow[0,1], x\longmapsto\left(\bigvee_{i\in I}\lambda_i\right)(x):=\sup_{i\in I}\{\lambda_i(x)\}$$

and

$$\bigwedge_{i\in I}\lambda_i : H\longrightarrow[0,1], x\longmapsto\left(\bigwedge_{i\in I}\lambda_i\right)(x):=\inf_{i\in I}\{\lambda_i(x)\}.$$

If I is a finite set, say $I=\{1,2,3,...,n\}$, then clearly

$\left(\bigvee\limits_{i\in I}\lambda_i\right)(x)=\max\{\lambda_1(x),\lambda_2(x),...,\lambda_n(x)\}$ and $\left(\bigwedge\limits_{i\in I}\lambda_i\right)(x)=\min\{\lambda_1(x),\lambda_2(x),...,\lambda_n(x)\}$.

Lemma 2.3. *Let $\lambda_1,\lambda_2,\lambda_3$ be fuzzy subsets of a set A, then $\lambda_1\vee(\lambda_2\wedge\lambda_3)=(\lambda_1\vee\lambda_2)\wedge(\lambda_1\vee\lambda_3)$ and $\lambda_1\wedge(\lambda_2\vee\lambda_3)=(\lambda_1\wedge\lambda_2)\vee(\lambda_1\wedge\lambda_3)$.*

Proof. Straightforward. □

If $A\subseteq H$, then the characteristic function λ_A of A is the fuzzy subset of H, defined as follows:

$$\lambda_A : H\longrightarrow[0,1], x\longmapsto\lambda_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A.\end{cases}$$

Proposition 2.4. *If A,B are subsets of a semihypergroup H, then $A\subseteq B$ if and only if $\lambda_A\leq\lambda_B$.*

Proof. Straightforward. □

Corollary 2.5. *Let A,B be subsets of a set X, then $A=B$ if and only if $\lambda_A=\lambda_B$.*

Proposition 2.6. *Let A,B be subsets of a set X, then $\lambda_{A\cap B}=\lambda_A\wedge\lambda_B$.*

Proof. Straightforward. □

Proposition 2.7. *Let $(H,\circ)$ be a semihypergroup and A,B be subsets of H. Then $\lambda_A\circ\lambda_B=\lambda_{A\circ B}$.*

Proof. Let $x \in H$. If $x \notin A \circ B$, then

$$\lambda_{A\circ B}(x) = 0 \tag{i}$$

This means that there exist no $y \in A$ and $z \in B$ such that $x \in y \circ z$.

If $X_x = \emptyset$, then

$$(\lambda_A \circ \lambda_B)(x) = 0 \tag{ii}$$

If $X_x \neq \emptyset$ and $(y, z) \in X_x$ then $x \in y \circ z$. Then $y \notin A$ or $z \notin B$. Thus either $\lambda_A(y) = 0$ or $\lambda_B(z) = 0$. So we have, $\min\{\lambda_A(y), \lambda_B(z)\} = 0$. Hence $(\lambda_A \circ \lambda_B)(x) = 0$.

Let $x \in A \circ B$, then $\lambda_{A\circ B}(x) = 1$. Thus $x \in a \circ b$, for some $a \in A$ and $b \in B$, so $(a, b) \in X_x$. Since $X_x \neq \emptyset$, we have

$$\begin{aligned}(\lambda_A \circ \lambda_B)(x) &= \bigvee_{(y,z)\in X_x} \{\lambda_A(y) \wedge \lambda_B(z)\} \\ &\geq \min\{\lambda_A(a), \lambda_B(b)\} = 1.\end{aligned}$$

Thus $(\lambda_A \circ \lambda_B)(x) = 1$. Hence $\lambda_A \circ \lambda_B = \lambda_{A\circ B}$. □

Proposition 2.8. *Let $\lambda_1, \lambda_2, \lambda_3$ are fuzzy subsets of H, then*

(i) $\lambda_1 \circ (\lambda_2 \vee \lambda_3) = (\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3); (\lambda_2 \vee \lambda_3) \circ \lambda_1 = (\lambda_2 \circ \lambda_1) \vee (\lambda_3 \circ \lambda_1)$.

(ii) $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3); (\lambda_2 \wedge \lambda_3) \circ \lambda_1 \leq (\lambda_2 \circ \lambda_1) \wedge (\lambda_3 \circ \lambda_1)$.

Proof. (i) Let $x \in H$, if $X_x = \emptyset$, then

$(\lambda_1 \circ (\lambda_2 \vee \lambda_3))(x) = 0 = (\lambda_1 \circ \lambda_2)(x) \vee (\lambda_1 \circ \lambda_3)(x)$.

If $X_x \neq \emptyset$, then

$$\begin{aligned}(\lambda_1 \circ (\lambda_2 \vee \lambda_3))(x) &= \bigvee_{(y,z)\in X_x} \{\lambda_1(y) \wedge (\lambda_2 \vee \lambda_3)(z)\} \\ &= \bigvee_{(y,z)\in X_x} \{\lambda_1(y) \wedge (\lambda_2(z) \vee \lambda_3(z))\} \\ &= \bigvee_{(y,z)\in X_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \vee (\lambda_1(y) \wedge \lambda_3(z))\} \\ &= \bigvee_{(y,z)\in X_x} \{(\lambda_1(y) \wedge \lambda_2(z)\} \vee \bigvee_{(y,z)\in X_x}\{ \ (\lambda_1(y) \wedge \lambda_3(z)\} \\ &= (\lambda_1 \circ \lambda_2)(x) \vee (\lambda_1 \circ \lambda_3)(x) \\ &= ((\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3))(x).\end{aligned}$$

Hence $\lambda_1 \circ (\lambda_2 \vee \lambda_3) = (\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3)$.

Similarly, we can prove that $(\lambda_2 \vee \lambda_3) \circ \lambda_1 = (\lambda_2 \circ \lambda_1) \vee (\lambda_3 \circ \lambda_1)$.

(ii) Suppose $X_x = \emptyset$, then $(\lambda_1 \circ (\lambda_2 \wedge \lambda_3))(x) = 0 = (\lambda_1 \circ \lambda_2)(x) \wedge (\lambda_1 \circ \lambda_3)(x)$ implies that $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3)$.

