NORM AND INNER PRODUCT ON FUZZY LINEAR SPACES OVER FUZZY FIELDS

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ABSTRACT. In this paper, we introduce the concepts of norm and inner product on fuzzy linear spaces over fuzzy fields and discuss some fundamental properties.

1. Introduction

Many authors introduced the notions of fuzzy norm and fuzzy inner product from different points of view. In 1984, Katsaras [8] defined a fuzzy norm on a linear space and Wu and Fang [15] introduced a notion of a fuzzy normed space. In 1991, Biswas [2] defined fuzzy norm and fuzzy inner product of elements in a linear space. Thereafter Felbin [5, 6], Cheng and Mordeson [3], Krishna and Sarma [11], Xiao and Zhu [16], Bang and Samanta [1] and Saadati and Vaezpour [12] discussed fuzzy norms on linear spaces; El-Abyad and El-Hamouly [4], Kohli and Kumar [10] and Goudarzi and Vaezpour [7] discussed fuzzy inner product spaces in different directions. All these introduce fuzzy norms/inner products on crisp linear spaces/modules over sets of fuzzy real numbers or variants of these. Gu Wenxiang and Lu Tu [14] introduced the notions of fuzzy fields and fuzzy linear spaces over fuzzy fields. This paper is (in contrast to the works mentioned above) an attempt to introduce the concepts of norm and inner product on fuzzy linear spaces.

In this paper we proceed as follows. Section 2 gives a brief summary of fuzzy fields and fuzzy linear spaces. In section 3, we introduce the idea of norm on fuzzy linear spaces over fuzzy fields with examples and some properties. The concept of inner product on fuzzy linear spaces is introduced in section 4. In section 5, interrelationship between norm and inner product on fuzzy linear space is discussed.

2. Preliminaries

This section gives a brief summary of fuzzy fields and fuzzy linear spaces over fuzzy fields.

Definition 2.1. [14] Let X be a field and F a fuzzy set in X with membership function μ_F. Suppose the following conditions hold:

(i) $\mu_F(x+y) \geq \min\{\mu_F(x), \mu_F(y)\}, x, y \in X$

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(ii) $\mu_F(-x) \geq \mu_F(x), x \in X$
(iii) $\mu_F(xy) \geq \min\{\mu_F(x), \mu_F(y)\}, x, y \in X$
(iv) $\mu_F(x^{-1}) \geq \mu_F(x), x(\neq 0) \in X$.

Then we call F a fuzzy field in X and denote it by (F, X). Also (F, X) is called a fuzzy field of X.

Proposition 2.2. [14] *If (F, X) is a fuzzy field of X, then*

(i) $\mu_F(0) \geq \mu_F(x), x \in X$
(ii) $\mu_F(1) \geq \mu_F(x), x(\neq 0) \in X$
(iii) $\mu_F(0) \geq \mu_F(1)$.

Proposition 2.3. [14] *Let X and Y be fields and f a homomorphism of X into Y. Suppose that (F, X) is a fuzzy field of X and (G, Y) is a fuzzy field of Y. Then*

(i) $(f(F), Y)$ *is a fuzzy field of Y*
(ii) $\left(f^{-1}(G), X\right)$ *is a fuzzy field of X.*

Proposition 2.4. *If (F, X) is a fuzzy field of X, then*

(i) $\mu_F(x) = \mu_F(-x), x \in X$
(ii) $\mu_F(x^{-1}) = \mu_F(x), x(\neq 0) \in X$.

Definition 2.5. [14] Let X be a field and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y with membership function μ_V. Suppose the following conditions hold:

(i) $\mu_V(x + y) \geq \min\{\mu_V(x), \mu_V(y)\}, x, y \in Y$
(ii) $\mu_V(-x) \geq \mu_V(x), x \in Y$
(iii) $\mu_V(\lambda x) \geq \min\{\mu_F(\lambda), \mu_V(x)\}, \lambda \in X, x \in Y$
(iv) $\mu_F(1) \geq \mu_V(0)$.

