# VAGUE RINGS AND VAGUE IDEALS

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ABSTRACT. In this paper, various elementary properties of vague rings are obtained. Furthermore, the concepts of vague subring, vague ideal, vague prime ideal and vague maximal ideal are introduced, and the validity of some relevant classical results in these settings are investigated.

## 1. **Introduction**

Fuzzy subgroups were introduced in [11] by Rosenfeld as a natural generalization of the concept of subgroup and have been widely studied. Following this, a new object related to groups called vague groups was introduced and studied in [2] by Demirci by forcing the operations of the group to be compatible with a given fuzzy equality. Although the theory of vague algebraic notions has been established in [3, 4, 5, 6, 8], the concepts of vague subring and vague ideal have not been studied yet. So, this work introduces some elementary properties of vague ring, vague subring, vague ideal, vague prime ideal and vague maximal ideal, and establishes some new results.

After this introductory Section, Section 2 is devoted to some definitions and properties related to vague groups and generalized vague subgroups that will be needed later. In Section 3, the definitions of vague ring and vague subring will be given and some basic properties of these concepts will be studied. In Section 4, the definitions of vague ideal, vague prime ideal and vague maximal ideal will be given, and some basic properties of these concepts will be investigated.

## 2. **Preliminaries**

The notions of fuzzy equality, strong fuzzy function, vague group and generalized vague subgroup and their fundamental properties are introduced in [1, 2, 3, 12, 13, 14]. Our aim in this section is to recall these notions and some of their elementary properties, which will be needed in this paper.

The symbols "$\wedge$" and "$\vee$" will always stand for the minimum and maximum operations between finitely many real numbers, respectively; and $X, Y, G$ will always stand for crisp and nonempty sets in this paper.

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**Definition 2.1.** [1] *A mapping* $E_X : X \times X \to [0,1]$ *is called a fuzzy equality on* $X$ *if the following conditions are satisfied:*
*(E.1)* $E_X(x,y) = 1 \Longleftrightarrow x = y\ ,\ \forall x, y \in X$,
*(E.2)* $E_X(x,y) = E_X(y,x)\ ,\ \forall x, y \in X\ $,
*(E.3)* $E_X(x,y) \wedge E_X(y,z) \leq E_X(x,z)\ ,\ \forall x, y, z \in X$.

For $x, y \in X$, the real number $E_X(x,y)$ shows the degree of the equality of $x$ and $y$. One can always define a fuzzy equality on $X$ with respect to (abbreviated to "w.r.t.") the classical equality of the elements of $X$. Indeed, the mapping $E_X^c : X \times X \to [0,1]$, defined by

$$E_X^c(x,y) = \begin{cases} 1 & , \quad if\ x = y \\ 0 & , \quad otherwise \end{cases}$$

is obviously a fuzzy equality on $X$.

**Definition 2.2.** [3] *Let* $E_X$ *and* $E_Y$ *be two fuzzy equalities on* $X$ *and* $Y$*, respectively. Then a fuzzy relation* $\tilde{\circ}$ *from* $X$ *to* $Y$ *(i.e., a fuzzy subset* $\tilde{\circ}$ *of* $X \times Y$*) is called a strong fuzzy function from* $X$ *to* $Y$ *w.r.t. the fuzzy equalities* $E_X$ *and* $E_Y$*, denoted by* $\tilde{\circ} : X \rightsquigarrow Y$*, if the characteristic function* $\mu_{\tilde{\circ}} : X \times Y \to [0,1]$ *of* $\tilde{\circ}$ *satisfies the following two conditions:*
*(F.1) For each* $x \in X$*, there exists* $y \in Y$ *such that* $\mu_{\tilde{\circ}}(x,y) = 1$,
*(F.2) For each* $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$\mu_{\tilde{\circ}}(x_1,y_1) \wedge \mu_{\tilde{\circ}}(x_2,y_2) \wedge E_X(x_1,x_2) \leq E_Y(y_1,y_2).$$

The concepts of vague binary operation on $X$ and transitivity of a vague binary operation are defined as follows.