If $X_x \neq \emptyset$, then

$$\begin{aligned}(\lambda_1 \circ (\lambda_2 \wedge \lambda_3))(x) &= \bigvee_{(y,z)\in X_x} \{\lambda_1(y) \wedge (\lambda_2 \wedge \lambda_3)(z)\} \\ &= \bigvee_{(y,z)\in X_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \wedge (\lambda_1(y) \wedge \lambda_3(z))\} \\ &= \bigvee_{(y,z)\in X_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \wedge (\lambda_1(y) \wedge \lambda_3(z))\} \\ &\leq \bigvee_{(y,z)\in X_x} \{\lambda_1(y) \wedge \lambda_2(z)\} \wedge \bigvee_{(y,z)\in X_x} \{\lambda_1(y) \wedge \lambda_3(z)\} \\ &= (\lambda_1 \circ \lambda_2)(x) \wedge (\lambda_1 \circ \lambda_3)(x).\end{aligned}$$

Therefore, $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3)$.
Similarly, we can show that $(\lambda_2 \wedge \lambda_3) \circ \lambda_1 \leq (\lambda_2 \circ \lambda_1) \wedge (\lambda_3 \circ \lambda_1)$. □

Definition 2.9. (cf. [9])Let μ be a fuzzy subset of H, then μ is called:

i) a *fuzzy right hyperideal* of H if $\mu(x) \leq \underset{\alpha\in x\circ y}{inf} \{\mu(\alpha)\}$, for every $x, y \in H$;

ii) a *fuzzy left hyperideal* of H if $\mu(y) \leq \underset{\alpha\in x\circ y}{inf} \{\mu(\alpha)\}$, for every $x, y \in H$;

iii) a *fuzzy hyperideal* of H (or fuzzy two-sided hyperideal) if it is both a fuzzy left hyperideal and a fuzzy right hyperideal.

Proposition 2.10. *(cf. [9])Let μ be a fuzzy hyperideal of H, then*

$$\max\{\mu(x_1), ..., \mu(x_n)\} \leq \inf_{\alpha\in x_1\circ x_2\circ ...x_n} \{\mu(\alpha)\}, \ \forall x_1, x_2, ..., x_n \in H.$$

Proposition 2.11. *(cf. [9])A non-empty subset A of H is a hyperideal of H if and only if the characteristic function λ_A of A is a fuzzy hyperideal of H.*

Proposition 2.12. *A fuzzy subset λ of a semihypergroup H is a fuzzy left (right) hyperideal of H if and only if for each $t \in [0,1]$, $U(\lambda;t) \neq \phi$ is a left (right) hyperideal of H, respectively.*

Proof. Suppose λ be a fuzzy left hyperideal of H, $x \in U(\lambda;t)$ and $y \in H$. Then $\lambda(x) \geq t$. Since λ is a fuzzy left hyperideal of H, so $\lambda(x) \leq \underset{\alpha\in y\circ x}{inf} \{\lambda(\alpha)\}$ for every $y \in H$. Hence $\lambda(\alpha) \geq t$ for all $\alpha \in y \circ x$, this implies $\alpha \in U(\lambda;t)$, that is $y \circ x \subseteq U(\lambda;t)$. Hence $U(\lambda;t)$ is a hyperideal of H.

Conversely, assume that $U(\lambda;t) \neq \emptyset$ is a left hyperideal of H. Let $x \in H$ such that $\lambda(x) > \underset{\alpha\in y\circ x}{inf} \{\lambda(\alpha)\}$ for all $y \in H$. Select $t \in [0,1]$ such that $\lambda(x) = t > \underset{\alpha\in y\circ x}{inf} \{\lambda(\alpha)\}$. Then $x \in U(\lambda;t)$ but $y \circ x \nsubseteq U(\lambda;t)$, a contradiction. Hence $\lambda(x) \leq \underset{\alpha\in y\circ x}{inf} \{\lambda(\alpha)\}$, that is λ is a fuzzy left hyperideal of H. □

Proposition 2.13. *Let λ, μ be fuzzy hyperideals of H, then $\lambda \wedge \mu$ and $\lambda \vee \mu$ are fuzzy hyperideals of H.*

Proof. Let λ and μ be fuzzy hyperideals of a semihypergroup H. Then for $x, y, z \in H$, $\inf\limits_{x \in y \circ z} \lambda(x) \geq \lambda(z)$ and $\inf\limits_{x \in y \circ z} \mu(x) \geq \mu(z)$.

Now, for each $x \in y \circ z$,

$$\begin{aligned} (\lambda \wedge \mu)(x) &= \lambda(x) \wedge \mu(x) \\ &\geq \lambda(z) \wedge \mu(z) \\ &= (\lambda \wedge \mu)(z). \end{aligned}$$

Hence, $\inf\limits_{x \in y \circ z} (\lambda \wedge \mu)(x) \geq (\lambda \wedge \mu)(z)$. Thus $\lambda \wedge \mu$ is a fuzzy left hyperideal of H.

Similarly we can show that $\lambda \wedge \mu$ is a fuzzy right hyperideal of H. Thus $\lambda \wedge \mu$ is a fuzzy hyperideal of H.

Similarly we can show that $\lambda \vee \mu$ is a fuzzy hyperideal of H. □

Lemma 2.14. *A fuzzy subset λ of H is a fuzzy left (resp. right) hyperideal of H if and only if $\lambda_H \circ \lambda \leq \lambda$ (resp. $\lambda \circ \lambda_H \leq \lambda$).*

Proof. Let λ be a fuzzy left hyperideal of H and $x \in H$, then

$$\begin{aligned} \lambda_H \circ \lambda(x) &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \\ &= \bigvee_{x \in y \circ z} \{\lambda(z)\} \quad (\because \lambda_H(y) = 1) \\ &\leq \bigvee_{x \in y \circ z} \lambda(x) \quad \text{since } \lambda(z) \leq \mathop{inf}_{\alpha \in y \circ z} \lambda(\alpha) \leq \lambda(\alpha) \text{ for each } \alpha \in y \circ z. \\ &= \lambda(x). \\ \text{Hence, } \lambda_H \circ \lambda(x) &\leq \lambda(x). \end{aligned}$$

Conversely, suppose that $\lambda_H \circ \lambda \leq \lambda$. We show that λ is a fuzzy left hyperideal of H. For $x \in H$,

$$\begin{aligned} \lambda(x) &\geq \lambda_H \circ \lambda(x) \\ &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \text{ for } y, z \in H \\ &= \bigvee_{x \in y \circ z} \{\lambda(z)\}, \text{ (because } \lambda_H(y) = 1) \\ &\geq \lambda(z) \text{ for each } z \text{ such that } x \in y \circ z. \end{aligned}$$

Thus $\inf\limits_{x \in y \circ z} \lambda(x) \geq \lambda(z)$. Hence λ is a fuzzy left hyperideal of H.