Then (V, Y) is called a fuzzy linear space over (F, X).

Proposition 2.6. *If (V, Y) is a fuzzy linear space over (F, X), then $\mu_V(-y) = \mu_V(y), y \in Y$.*

Proposition 2.7. [14] *If (V, Y) is a fuzzy linear space over (F, X), then*

(i) $\mu_F(0) \geq \mu_V(0)$
(ii) $\mu_V(0) \geq \mu_V(x), x \in Y$
(iii) $\mu_F(0) \geq \mu_V(x), x \in Y$.

Proposition 2.8. [14] *Let (F, X) be a fuzzy field of X and Y a linear space over X. Let V be a fuzzy set of Y. Then (V, Y) is a fuzzy linear space over (F, X) if and only if*

(i) $\mu_V(\lambda x + \mu y) \geq min\{\mu_F(\lambda) \wedge \mu_V(x), \mu_F(\mu) \wedge \mu_V y)\}, \lambda, \mu \in X$ *and* $x, y \in Y$.
(ii) $\mu_F(1) \geq \mu_V(x), x \in Y$.

Note 2.9. Condition (i) in proposition 2.8 can be restated as
$\mu_V(\lambda x + \mu y) \geq \min\{\mu_F(\lambda), \mu_F(\mu), \mu_V(x), \mu_V(y)\}$.

Proposition 2.10. [14] *Let Y and Z be linear spaces over the field X and f a linear transformation of Y into Z. Let (F,X) be a fuzzy field of X and (W,Z) be a fuzzy linear space over (F,X). Then $\big(f^{-1}(W),Y\big)$ is a fuzzy linear space over (F,X).*

Proposition 2.11. [14] *Let Y and Z be linear spaces over the field X and f a linear transformation of Y into Z. Let (F,X) be a fuzzy field of X and (V,Y) be a fuzzy linear space over (F, X). Then $(f(V),Z)$ is a fuzzy linear space over (F,X).*

Proposition 2.12. *Let (F,X) be a fuzzy field of X and let $(V_1,Y_1),(V_2,Y_2),\cdots,$ (V_n,Y_n) be fuzzy linear spaces over (F,X). Then $(V_1\times V_2\times\cdots\times V_n, Y_1\times Y_2\times\cdots\times Y_n)$ is a fuzzy linear space over (F,X).*

Proof. Let $V = V_1 \times V_2 \times \cdots \times V_n$.

Let $x=(x_1,x_2,\cdots,x_n), y=(y_1,y_2,\cdots,y_n)\in Y_1\times Y_2\times\cdots\times Y_n$ and $\alpha,\beta\in X$.

(i)
$$\begin{aligned}
\mu_V(\alpha x+\beta y) &= \mu_{V_1\times V_2\times\cdots\times V_n}(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2,\cdots,\alpha x_n+\beta y_n)\\
&= \min_{j=1,2,\cdots,n}\mu_{V_j}(\alpha x_j+\beta y_j)\\
&\geq \min_{j=1,2,\cdots,n}\min\{\mu_F(\alpha),\mu_{V_j}(x_j),\mu_F(\beta),\mu_{V_j}(y_j)\}\\
&= \min\left\{\mu_F(\alpha),\min_{j=1,2,\cdots,n}\mu_{V_j}(x_j),\mu_F(\beta),\min_{j=1,2,\cdots,n}\mu_{V_j}(y_j)\right\}\\
&= \min\{\mu_F(\alpha),\mu_V(x),\mu_F(\beta),\mu_V(y)\}.
\end{aligned}$$

(ii) $\mu_F(1)\geq\mu_{V_j}(x_j)$ for all $j=1,2,\cdots,n$. So $\mu_F(1)\geq\min_j\mu_{V_j}(x_j)=\mu_V(x)$ for all $x\in Y_1\times Y_2\times\cdots\times Y_n$. Hence $(V_1\times V_2\times\cdots\times V_n, Y_1\times Y_2\times\cdots Y_n)$ is a fuzzy linear space over (F,X). □

3. Norm on Fuzzy Linear Spaces

In this paper, $\mathbb{K}$ denotes either $\mathbb{R}$, the set of all real numbers or $\mathbb{C}$, the set of all complex numbers.