**Definition 2.3.** [2, 3]

(i) *A strong fuzzy function* $\tilde{\circ} : X \times X \rightsquigarrow X$ *w.r.t. a fuzzy equality* $E_{X\times X}$ *on* $X \times X$ *and a fuzzy equality* $E_X$ *on* $X$ *is called a vague binary operation on* $X$ *w.r.t.* $E_{X\times X}$ *and* $E_X$*. (For all* $(x_1,x_2) \in X \times X$, $x_3 \in X$, $\mu_{\tilde{\circ}}((x_1,x_2),x_3)$ *will be denoted by* $\mu_{\tilde{\circ}}(x_1,x_2,x_3)$ *for the sake of simplicity.)*
(ii) *A vague binary operation* $\tilde{\circ}$ *on* $X$ *w.r.t.* $E_{X\times X}$ *and* $E_X$ *is said to be transitive of the first order if* $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(c,d) \leq \mu_{\tilde{\circ}}(a,b,d)$ *for all* $a, b, c, d \in X$.
(iii) *A vague binary operation* $\tilde{\circ}$ *on* $X$ *w.r.t.* $E_{X\times X}$ *and* $E_X$ *is said to be transitive of the second order if* $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(b,d) \leq \mu_{\tilde{\circ}}(a,d,c)$ *for all* $a, b, c, d \in X$.
(iv) *A vague binary operation* $\tilde{\circ}$ *on* $X$ *w.r.t.* $E_{X\times X}$ *and* $E_X$ *is said to be transitive of the third order if* $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(a,d) \leq \mu_{\tilde{\circ}}(d,b,c)$ *for all* $a, b, c, d \in X$.

**Definition 2.4.** [2] *Let* $\tilde{\circ}$ *be a vague binary operation on* $G$ *w.r.t. a fuzzy equality* $E_{G\times G}$ *on* $G \times G$ *and a fuzzy equality* $E_G$ *on* $G$*. Then*

(i) $G$ *together with* $\tilde{\circ}$*, denoted by* $< G, \tilde{\circ}, E_{G\times G}, E_G >$ *or simply* $< G, \tilde{\circ} >$*, is called a vague semigroup if the characteristic function* $\mu_{\tilde{\circ}} : G \times G \times G \to [0,1]$

*of* $\tilde{\circ}$ *fulfills the condition: For all* $a, b, c, d, m, q, w \in G$,

$$\mu_{\tilde{\circ}}(b, c, d) \wedge \mu_{\tilde{\circ}}(a, d, m) \wedge \mu_{\tilde{\circ}}(a, b, q) \wedge \mu_{\tilde{\circ}}(q, c, w) \leq E_G(m, w).$$

(ii) *A vague semigroup* $< G, \tilde{\circ} >$ *is called a vague monoid if there exists a two-sided identity element* $e_{\tilde{\circ}} \in G$*, that is an element* $e_{\tilde{\circ}}$ *satisfying* $\mu_{\tilde{\circ}}(e_{\tilde{\circ}}, a, a) \wedge \mu_{\tilde{\circ}}(a, e_{\tilde{\circ}}, a) = 1$ *for each* $a \in G$.

(iii) *A vague monoid* $< G, \tilde{\circ} >$ *is called a vague group if for each* $a \in G$*, there exists a two-sided inverse element* $a^{-1} \in G$*, that is an element* $a^{-1}$ *satisfying* $\mu_{\tilde{\circ}}(a^{-1}, a, e_{\tilde{\circ}}) \wedge \mu_{\tilde{\circ}}(a, a^{-1}, e_{\tilde{\circ}}) = 1$.

(iv) *A vague semigroup* $< G, \tilde{\circ} >$ *is said to be commutative (Abelian) if*

$$\mu_{\tilde{\circ}}(a, b, m) \wedge \mu_{\tilde{\circ}}(b, a, w) \leq E_G(m, w)$$

*for each* $a, b, m, w \in G$.

In the rest of this paper, the notation $< G, \tilde{\circ} >$ always stands for the vague group $< G, \tilde{\circ} >$ w.r.t. a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality $E_G$ on $G$.

**Proposition 2.5.** [2] *For a given vague group* $< G, \tilde{\circ} >$*, there exists a unique binary operation in the classical sense, denoted by* $\circ$*, on* $G$ *such that* $< G, \circ >$ *is a group in the classical sense.*

The binary operation "$\circ$" in Proposition 2.5 is explicitly given by the equivalence

$$a \circ b = c \iff \mu_{\tilde{\circ}}(a, b, c) = 1, \quad \forall a, b, c \in G. \tag{1}$$

The binary operation "$\circ$", defined by the equivalence (2), is called the ordinary description of $\tilde{\circ}$, and is denoted by $\circ = ord(\tilde{\circ})$ in [3, 5, 6].