Similarly, we can prove the case of fuzzy right hyperideal. □

Lemma 2.15. *If λ is a fuzzy left hyperideal and μ a fuzzy right hyperideal of H, then $\lambda \circ \mu$ is a fuzzy hyperideal of H and $\lambda \circ \mu \leq \lambda \wedge \mu$.*

Proof. Let λ be a fuzzy left hyperideal and μ a fuzzy right hyperideal of H, then

$$\begin{aligned} \lambda_H \circ (\lambda \circ \mu) &= (\lambda_H \circ \lambda) \circ \mu \\ &\leq \lambda \circ \mu \text{ (by Lemma 2.14)} \end{aligned}$$

Hence $\lambda \circ \mu$ is a fuzzy left hyperideal of H.

Also,

$$\begin{aligned}(\lambda \circ \mu) \circ \lambda_H &= \lambda \circ (\mu \circ \lambda_H) \\ &\leq \lambda \circ \mu \ \text{(by Lemma 2.14)}\end{aligned}$$

So, $\lambda \circ \mu$ is a fuzzy right hyperideal of H.
Thus $\lambda \circ \mu$ is a fuzzy hyperideal of H.
Let $x \in H$. If $X_x = \emptyset$, then $(\lambda \circ \mu)(x) = 0 \leq (\lambda \wedge \mu)(x)$.
If $X_x \neq \emptyset$, then $(\lambda \circ \mu)(x) = \bigvee_{(y,z) \in X_x} \min\{\lambda(y), \mu(z)\}$.

For $(y, z) \in X_x$ we have $x \in y \circ z$, so we have $\lambda(x) \geq \inf_{\alpha \in y \circ z} \lambda(\alpha) \geq \lambda(y)$ and $\mu(x) \geq \inf_{\alpha \in y \circ z} \mu(\alpha) \geq \mu(z)$. Hence

$$\begin{aligned}(\lambda \circ \mu)(x) &= \bigvee_{(y,z) \in X_x} \min\{\lambda(y), \mu(z)\} \\ &\leq \min\{\lambda(x), \mu(x)\} \\ &= (\lambda \wedge \mu)(x).\end{aligned}$$

Thus, $\lambda \circ \mu \leq \lambda \wedge \mu$. □

3. Prime and Semiprime Hyperideals

In this section we define prime and semiprime hyperideals of a semihypergroup and characterize those semihypergroups for which each hyperideal is semiprime.

Theorem 3.1. *Let $(H, \circ)$ be a semihypergroup with identity. Then the following conditions are equivalent:*

(1) H is semisimple;
(2) $A \cap B = A \circ B$, for every hyperideals A and B of H;
(3) $A = A \circ A$, for all two sided hyperideal A of H;
(4) $< a >=< a > \circ < a >$.

Proof. (1) $\Longrightarrow$ (2). Let $a \in A \cap B$, then $a \in A$ and $a \in B$. Since H is semisimple, there exist $x, y, z \in H$ such that

$$a \in x \circ a \circ y \circ a \circ z = (x \circ a \circ y) \circ (a \circ z) \subseteq A \circ B.$$

Thus $A \cap B \subseteq A \circ B$.

On the other hand $A \circ B \subseteq A$ (because A is a hyperideal of H) and $A \circ B \subseteq B$ (because B is a hyperideal of H), so we have $A \circ B \subseteq A \cap B$. Hence, $A \cap B = A \circ B$.

(2) $\Longrightarrow$ (3). Take $B = A$, then by hypothesis $A \cap A = A \circ A$. This implies that $A = A \circ A$.

(3) $\Longrightarrow$ (4). Obvious.

(4) $\Longrightarrow$ (1). As $a \in< a >=< a > \circ < a >$, so

$$\begin{aligned}a &\in (H \circ a \circ H) \circ (H \circ a \circ H) \\ &= H \circ a \circ (H \circ H) \circ a \circ H \\ &\subseteq H \circ a \circ H \circ a \circ H.\end{aligned}$$

This implies that $a \in x \circ a \circ y \circ a \circ z$ for some $x, y, z \in H$. Hence H is semisimple. $\square$

Definition 3.2. A hyperideal I of a semihypergroup H is called a *prime (semiprime) hyperideal* of H if for all hyperideals A, B of H, $A \circ B \subseteq I$ ($A \circ A \subseteq I$) implies $A \subseteq I$ or $B \subseteq I$ ($A \subseteq I$), respectively.

Definition 3.3. A hyperideal I of a semihypergroup H is called an *irreducible hyperideal* if for all hyperideals I_1, I_2 of H, $I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$.

Proposition 3.4. *A hyperideal A of a semihypergroup H is prime if and only if A is semiprime and irreducible.*

Proof. Let A be a prime hyperideal of H. Then clearly A is semiprime. Let B and C be any hyperideals of H such that $B \cap C = A$. Since $B \circ C \subseteq B \cap C = A$ and A is a prime hyperideal of H, so $B \subseteq A$ or $C \subseteq A$. On the other hand $A \subseteq B$ and $A \subseteq C$ (since $B \cap C = A$). Hence $B = A$ or $C = A$.

Conversely, let A be an irreducible semiprime hyperideal of H. Let B and C be any hyperideals of H such that $B \circ C \subseteq A$. Since $(B \cap C) \circ (B \cap C) \subseteq B \circ C \subseteq A$ and A is semiprime, so $B \cap C \subseteq A$. But $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = A$ and A is irreducible, so we have, $A \cup B = A$ or $A \cup C = A$. Hence $B \subseteq A$ or $C \subseteq A$. Thus A is prime. $\square$

Proposition 3.5. *Let I be a hyperideal of H and $a \in H$ such that $a \notin I$. Then there exists an irreducible hyperideal A of H such that $I \subseteq A$ and $a \notin A$.*

Proof. Let Ω be the collection of all hyperideals of H which contain I but do not contain "a". Then Ω is non-empty, because $I \in \Omega$. The collection Ω is partially ordered under inclusion. As every totally ordered subset of Ω is bounded above, so by Zorn's Lemma, there exists a maximal element, say, A in Ω. We show that A is an irreducible hyperideal of H. Let C and D be two hyperideals of H such that $C \cap D = A$. If both C and D properly contain A then $C \cap D$ contains A properly, which is a contradiction. Hence $C = A$ or $D = A$, that is A is irreducible. $\square$

In the next theorem we characterize those semihypergroups in which each hyperideal is semiprime.

Theorem 3.6. *Let H be a semihypergroup. Then the following conditions are equivalent:*

(1) H is semisimple,
(2) $A \cap B = A \circ B$, for all hyperideals A and B of H,
(3) $A = A \circ A$, for all hyperideal A of H,
(4) Each hyperideal of H is semiprime,
(5) Each hyperideal of H is the intersection of prime hyperideals of H which contain it.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) Follow from the Theorem 3.1.