Definition 3.1. Let $(F,\mathbb{K})$ be a fuzzy field of $\mathbb{K}$, X be a linear space over $\mathbb{K}$ and (V,X) be a fuzzy linear space over $(F,\mathbb{K})$. A norm on (V,X) is a function $\|\ \|: X\to[0,\infty)$ such that

(i) $\mu_F(\|x\|)\geq\mu_V(x)$ for all $x\in X$
(ii) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$
(iii) $\|x+y\|\leq\|x\|+\|y\|$ for all $x,y\in X$
(iv) $\|kx\|=|k|\ \|x\|$ for all $k\in\mathbb{K}$ and for all $x\in X$.

A normed fuzzy linear space is a fuzzy linear space with a norm on it.

Example 3.2. Let $(F,\mathbb{R})$ be a fuzzy field of $\mathbb{R}$. The absolute value function $|\ |$ is a norm on $(F,\mathbb{R})$.

Proof.
$$\begin{aligned}
\mu_F(|x|) &= \begin{cases}\mu_F(x) & \text{if } x\geq 0\\ \mu_F(-x) & \text{if } x<0.\end{cases}\\
&= \mu_F(x).
\end{aligned}$$
□

Example 3.3. Let $(F, \mathbb{R})$ be a fuzzy field of $\mathbb{R}$. The norms $\| \|_1$ and $\| \|_\infty$ on $\mathbb{R}^n$ defined by $\|x\|_1 = |x_1| + \cdots + |x_n|$ and $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ are norms on the fuzzy linear space $(\underbrace{F \times F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$.

Proof. (i)

$$\begin{aligned} \mu_F(\|x\|_1) &= \mu_F(|x_1| + \cdots + |x_n|) \\ &\geq \min\{\mu_F(|x_1|), \ldots, \mu_F(|x_n|)\} \\ &= \min\{\mu_F(x_1), \ldots, \mu_F(x_n)\} \\ &= \mu_{F\times\cdots\times F}(x). \end{aligned}$$

Therefore $\| \|_1$ is a norm on $(\underbrace{F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$.

(ii)

$$\begin{aligned} \mu_F(\|x\|_\infty) &= \mu_F(\max\{|x_1|, \ldots, |x_n|\}) \\ &= \mu_F(|x_i|) \\ &= \mu_F(x_i) \\ &\geq \min\{\mu_F(x_1), \ldots, \mu_F(x_n)\} \\ &= \mu_{F\times\cdots\times F}(x). \end{aligned}$$

Therefore $\| \|_\infty$ is a norm on $(\underbrace{F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$. □

Proposition 3.4. *If (V, X) is a normed fuzzy linear space over $(F, \mathbb{K})$, then $\mu_F(\|x\|^2) \geq \mu_F(\|x\|)$ for all $x \in X$.*

Proof. For all $k_1, k_2 \in \mathbb{K}, \mu_F(k_1k_2) \geq \min\{\mu_F(k_1), \mu_F(k_2)\}$ and so for all $k \in \mathbb{K}, \mu_F(k^2) = \mu_F(kk) \geq \mu_F(k)$. Hence $\mu_F(\|x\|^2) \geq \mu_F(\|x\|)$ for all $x \in X$. □

Proposition 3.5. *Let $(V_j, X_j, \| \|_j)$ be normed fuzzy linear spaces over the fuzzy field $(F, \mathbb{K})$ for $j = 1, 2, \ldots,$ n. The norms $\| \|$ defined by $\|x\| = \|x_1\|_1 + \|x_2\|_2 + \cdots + \|x_n\|_n$ and $\| \|_\infty$ defined by $\|x\|_\infty = \max\{\|x_1\|_1, \|x_2\|_2, \ldots, \|x_n\|_n\}$ $(x = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n)$ are norms on the fuzzy linear space $(V_1 \times V_2 \times \cdots \times V_n, X_1 \times X_2 \times \cdots \times X_n)$.*

Proof. (i)

$$\begin{aligned} \mu_F(\|x\|) &= \mu_F(\|x_1\|_1 + \|x_2\|_2 + \cdots + \|x_n\|_n) \\ &\geq \min\{\mu_F(\|x_1\|_1), \mu_F(\|x_2\|_2), \ldots, \mu_F(\|x_n\|_n)\} \\ &\geq \min\left\{\mu_{V_1}(x_1), \mu_{V_2}(x_2), \ldots, \mu_{V_n}(x_n)\right\} \\ &= \mu_{V_1\times V_2\times\cdots\times V_n}(x). \end{aligned}$$

Therefore $\| \|$ is a norm on $(V_1 \times V_2 \times \cdots \times V_n, X_1 \times X_2 \times \cdots \times X_n)$.

(ii)

$$\begin{aligned} \mu_F(\|x\|_\infty) &= \mu_F(\max\{\|x_1\|_1, \|x_2\|_2, \ldots, \|x_n\|_n\}) \\ &= \mu_F(\|x_r\|_r) \\ &\geq \mu_{V_r}(x_r) \\ &\geq \min\left\{\mu_{V_1}(x_1), \mu_{V_2}(x_2), \ldots, \mu_{V_n}(x_n)\right\} \\ &= \mu_{V_1\times V_2\times\cdots\times V_n}(x). \end{aligned}$$

Therefore $\| \|_\infty$ is a norm on $(V_1 \times V_2 \times \cdots \times V_n, X_1 \times X_2 \times \cdots \times X_n)$. □

Lemma 3.6. *Let (V, X) be a fuzzy linear space over a fuzzy field $(F, \mathbb{K})$, Y be a linear space over $\mathbb{K}$ and $T : X \to Y$ be an injective linear transformation. Then $\mu_{T(V)}(T(x)) = \mu_V(x)$ for all $x \in X$.*

Proof.

$$\begin{aligned}
\mu_{T(V)}(T(x)) &= \sup\{\mu_V(z) : z \in X, T(z) = T(x)\} \\
&= \sup\{\mu_V(z) : z \in X, z = x\} \\
&= \mu_V(x).
\end{aligned}$$

□

Proposition 3.7. *Let (V, X) be a fuzzy linear space over a fuzzy field $(F, \mathbb{K})$, Y be a linear space over $\mathbb{K}$ and T be an isomorphism of X onto Y. (V, X) is a normed fuzzy linear space over $(F, \mathbb{K})$ if and only if $(T(V), Y)$ is a normed fuzzy linear space over $(F, \mathbb{K})$.*

Proof. $(\Rightarrow)$ Let $\| \ \|_X$ be a norm on (V, X). Consider the norm $\| \ \|_Y$ on Y defined by $\|y\|_Y = \|x\|_X$, where $y = Tx$.
$\mu_F(\|y\|_Y) = \mu_F(\|x\|_X) \geq \mu_V(x) = \mu_{T(V)}(T(x)) = \mu_{T(V)}(y)$. Therefore $\| \ \|_Y$ is a norm on $(T(V), Y)$.

$(\Leftarrow)$ Assume that $\| \ \|_Y$ is a norm on $(T(V), Y)$. Consider the norm $\| \ \|_X$ on X defined by $\|x\|_X = \|Tx\|_Y$.
$\mu_F(\|x\|_X) = \mu_F(\|Tx\|_Y) \geq \mu_{T(V)}(T(x)) = \mu_V(x)$. Therefore $\| \ \|_X$ is a norm on $(V, X))$. □