(3) $\Longrightarrow$ (4) Let A and I be hyperideals of H such that $A \circ A \subseteq I$. By hypothesis $A \circ A = A$, so $A \subseteq I$. Hence each hyperideal of H is semiprime.

(4) $\Longrightarrow$ (5) Let A be a hyperideal of H, then obviously A is contained in the intersection of all irreducible hyperideals of H which contain A. If $a \notin A$, then by Proposition 3.5, there exists an irreducible hyperideal of H which contains A but does not contain a. Hence A is the intersection of all irreducible hyperideals of H which contain it. By hypothesis, each hyperideal of H is semiprime, so each hyperideal of H is the intersection of all irreducible semiprime hyperideals of H which contain it. By Proposition 3.4, each irreducible semiprime hyperideal of H is prime. Hence each hyperideal of H is the intersection of prime hyperideals of H which contain it.

(5) $\Longrightarrow$ (3) Let A be a proper hyperideal of H, then $A \circ A$ is a hyperideal of H. By hypothesis,

$$A \circ A = \bigcap_{\alpha}\{A_\alpha : A_\alpha \text{ are prime hyperideal of } H \text{ containing } A \circ A\}.$$

This implies that $A \circ A \subseteq A_\alpha$ for each α. Since each A_α is prime, so $A \subseteq A_\alpha$ for each α and hence $A \subseteq \bigcap_{\alpha} A_\alpha = A \circ A$. But $A \circ A \subseteq A$ always. Hence $A \circ A \subseteq A$. □

Next proposition shows that in a semisimple semihypergroup the concepts of prime hyperideal and irreducible hyperideal coincide.

Proposition 3.7. *Let T be a hyperideal of a semisimple semihypergroup H, then the following are equivalent:*

(1) T is a prime hyperideal of H;
(2) T is an irreducible hyperideal of H.

Proof. (1) $\Longrightarrow$ (2) Suppose T is a prime hyperideal of H. Let A, B be any hyperideals of H such that $A \cap B = T$. Then $T \subseteq A$ and $T \subseteq B$. Since $A \cap B \supseteq A \circ B$, so $A \circ B \subseteq T$. Since T is prime, so $A \subseteq T$ or $B \subseteq T$. Thus $A = T$ or $B = T$.

(2) $\Longrightarrow$ (1) Suppose T is an irreducible hyperideal of H. Let A, B be any hyperideals of H such that $A \circ B \subseteq T$. Since H is semisimple, so we have $A \cap B = A \circ B \subseteq T$. Then $(A \cap B) \cup T = T$. But $(A \cap B) \cup T = (A \cup T) \cap (B \cup T)$. Hence $(A \cup T) \cap (B \cup T) = T$. Since T is irreducible, so $A \cup T = T$ or $B \cup T = T$. Thus we have $A \subseteq T$ or $B \subseteq T$. □

Theorem 3.8. *The following conditions are equivalent for a semihypergroup H:*

(1) Each hyperideal of H is prime,
(2) H is semisimple and the set of hyperideals of H is a chain.

Proof. (1) $\Longrightarrow$ (2) Suppose each hyperideal of H is prime. Then by Theorem 3.6, H is semisimple. Let A, B be hyperideals of H, then $A \circ B \subseteq A \cap B$. By hypothesis each hyperideal of H is prime, so $A \cap B$ is prime. Thus $A \subseteq A \cap B$ or $B \subseteq A \cap B$, that is $A \subseteq B$ or $B \subseteq A$.

(2) $\Longrightarrow$ (1) Suppose H is semisimple and the set of hyperideals of H is a chain. Let A, B, C be hyperideals of H such that $A \circ B \subseteq C$. Since H is semisimple, so $A \circ B = A \cap B$. Since the set of hyperideals is a chain, so either $A \subseteq B$ or $B \subseteq A$, that is either $A \cap B = A$ or $A \cap B = B$. Thus either $A \subseteq C$ or $B \subseteq C$. □

Example 3.9. Let $(S, \cdot)$ be any semigroup. Define a hyperoperation $\circ$ on S by:

$$a \circ b = \{a, b, ab\} \text{ for all } a, b \in S.$$

Then for all $a, b, c \in S$, we have

$$\begin{aligned} a \circ (b \circ c) &= a \circ \{b, c, bc\} = (a \circ b) \cup (a \circ c) \cup (a \circ (bc)) \\ &= \{a, b, ab\} \cup \{a, c, ac\} \cup \{a, bc, a(bc)\} \\ &= \{a, b, c, ab, ac, bc, a(bc)\} \\ (a \circ b) \circ c &= \{a, b, ab\} \circ c = (a \circ c) \cup (b \circ c) \cup ((ab) \circ c) \\ &= \{a, c, ac\} \cup \{b, c, bc\} \cup \{ab, c, (ab)c\} \\ &= \{a, b, c, ab, ac, bc, (ab)c\} \end{aligned}$$

Thus $a \circ (b \circ c) = (a \circ b) \circ c$. This implies $(S, \circ)$ is a semihypergroup.

The only hyperideal of such a semihypergroup is S itself. Since $S \circ S = S$, so $(S, \circ)$ is semisimple.

Example 3.10. Let $(S, \leq, \cdot)$ be any ordered semigroup. Define a hyperoperation $\circ$ on S by:

$$a \circ b = \{x \in S : x \leq ab\} = (ab] \text{ for all } a, b \in S.$$

Then for all $a, b, c \in S$, we claim that $a \circ (b \circ c) = (a(bc)]$. Let $t \in a \circ (b \circ c)$, then $t \in a \circ x$ for some $x \in (b \circ c)$. This implies $t \leq ax$ and $x \leq bc$. Hence $t \leq a(bc)$, so $a \circ (b \circ c) \subseteq (a(bc)]$. Let $s \in (a(bc)]$, then $s \leq a(bc)$. So $s \in a \circ (bc) \subseteq \bigcup\limits_{x \in b \circ c} a \circ x = a \circ (b \circ c)$. Thus $(a(bc)] \subseteq a \circ (b \circ c)$. Consequently $(a(bc)] = a \circ (b \circ c)$. Similarly we can show that $((ab)c] = (a \circ b) \circ c$. Hence $(a \circ b) \circ c = a \circ (b \circ c)$. Thus $(S, \circ)$ is a semihypergroup.

Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the following multiplication table and order relation

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq = \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$$

Then the hyperoperation $\circ$ is defined in the following table

$\circ$	a	b	c	d	e
a	a	$\{a, b, d\}$	a	$\{a, b, d\}$	$\{a, b, d\}$
b	a	b	a	$\{a, b, d\}$	$\{a, b, d\}$
c	a	$\{a, b, d\}$	$\{a, c\}$	$\{a, b, d\}$	$\{a, b, c, d, e\}$
d	a	$\{a, b, d\}$	a	$\{a, b, d\}$	$\{a, b, d\}$
e	a	$\{a, b, d\}$	$\{a, c\}$	$\{a, b, d\}$	$\{a, b, c, d, e\}$

Then $(S, \circ)$ is a semihypergroup and the only hyperideals of S are $\{a, b, d\}$ and S. Both the hyperideals are idempotent, so $(S, \circ)$ is a semisimple semihypergroup. Also both the hyperideals are prime.

Example 3.11. Consider the ordered semigroup $S = \{a, b, c, d\}$ with the following multiplication table and order relation:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b)\}$$

Then the hyperoperation $\circ$ on S is defined by the following table

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a,b\}$	a
d	a	a	$\{a,b\}$	$\{a,b\}$

Then $(S,\circ)$ is a semihypergroup and the hyperideals of $(S,\circ)$ are $\{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, S$. No hyperideal is prime or semiprime. But the proper hyperideals $\{a,b,c\}, \{a,b,d\}$ are irreducible.

Example 3.12. Consider the ordered semigroup $S = \{a,b,c,d\}$ with the following multiplication table and order relation:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	a	a
d	a	d	a	a

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d)\}$$

Then the hyperoperation $\circ$ on S is defined by the following table

$\circ$	a	b	c	d
a	a	a	a	a
b	a	$\{a,b\}$	$\{a,c\}$	a
c	a	a	a	a
d	a	$\{a,d\}$	a	a

Then $(S,\circ)$ is a semihypergroup and the hyperideals of $(S,\circ)$ are $\{a\}, \{a,c\}, \{a,d\}, \{a,c,d\}, S$. The hyperideal $\{a,c,d\}$ is prime and all other hyperideals are neither prime nor semiprime.

4. **Prime Fuzzy Hyperideals and Semiprime Fuzzy Hyperideals**

In this section we define prime and semiprime fuzzy hyperideals of a semihypergroup and characterize those semihypergroups for which each fuzzy hyperideal is semiprime.

Definition 4.1. A fuzzy hyperideal λ of a semihypergroup H is called *prime (semiprime) fuzzy hyperideal* of H if for fuzzy hyperideals μ and ν of H, $\mu \circ \nu \leq \lambda$ $(\mu \circ \mu \leq \lambda)$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$ $(\mu \leq \lambda)$.

Proposition 4.2. *Let I be a hyperideal of a semihypergroup H, then the following hold:*

(1) I is a prime hyperideal of H if and only if the characteristic function λ_I of I is a prime fuzzy hyperideal of H.

(2) I is a semiprime hyperideal of H if and only if the characteristic function λ_I of I is a semiprime fuzzy hyperideal of H.

Proof. (1) Suppose I is a prime hyperideal of H. Then by Proposition 2.11, λ_I is a fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H, such that $\mu \circ \nu \leq \lambda_I$ but $\mu \nleq \lambda_I$ and $\nu \nleq \lambda_I$. Then there exist $y, z \in H$ such that $\mu(y) \neq 0$ and $\nu(z) \neq 0$, but $\lambda_I(y) = 0$ and $\lambda_I(z) = 0$. Then $y, z \notin I$. Since I is a prime hyperideal of H, we have $y \circ z \nsubseteq I$. Hence there exists $x \in y \circ z$ such that $x \notin I$. This implies $\lambda_I(x) = 0$. Hence $(\mu \circ \nu)(x) = 0$. Since $\mu(y) \neq 0$ and $\nu(z) \neq 0$, we have $\min\{\mu(y), \nu(z)\} \neq 0$. Since $x \in y \circ z$ so $(y, z) \in X_x$. Since $X_x \neq \emptyset$ we have, $(\mu \circ \nu)(x) = \bigvee_{(y,z) \in X_x} \min\{\mu(y), \nu(z)\} \neq 0$. Thus $(\mu \circ \nu)(x) > 0$, a contradiction. Hence $\mu \circ \nu \leq \lambda_I$ implies that $\mu \leq \lambda_I$ or $\nu \leq \lambda_I$. Thus λ_I is a prime fuzzy hyperideal of H.

Conversely, suppose that λ_I be a prime fuzzy hyperideal of H. Let A, B be any hyperideals of H such that $A \circ B \subseteq I$. Then by Proposition 2.11, λ_A, λ_B are fuzzy hyperideals of H. Since $A \circ B \subseteq I$, so by Proposition 2.4, $\lambda_{A \circ B} \leq \lambda_I$. Thus by Proposition 2.7, $\lambda_{A \circ B} = \lambda_A \circ \lambda_B \leq \lambda_I$, so we have $\lambda_A \leq \lambda_I$ or $\lambda_B \leq \lambda_I$. By Proposition 2.4, we get $A \subseteq I$ or $B \subseteq I$. Hence I is a prime hyperideal of H.

Similarly we can prove part (2). □

Proposition 4.3. *Let $\{\lambda_i : i \in I\}$ be a family of prime fuzzy hyperideals of a semihypergroup H, then $\underset{i \in I}{\wedge} \lambda_i$ is a semiprime fuzzy hyperideal of H.*

Proof. Straightforward. □

Theorem 4.4. *A semihypergroup H is semisimple if and only if for all fuzzy hyperideals λ and μ of H, $\lambda \circ \mu = \lambda \wedge \mu$.*

Proof. Let λ and μ be fuzzy hyperideals of a semihypergroup H. Suppose H is semisimple and $a \in H$. Then there exist $x, y, z \in H$ such that $a \in x \circ a \circ y \circ a \circ z = (x \circ a \circ y) \circ (a \circ z)$. So for each $\alpha \in x \circ a \circ y$ and $\beta \in a \circ z, (\alpha, \beta) \in X_a$, i.e. $a \in \alpha \circ \beta \subseteq (x \circ a \circ y) \circ (a \circ z)$, we have $X_a \neq \emptyset$. Hence

$$\begin{aligned} (\lambda \circ \mu)(a) &= \bigvee_{(\alpha,\beta) \in X_a} \min\{\lambda(\alpha), \mu(\beta)\} \\ &\geq \min\{\lambda(\alpha), \mu(\beta)\}. \end{aligned}$$

As λ and μ are fuzzy hyperideals of H, we have $\lambda(\alpha) \geq \lambda(\alpha_1) \geq \lambda(a)$, for each $\alpha \in \alpha_1 \circ y$ and $\alpha_1 \in x \circ a$ and $\mu(\beta) \geq \mu(a)$, for every $\beta \in a \circ z$.

Hence $\min\{\lambda(\alpha), \mu(\beta)\} \geq \min\{\lambda(a), \mu(a)\}$.