Proposition 3.8. *Let X be a linear space over $\mathbb{K}$, (W, Y) a fuzzy linear space over a fuzzy field $(F, \mathbb{K})$ and $T : X \to Y$ be an injective linear transformation. If (W, Y) is a normed fuzzy linear space over $(F, \mathbb{K})$, then $\big(T^{-1}(W), X\big)$ is a normed fuzzy linear space over $(F, \mathbb{K})$.*

Proof. Assume that $\| \ \|_Y$ is a norm on (W, Y). Consider the norm $\| \ \|_X$ on X given by $\|x\|_X = \|Tx\|_Y$.
$\mu_F(\|x\|_X) = \mu_F(\|Tx\|_Y) \geq \mu_W(Tx) = \mu_{T^{-1}(W)}(x)$.

So $\| \ \|_X$ is a norm on $\big(T^{-1}(W), X\big)$. □

Proposition 3.9. *Let (V, X) be a normed fuzzy linear space over a fuzzy field $(F, \mathbb{K})$ and $T : X \to X$ be an injective linear transformation. Then $\big(T^{-1}(V), X\big)$ is a normed fuzzy linear space over $(F, \mathbb{K})$.*

4. **Inner Product on Fuzzy Linear Spaces**

Definition 4.1. An inner product on a fuzzy linear space (V, X) over a fuzzy field $(F, \mathbb{K})$ is a function $\langle \ , \ \rangle : X \times X \to \mathbb{K}$ such that for all $x, y, z \in X$ and $k \in \mathbb{K}$,

(i) $\mu_F(\langle x, y\rangle) \geq \mu_{V\times V}(x, y)$
(ii) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$
(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle kx, y\rangle = k\langle x, y\rangle$
(iv) $\langle y, x\rangle = \overline{\langle x, y\rangle}$.

Example 4.2. Let F be a fuzzy field of $\mathbb{R}$. The inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ defined by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is an inner product on a fuzzy linear space $(\underbrace{F \times F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$.

Proof. Let $V = (\underbrace{F \times F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$.

$$\begin{aligned}
\mu_F(\langle x, y\rangle) &= \mu_F(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \\
&\geq \min\{\mu_F(x_1y_1), \mu_F(x_2y_2), \ldots, \mu_F(x_ny_n)\} \\
&\geq \min\{\min\{\mu_F(x_1), \mu_F(y_1)\}, \ldots, \min\{\mu_F(x_n), \mu_F(y_n)\}\} \\
&= \min\{\min\{\mu_F(x_1), \ldots, \mu_F(x_n)\}, \min\{\mu_F(y_1), \ldots, \mu_F(y_n)\}\} \\
&= \min\{\mu_V(x), \mu_V(y)\} \\
&= \mu_{V\times V}(x, y).
\end{aligned}$$

So $\langle\ ,\ \rangle$ is an inner product on $(\underbrace{F \times F \times \cdots \times F}_{n\ times}, \mathbb{R}^n)$. □

Proposition 4.3. *If $\langle\ ,\ \rangle$ is an inner product on the fuzzy linear space (V, X) over the fuzzy field $(F, \mathbb{K})$, then for all $x, y, z \in X$ and $k \in \mathbb{K}$,*

(i) $\mu_F(\langle x + y, z\rangle) \geq \mu_{V\times V}(x + y, z)$
(ii) $\mu_F\left(\overline{\langle x, y\rangle}\right) \geq \mu_{V\times V}(y, x)$
(iiii) $\mu_F(\lambda\langle x, y\rangle) \geq \mu_{V\times V}(\lambda x, y)$.

Proposition 4.4. *If $\langle\ ,\ \rangle$ is an inner product on the fuzzy linear space (V, X) over the fuzzy field $(F, \mathbb{K})$, then*

(i) $\mu_F(\langle x + y, z\rangle) \geq \min\{\mu_V(x), \mu_V(y), \mu_V(z)\}$
(ii) $\mu_F(\langle kx, y\rangle) \geq \min\{\mu_F(k), \mu_V(x), \mu_V(y)\}$.

Proof. (i)

$$\begin{aligned}
\mu_F(\langle x + y, z\rangle) &\geq \mu_{V\times V}(x + y, z) \\
&= \min\{\mu_V(x + y), \mu_V(z)\} \\
&\geq \min\{\min\{\mu_V(x), \mu_V(y)\}, \mu_V(z)\} \\
&= \min\{\mu_V(x), \mu_V(y), \mu_V(z)\}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\mu_F(\langle kx, y\rangle) &\geq \mu_{V\times V}(kx, y) \\
&= \min\{\mu_V(kx), \mu_V(y)\} \\
&\geq \min\{\min\{\mu_F(k), \mu_V(x)\}, \mu_V(y)\} \\
&= \min\{\mu_F(k), \mu_V(x), \mu_V(y)\}.
\end{aligned}$$

□

Proposition 4.5. *Let (V, X) be a fuzzy linear space over a fuzzy field $(F, \mathbb{K})$, Y a linear space over $\mathbb{K}$ and T an isomorphism of X onto Y. Then there exists an inner product on (V, X) if and only if there exists an inner product on $(T(V), Y)$.*

Proof. ($\Rightarrow$) Let $\langle\ ,\ \rangle_X$ be an inner product on (V, X). Consider the inner product $\langle\ ,\ \rangle_Y$ on Y defined by $\langle y_1, y_2\rangle_Y = \langle x_1, x_2\rangle_X$ where $y_1 = Tx_1$ and $y_2 = Tx_2$.
$\mu_F(\langle y_1, y_2\rangle_Y) = \mu_F(\langle x_1, x_2\rangle_X) \geq \mu_{V\times V}(x_1, x_2) = \mu_{T(V)\times T(V)}(Tx_1, Tx_2) = \mu_{T(V)\times T(V)}(y_1, y_2)$. So $\langle\ ,\ \rangle_Y$ is an inner product on $(T(V), Y)$.

($\Leftarrow$) Assume that $\langle\ ,\ \rangle_Y$ is an inner product on $(T(V), Y)$. Consider the inner product $\langle\ ,\ \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.
$\mu_F(\langle x_1, x_2\rangle_X) = \mu_F(\langle Tx_1, Tx_2\rangle_Y) \geq \mu_{T(V)\times T(V)}(Tx_1, Tx_2) = \mu_{V\times V}(x_1, x_2)$.
So $\langle\ ,\ \rangle_X$ is an inner product on (V, X). □

Proposition 4.6. *Let X be a linear space over $\mathbb{K}$, (W, Y) be a fuzzy linear space over a fuzzy field $(F, \mathbb{K})$ and $T : X \to Y$ be an injective linear transformation. If there exists an inner product on (W, Y), then there exists an inner product on $\big(T^{-1}(W), X\big)$.*

Proof. Let $\langle\ ,\ \rangle_Y$ be an inner product on $(W,\ Y)$. Consider the inner product $\langle\ ,\ \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.
$\mu_F(\langle x_1, x_2\rangle_X) = \mu_F(\langle Tx_1, Tx_2\rangle_Y) \geq \mu_{W\times W}(Tx_1, Tx_2) = \min\{\mu_W(Tx_1), \mu_W(Tx_2)\}$
$= \min\{\mu_{T^{-1}(W)}(x_1), \mu_{T^{-1}(W)}(x_2)\} = \mu_{T^{-1}(W)\times T^{-1}(W)}(x_1, x_2)$.
Therefore$\langle\ ,\ \rangle_X$ is an inner product on $\big(T^{-1}(W), X\big)$. □

Proposition 4.7. *Let (V, X) be a fuzzy linear space over $(F, \mathbb{K})$ and $T : X \to X$ be an injective linear transformation. If there exists an inner product on (V, X), then there exists an inner product on $\big(T^{-1}(V), X\big)$.*

5. **Relationship Between Norm and Inner Product on Fuzzy Linear Spaces**

Theorem 5.1. *Let (V, X) be a fuzzy linear space over $(F, \mathbb{K})$. A norm on (V, X) satisfying the parallelogram law induces an inner product on (V, X) if $\mu_F(4), \mu_F(i) \geq \mu_V(x)$ for all $x \in X$.*