Thus $(\lambda \circ \mu)(a) \geq (\lambda \wedge \mu)(a)$.

On the other hand, by Lemma 2.15, we have $(\lambda \circ \mu)(a) \leq (\lambda \wedge \mu)(a)$.

Hence, $(\lambda \circ \mu)(a) = (\lambda \wedge \mu)(a)$.

Conversely, assume that $\lambda \circ \mu = \lambda \wedge \mu$. Let A, B be hyperideals of H, then by Proposition 2.11, λ_A and λ_B are fuzzy hyperideals of H. Hence by hypothesis,

$\lambda_A \circ \lambda_B = \lambda_A \wedge \lambda_B$. By Proposition 2.7, $\lambda_A \circ \lambda_B = \lambda_{A \circ B}$, thus $\lambda_{A \circ B} = \lambda_A \wedge \lambda_B = \lambda_{A \cap B}$ which implies $A \cap B = A \circ B$. Thus by Theorem 3.1, H is semisimple. □

Corollary 4.5. *A semihypergroup H is semisimple if and only if for each fuzzy hyperideal λ of H, we have $\lambda \circ \lambda = \lambda$.*

Proof. Let λ be a fuzzy hyperideal of a semisimple semihypergroup H. Then by Theorem 4.4, $\lambda \circ \lambda = \lambda \wedge \lambda = \lambda$.

Conversely, assume that $\lambda \circ \lambda = \lambda$ for each fuzzy hyperideal λ of H. Let A be any hyperideal of H, then by Proposition 2.11, λ_A is a fuzzy hyperideal of H. By hypothesis, $\lambda_A \circ \lambda_A = \lambda_A$. By Proposition 2.7, $\lambda_A \circ \lambda_A = \lambda_{A \circ A} = \lambda_A$. Hence $A = A \circ A$. Thus by Theorem 3.1, H is semisimple. □

Definition 4.6. A fuzzy hyperideal λ of a semihypergroup H is called an *irreducible fuzzy hyperideal* if for each fuzzy hyperideals μ and ν of H, $\mu \wedge \nu = \lambda$ implies $\mu = \lambda$ or $\nu = \lambda$.

Proposition 4.7. *A non-empty subset I of a semihypergroup H is an irreducible hyperideal of H if and only if the characteristic function λ_I of I is an irreducible fuzzy hyperideal of H.*

Proof. Suppose I is an irreducible hyperideal of H, then λ_I the characteristic function of I is a fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda_I$ with $\mu \neq \lambda_I$ and $\nu \neq \lambda_I$. Then there exist $x, y \in H$ such that $\mu(x) \neq 0$ and $\nu(y) \neq 0$ but $\lambda_I(x) = 0$ and $\lambda_I(y) = 0$. Hence $x \notin I$ and $y \notin I$. Since I is an irreducible hyperideal of H, we have $< x > \cap < y > \neq I$. Thus there exists $a \in < x > \cap < y >$ such that $a \notin I$. Hence $\lambda_I(a) = 0$. Thus $(\mu \wedge \nu)(a) = 0$. Since $\mu(x) \neq 0$ and $\nu(y) \neq 0$, we have $\min\{\mu(x), \nu(y)\} \neq 0$.

As $a \in < x >= x \cup H \circ x \cup x \circ H \cup H \circ x \circ H$, therefore $a = x$ or there exist $h, h_1 \in H$ such that $a \in h \circ x$ or $a \in x \circ h$ or $a \in h \circ x \circ h_1$.

If $a = x$, then $\mu(a) = \mu(x)$.

If $a \in h \circ x$, then $\mu(a) \geq \mu(x)$ because μ is a fuzzy left hyperideal of H.

If $a \in x \circ h$, then $\mu(a) \geq \mu(x)$ because μ is a fuzzy right hyperideal of H.

If $a \in h \circ x \circ h_1 = \bigcup\limits_{g \in h \circ x} g \circ h_1$, then $\alpha \in g_1 \circ h_1$ for some $g_1 \in h \circ x$ and since μ is a fuzzy (two-sided) hyperideal of H,

$$\mu(a) \geq \mu(g_1) \geq \mu(x).$$

Also $a \in < y >= y \cup H \circ y \cup y \circ H \cup H \circ y \circ H$. Then $a = y$ or there exist $h, h_1 \in H$, such that $a \in h \circ y$ or $a \in y \circ h$ or $a \in h \circ y \circ h_1$.

If $a = y$ then $\nu(a) = \nu(y)$.

If $a \in h \circ y$, then $\nu(a) \geq \nu(y)$ because ν is a fuzzy left hyperideal of H.

If $a \in y \circ h$ then $\nu(a) \geq \nu(y)$ because ν is a fuzzy right hyperideal of H.

If $a \in h \circ y \circ h_1 = \bigcup\limits_{i \in h \circ y} i \circ h_1$ then for each $\beta \in i \circ h_1$,

$$\nu(a) \geq \nu(\beta) \geq \nu(i) \geq \nu(y).$$

Hence, $\min\{\mu(a), \nu(a)\} \geq \min\{\mu(x), \nu(y)\} \neq 0$ i.e. $(\mu \wedge \nu)(a) \neq 0$ which is a contradiction. Thus $\mu \wedge \nu = \lambda_I$ implies that $\mu = \lambda_I$ or $\nu = \lambda_I$.

Conversely, assume that λ_I is an irreducible fuzzy hyperideal of H. Let A, B be any hyperideals of H such that $A \cap B = I$. Then λ_A and λ_B are fuzzy hyperideals of H and $\lambda_{A\cap B} = \lambda_I$. Since $\lambda_{A\cap B} = \lambda_A \wedge \lambda_B$, so $\lambda_A \wedge \lambda_B = \lambda_I$. Thus by hypothesis, $\lambda_A = \lambda_I$ or $\lambda_B = \lambda_I$. Hence $A = I$ or $B = I$. Thus I is an irreducible hyperideal of H. □

Proposition 4.8. *A fuzzy hyperideal λ of a semihypergroup H is prime fuzzy hyperideal if and only if λ is semiprime and irreducible fuzzy hyperideal.*

Proof. Let λ be a prime fuzzy hyperideal of H. Then λ is semiprime. Let μ and ν be any fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda$. Since $\mu \circ \nu \leq \mu \wedge \nu = \lambda$ and λ is a prime fuzzy hyperideal of H, we have $\mu \leq \lambda$ or $\nu \leq \lambda$. On the other hand $\mu \wedge \nu = \lambda$ implies that $\lambda \leq \mu$ and $\lambda \leq \nu$. Hence $\mu = \lambda$ or $\nu = \lambda$.