Proof. If $\|\ \|$ is a norm on (V, X) satisfying the parallelogram law, then $\mu_F(\|x\|) \geq \mu_V(x)$ for all $x \in X$ and $\|\ \|$ induces the inner product $\langle\ ,\ \rangle$ on X given by

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

$$\begin{aligned}
\mu_F(\langle x, y\rangle) &= \\
&= \mu_F\left(\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)\right) \\
&\geq \min\left\{\mu_F(1/4), \mu_F(\|x+y\|^2), \mu_F(-\|x-y\|^2), \mu_F(i), \mu_F(\|x+iy\|^2), \mu_F(-\|x-iy\|^2)\right\} \\
&= \min\left\{\mu_F(4), \mu_F(i), \mu_F(\|x+y\|^2), \mu_F(\|x-y\|^2), \mu_F(\|x+iy\|^2), \mu_F(\|x-iy\|^2)\right\} \\
&\geq \min\{\mu_F(4), \mu_F(i), \mu_F(\|x+y\|), \mu_F(\|x-y\|), \mu_F(\|x+iy\|), \mu_F(\|x-iy\|)\} \\
&\geq \min\{\mu_F(4), \mu_F(i), \mu_V(x+y), \mu_V(x-y), \mu_V(x+iy), \mu_V(x-iy)\} \\
&\geq \min\{\mu_F(4), \mu_F(i), \mu_V(x), \mu_V(y)\} \\
&= \min\{\mu_V(x), \mu_V(y)\} \text{ if } \mu_F(4), \mu_F(i) \geq \mu_V(x) \text{ for all } x \in X \\
&= \mu_{V\times V}(x, y).
\end{aligned}$$

Hence $\|\ \|$ induces an inner product on (V,X) if $\mu_F(4), \mu_F(i) \geq \mu_V(x)$ for all $x \in X$. $\square$

Proposition 5.2. *Let (V,X) be a normed fuzzy linear space over $(F,\mathbb{K})$, Y a linear space over $\mathbb{K}$ and T an isomorphism of X onto Y. Suppose that $\mu_F(4), \mu_F(i) \geq \mu_V(x)$ for all $x \in X$. The norm on (V,X) induces an inner product on (V,X) if and only if the norm on $(T(V),Y)$ induces an inner product on $(T(V),Y)$.*

Proof. If the norm $\|\ \|_X$ on (V,X) induces an inner product on (V,X), then $\|\ \|_X$ satisfies the parallelogram law and by proposition 3.7, $\|\ \|_Y$ defined by $\|y\|_Y = \|x\|_X$, where $y = Tx$, is a norm on $(T(V),Y)$.
If $y_1, y_2 \in Y$ and $y_1 = Tx_1, y_2 = Tx_2$, then
$\|y_1 + y_2\|_Y^2 + \|y_1 - y_2\|_Y^2 = \|x_1 + x_2\|_X^2 + \|x_1 - x_2\|_X^2 = 2\left(\|x_1\|_X^2 + \|x_2\|_X^2\right) = 2\left(\|y_1\|_Y^2 + \|y_2\|_Y^2\right)$. That is $\|\ \|_Y$ satisfies the parallelogram law.
Also $\mu_F(4), \mu_F(i) \geq \mu_V(x) = \mu_{T(V)}(Tx) = \mu_{T(V)}(y)$ for all $y \in Y$. Hence $\|\ \|_Y$ induces an inner product on $(T(V),Y)$.

Similarly the converse holds. $\square$

Note 5.3. An inner product $\langle\ ,\ \rangle$ on a fuzzy linear space (V,X) over $(F,\mathbb{K})$ induces a norm $\|\ \|$ on X given by $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$ and $\|\ \|$ is a norm on (V,X) only if $\mu_F(\|x\|) \geq \mu_V(x)$ for all $x \in X$.