Conversely, let λ be an irreducible semiprime fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since $(\mu \wedge \nu) \circ (\mu \wedge \nu) \leq \mu \circ \nu \leq \lambda$ and λ is semiprime, so we have $\mu \wedge \nu \leq \lambda$. By Lemma 2.3, we have $\lambda \vee (\mu \wedge \nu) = (\lambda \vee \mu) \wedge (\lambda \vee \nu)$. But $\lambda \vee (\mu \wedge \nu) = \lambda$. Thus $(\lambda \vee \mu) \wedge (\lambda \vee \nu) = \lambda$. Since λ is irreducible, so we have $\lambda \vee \mu = \lambda$ or $\lambda \vee \nu = \lambda$. Hence $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is a prime fuzzy hyperideal of H. □

Proposition 4.9. *Let λ be a fuzzy hyperideal of a semihypergroup H with $\lambda(a) = t$, where $a \in H$ and $t \in (0, 1]$, then there exists an irreducible fuzzy hyperideal μ of H such that $\lambda \leq \mu$ and $\mu(a) = t$.*

Proof. Let $X = \{\nu : \nu$ is a fuzzy hyperideal of H, $\nu(a) = t$ and $\lambda \leq \nu\}$. Then $X \neq \emptyset$, because $\lambda \in X$. The collection X is partially ordered under inclusion. Suppose Y be a totally ordered subset of X, say $Y = \{\nu_i : i \in I\}$. Then by Proposition 2.13, $(\bigvee_{i\in I} \nu_i)$ is a fuzzy hyperideal of H. As $\lambda \leq \nu_i$ for each $i \in I$, so $\lambda \leq \bigvee_{i\in I} \nu_i$. Also $\left(\bigvee_{i\in I} \nu_i\right)(a) = \bigvee_{i\in I} (\nu_i(a)) = t$. Thus $\bigvee_{i\in I} \nu_i$ is the least upper bound of Y. So by Zorn's Lemma, there exists a fuzzy hyperideal μ of H which is maximal with respect to the property that $\lambda \leq \mu$ and $\mu(a) = t$.

Now we show that μ is an irreducible fuzzy hyperideal of H. Suppose $\mu = \mu_1 \wedge \mu_2$, where μ_1 and μ_2 are fuzzy hyperideals of H. Then $\mu \leq \mu_1$ and $\mu \leq \mu_2$. We claim that $\mu = \mu_1$ or $\mu = \mu_2$. Suppose, on contrary that $\mu \neq \mu_1$ and $\mu \neq \mu_2$. Since μ is maximal with respect to the property that $\mu(a) = t$ and $\mu \neq \mu_1$ and $\mu \neq \mu_2$, it follows that $\mu_1(a) \neq t$ and $\mu_2(a) \neq t$. Hence $t = \mu(a) = (\mu_1 \wedge \mu_2)(a) \neq t$, which is a contradiction. Hence either $\mu = \mu_1$ or $\mu = \mu_2$. Thus μ is an irreducible fuzzy hyperideal of H. □

Theorem 4.10. *Let H be a semihypergroup. Then the following conditions are equivalent:*

(1) H is semisimple,
(2) $\lambda \circ \mu = \lambda \wedge \mu$, for every fuzzy hyperideals λ and μ of H,
(3) $\lambda \circ \lambda = \lambda$, for all fuzzy hyperideal λ of H,
(4) Each fuzzy hyperideal of H is semiprime,

(5) Each fuzzy hyperideal of H is the intersection of prime fuzzy hyperideals of H which contain it.

Proof. (1) $\Longleftrightarrow$ (2) Follows from the Theorem 4.4.

(1) $\Longleftrightarrow$ (3) Follows from the Corollary 4.5.

(3) $\Longrightarrow$ (4) Let λ and μ be fuzzy hyperideals of H such that $\lambda \circ \lambda \leq \mu$. By hypothesis $\lambda \circ \lambda = \lambda$. So $\lambda \leq \mu$. Hence each fuzzy hyperideal of H is semiprime.

(4) $\Longrightarrow$ (5) Let λ be a proper fuzzy hyperideal of H and $\{\lambda_i : i \in I\}$ the collection of all irreducible fuzzy hyperideals of H which contain λ. Proposition 4.9, guarantees the existence of such fuzzy hyperideals. Hence $\lambda \leq \underset{i\in I}{\wedge} \lambda_i$. Let $x \in H$, then by Proposition 4.9, there exists an irreducible fuzzy hyperideal λ_α of H such that $\lambda \leq \lambda_\alpha$ and $\lambda(x) = \lambda_\alpha(x)$. Thus $\lambda_\alpha \in \{\lambda_i : i \in I\}$. Hence $\underset{i\in I}{\wedge} \lambda_i \leq \lambda_\alpha$. So $\underset{i\in I}{\wedge} \lambda_i(x) \leq \lambda_\alpha(x) = \lambda(x)$. Thus $\underset{i\in I}{\wedge} \lambda_i \leq \lambda_\alpha$. Consequently, $\underset{i\in I}{\wedge} \lambda_i = \lambda$. By hypothesis, each fuzzy hyperideal of H is semiprime, so each fuzzy hyperideal of H is the intersection of all irreducible fuzzy semiprime hyperideals of H which contain it. By Proposition 4.8, each fuzzy irreducible semiprime hyperideal is prime, therefore each fuzzy hyperideal is the intersection of all prime fuzzy hyperideals of H which contain it.

(5) $\Longrightarrow$ (3) Let λ be a proper fuzzy hyperideal of H. Then $\lambda \circ \lambda$ is also a fuzzy hyperideal of H. Since λ is fuzzy hyperideal of H, so $\lambda \circ \lambda \leq \lambda$. By hypothesis, we have $\lambda \circ \lambda = \underset{i\in I}{\wedge} \lambda_i$ where λ_i are prime fuzzy hyperideals of H. Thus $\lambda \circ \lambda \leq \lambda_i$ for all $i \in I$. Hence $\lambda \leq \lambda_i$ for all $i \in I$. Thus $\lambda \leq \underset{i\in I}{\wedge} \lambda_i = \lambda \circ \lambda$. Hence $\lambda \circ \lambda = \lambda$. $\square$

Proposition 4.11. *Let H be a semisimple semihypergroup and λ be a fuzzy hyperideal of H. Then the following are equivalent:*

(1) λ is a prime fuzzy hyperideal of H,

(2) λ is an irreducible fuzzy hyperideal of H.

Proof. (1) $\Longrightarrow$ (2) Suppose λ is a prime fuzzy hyperideal of H. Let μ, ν be fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda$. Since $\mu \circ \nu \leq \mu \wedge \nu \leq \lambda$, so $\mu \leq \lambda$ or $\nu \leq \lambda$. On the other hand, $\mu \wedge \nu = \lambda$ implies that $\lambda \leq \mu$ and $\lambda \leq \nu$. Thus $\mu = \lambda$ or $\nu = \lambda$.

(2) $\Longrightarrow$ (1) Suppose λ is an irreducible fuzzy hyperideal of H. Let μ, ν be fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since H is semisimple, so by Theorem 4.4, $\mu \circ \nu = \mu \wedge \nu$. Hence $\mu \wedge \nu \leq \lambda$, thus $(\mu \wedge \nu) \vee \lambda = \lambda$. But, $(\mu \wedge \nu) \vee \lambda = (\mu \vee \lambda) \wedge (\nu \vee \lambda)$, so $(\mu \vee \lambda) \wedge (\nu \vee \lambda) = \lambda$. Since λ is an irreducible fuzzy hyperideal of H, so we have $\mu \vee \lambda = \lambda$ or $\nu \vee \lambda = \lambda$. Hence $\mu \leq \lambda$ or $\nu \leq \lambda$. $\square$

Theorem 4.12. *Let H be a semihypergroup. Then every fuzzy hyperideal of H is prime if and only if the set of fuzzy hyperideals of H form a chain and H is semisimple.*

Proof. Suppose that every fuzzy hyperideal of H is prime then by Theorem 4.4, H is semisimple.

Let λ, μ be any fuzzy hyperideals of H, then $\lambda \wedge \mu$ is a fuzzy hyperideal of H. By hypothesis, $\lambda \wedge \mu$ is a prime fuzzy hyperideal of H. Since $\lambda \circ \mu \leq \lambda \wedge \mu$, so $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. Hence $\lambda \leq \mu$ or $\mu \leq \lambda$.

Conversely, assume that H is semisimple semihypergroup and the set of fuzzy hyperideals of H form a chain. Let λ, μ and ν be any fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since the set of fuzzy hyperideals of H form a chain, so $\mu \leq \nu$ or $\nu \leq \mu$. If $\mu \leq \nu$ then $\mu = \mu \circ \mu \leq \mu \circ \nu \leq \lambda$. If $\nu \leq \mu$ then $\nu = \nu \circ \nu \leq \mu \circ \nu \leq \lambda$. Thus every fuzzy hyperideal of H is prime. □

Example 4.13. Consider the semihypergroup given in Example 3.9. By Proposition 2.12, the only fuzzy hyperideals of S are the constant functions. Since S is semisimple so every fuzzy hyperideal is idempotent. Since the set of fuzzy hyperideals of S is a chain, so every fuzzy hyperideal is prime.

Example 4.14. Consider the semihypergroup S given in Example 3.10. By Proposition 2.12, the only fuzzy hyperideals of S are of the form $\lambda(a) = \lambda(b) = \lambda(d) \geq \lambda(c) = \lambda(e)$. Since S is semisimple so every fuzzy hyperideal is semiprime and also idempotent, that is $\lambda \circ \lambda = \lambda$.

Now consider the fuzzy hyperideals

$$\begin{gathered}
\lambda(a) = \lambda(b) = \lambda(d) = 0.6 \text{ and } \lambda(c) = \lambda(e) = 0.2 \\
\mu(a) = \mu(b) = \mu(d) = 0.5 \text{ and } \mu(c) = \mu(e) = 0.3 \\
\nu(a) = \nu(b) = \nu(d) = 0.55 \text{ and } \nu(c) = \nu(e) = 0.25
\end{gathered}$$

then $\lambda \circ \mu = \lambda \wedge \mu$ because S is semisimple. $(\lambda \wedge \mu)(a) = (\lambda \wedge \mu)(b) = (\lambda \wedge \mu)(d) = 0.5$ and $(\lambda \wedge \mu)(c) = (\lambda \wedge \mu)(e) = 0.2$. Thus $\lambda \circ \mu = \lambda \wedge \mu \leq \nu$ but neither $\lambda \nleq \nu$ nor $\mu \nleq \nu$. Hence ν is not prime fuzzy hyperideal of S.

References

[1] J. Ahsan, K. Saifullah and M. F. Khan, *Semigroups characterized by thier fuzzy ideals*, Fuzzy systems and Math., **9** (1995) 29-32.

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, AMS, Math., Surveys, Providence, R. I., **1&2(7)** (1961/67).

[3] P. Corsini, *Join spaces, power sets, fuzzy sets*, Proceedings of the 5th International Congress on Aigebraic Hyperstructures and Applications 1993, Isai, Romania, Hadronic Press, 1994.

[4] P. Corsini, *New themes of research on hyperstructures associated with fuzzy sets.*, J. of Basic Science, Mazandaran, Iran, **2(2)** (2003), 25-36.

[5] P. Corsini, *A new connection between hypergroups and fuzzy sets.*, Southeast Bul. of Math., **27** (2003), 221-229.

[6] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory,* Kluwer Academic Publishers, Dordrecht, Hardbound, 2003.

[7] I. Cristea, *Hyperstructures and fuzzy sets endowed with two membership functions*, Fuzzy Sets and Systems, **160** (2009), 1114-1124.

[8] I. Cristea, *About the fuzzy grade of the direct product of two hypergroupoids*, Iranian Journal of Fuzzy Systems, **7(2)** (2010), 95-108.

[9] B. Davvaz, *Fuzzy hyperideals in semihypergroups*, Italian J. Pure and Appl. Math., **8** (2000), 67-74.

[10] B. Davvaz, *Strong regularity and fuzzy strong regularity in semihypergroups.*, Korean J. Comput. & Appl. Math., **7(1)** (2000), 205-213.

[11] A. Hasankhani, *Ideals in a semihypergroup and Green's relations.*, Ratio Mathematica, **13** (1999), 29-36.

[12] A. Kehagias, *Lattice-fuzzy meet and join hyperoperations*, Proceedings of the 8th International Congress on AHA and Appl., Samothraki, Greece, (2003), 171-182.

[13] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Paul., **28** (1979) 17-21.

[14] V. Leoreanu, *About hyperstructures associated with fuzzy sets of type 2.*, Italian J. of Pure and Appl. Math., **17** (2005), 127-136.

[15] F. Marty, *Sur une generalization de la notion de groupe*, 8^{iem} Congress Math. Scandinaves, Stockholm, (1934), 45-49.

[16] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy semigroups,* Springer, 2003.

[17] M. Stefanescu and I. Cristea, *On the fuzzy grade of hypergroups*, Fuzzy Sets and Systems, **159(9)** (2008), 1097-1106

[18] I. Tofan and A. C. Volf, *On some connections between hyperstructures and fuzzy sets*, Italian J. of Pure and Appl. Math., **7** (2000), 63-68.

[19] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

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