Proposition 5.4. *Let (V,X) be a fuzzy linear space over $(F,\mathbb{K})$ with an inner product on it, Y a linear space over $\mathbb{K}$ and T an isomorphism of X onto Y. The inner product on (V,X) induces a norm on (V,X) if and only if the inner product on $(T(V),Y)$ induces a norm on $(T(V),Y)$.*

Proof. If the inner product $\langle\ ,\ \rangle_X$ on (V,X) induces a norm $\|\ \|_X$ on (V,X), then $\mu_F(\|x\|_X) \geq \mu_V(x)$ for all $x \in X$ and, by proposition 4.5, $\langle\ ,\ \rangle_Y$ defined by $\langle y_1, y_2\rangle_Y = \langle x_1, x_2\rangle_X$ $(y_1 = Tx_1, y_2 = Tx_2)$ is an inner product on $(T(V),Y)$.

$\langle\ ,\ \rangle_Y$ induces the norm $\|\ \|_Y$ on Y given by $\|y\|_Y = \|x\|_X$ (y = Tx).
$\mu_F(\|y\|_Y) = \mu_F(\|x\|_X) \geq \mu_V(x) = \mu_{T(V)}(Tx) = \mu_{T(V)}(y)$ and hence $\|\ \|_Y$ is a norm on $(T(V),Y)$.

Similarly the converse holds. $\square$

Proposition 5.5. *In proposition 3.8, if $\mu_F(4), \mu_F(i) \geq \mu_W(y)$ for all $y \in Y$ and if $\|\ \|_Y$ induces an inner product on (W,Y), then $\|\ \|_X$ induces an inner product on $\left(T^{-1}(W),X\right)$.*

Proof. Since $\|\ \|_Y$ induces an inner product on (W,Y), $\|\ \|_Y$ satisfies the parallelogram law $\|y_1 + y_2\|_Y^2 + \|y_1 - y_2\|_Y^2 = 2\left(\|y_1\|_Y^2 + \|y_2\|_Y^2\right)$. Therefore if $x_1, x_2 \in X$, then $\|x_1 + x_2\|_X^2 + \|x_1 - x_2\|_X^2 = \|Tx_1 + Tx_2\|_Y^2 + \|Tx_1 - Tx_2\|_Y^2 =$
$2\left(\|Tx_1\|_Y^2 + \|Tx_2\|_Y^2\right) = 2\left(\|x_1\|_X^2 + \|x_2\|_X^2\right)$.
That is $\|\ \|_X$ satisfies the parallelogram law.
Also, for all $x \in X$, $\mu_F(4), \mu_F(i) \geq \mu_W(Tx) = \mu_{T^{-1}(W)}(x)$. Hence $\|\ \|_X$ induces an inner product on $\left(T^{-1}(W),X\right)$. $\square$

Proposition 5.6. *In proposition 4.6, if $\langle\ ,\ \rangle_Y$ induces a norm on (W,Y), then $\langle\ ,\ \rangle_X$ induces a norm on $\left(T^{-1}(W),X\right)$.*

Proof. Assume that $\langle\ ,\ \rangle_Y$ induces a norm $\|\ \|_Y$ given by $\|y\|_Y = \langle y, y\rangle_Y^{1/2}$ with $\mu_F(\|y\|_Y) \geq \mu_W(y), y \in Y$.
Consider the norm $\|\ \|_X$ on X induced by $\langle\ ,\ \rangle_X$ given by$\|x\|_X = \langle x, x\rangle_X^{1/2}$.
$\|x\|_X = \langle x, x\rangle_X^{1/2} = \langle Tx, Tx\rangle_Y^{1/2} = \|Tx\|_Y$.
Therefore $\mu_F(\|x\|_X) = \mu_F(\|Tx\|_Y) \geq \mu_W(Tx) = \mu_{T^{-1}(W)}(x)$.
Therefore $\|\ \|_X$ is a norm on $\big(T^{-1}(W), X\big)$. □

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