If $\tilde{\circ}$ is a vague binary operation on $G$ w.r.t. a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality $E_G$ on $G$, in the rest of this paper the ordinary description of $\tilde{\circ}$ will be denoted by $\circ$. In this case, from [3, 5] we have the following property

$$\mu_{\tilde{\circ}}(a, b, a \circ b) = 1 \quad \text{and} \quad \mu_{\tilde{\circ}}(a, b, c) \leq E_G(a \circ b, c)\ ,\ \forall a, b, c \in G. \tag{2}$$

**Theorem 2.6.** [2] *Let* $< G, \tilde{\circ} >$ *be a vague group.*

(i) *If the vague binary operation* $\tilde{\circ}$ *is transitive of the second order, then* $E_G(a, b) = E_G(a^{-1}, b^{-1})$ *for all* $a, b \in G$.

(ii) $\mu_{\tilde{\circ}}(b^{-1}, a^{-1}, u) \wedge \mu_{\tilde{\circ}}(a, b, v) \leq E_G(u, v^{-1}) \wedge E_G(v, u^{-1})$ *for all* $a, b, u, v \in G$.

For a given fuzzy equality $E_G$ on $G$ and for a crisp subset $A$ of $G$, the restriction of the mapping $E_G$ to $A \times A$, denoted by $E_A$, is obviously a fuzzy equality on $A$.

**Definition 2.7.** [13] *Let* $< G, \tilde{\circ} >$ *be a vague group and* $A$ *be a nonempty, crisp subset of* $G$*. Let* $\tilde{\odot}$ *be a vague binary operation on* $A$ *such that*

$$\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c),\ \forall a, b, c \in A.$$

*If* $< A, \tilde{\odot} >$ *is itself a vague group w.r.t. the fuzzy equalities* $E_{A\times A}$ *on* $A \times A$ *and* $E_A$ *on* $A$*, then* $< A, \tilde{\odot} >$ *is said to be a generalized vague subgroup of* $< G, \tilde{\circ} >$*, denoted by* $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.

For a given vague group $< G, \tilde{\circ} >$, because of the uniqueness of the identity and the inverse of an element of $< G, \tilde{\circ} >$, it can be easily seen that if $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$, then the identity of $< A, \tilde{\odot} >$ and the inverse of $x \in A$ w.r.t. $< A, \tilde{\odot} >$ are the identity of $< G, \tilde{\circ} >$ and the inverse of $x \in A$ w.r.t. $< G, \tilde{\circ} >$, i.e., $e_A = e_G$ and $x_A^{-1} = x_G^{-1}$, respectively.

**Example 2.8.** *Let $\alpha \in [0, 1)$ be a fixed number, and set $x^\bullet = \frac{1}{Max(x,1)} \wedge Min(x, 1)$ for any $x \in \mathbb{R}^+$. For $x, y, u, v, z \in \mathbb{R}^+$, considering the fuzzy equalities*

$$E_{\mathbb{R}^+}(x, y) = \begin{cases} 1 & , \quad if\ x = y \\ \alpha \vee (\frac{1}{x^\bullet} \wedge \frac{1}{y^\bullet}) & , \quad otherwise \end{cases}$$

*on $\mathbb{R}^+$ and*

$$E_{\mathbb{R}^+ \times \mathbb{R}^+}((x, y), (u, v)) = \begin{cases} 1 & , \quad if\ (x, y) = (u, v) \\ \alpha \vee [(x^2)^\bullet \wedge (y^2)^\bullet \wedge (u^2)^\bullet \wedge (v^2)^\bullet] & , \ otherwise \end{cases}$$

*on $\mathbb{R}^+ \times \mathbb{R}^+$. And, for $x, y, u, v, z \in \mathbb{Q}^+$, considering the fuzzy equalities $E_{\mathbb{Q}^+}(x, y) = E_{\mathbb{R}^+}(x, y)$ on $\mathbb{Q}^+$ and*

$$E_{\mathbb{Q}^+ \times \mathbb{Q}^+}((x, y), (u, v)) = E_{\mathbb{R}^+ \times \mathbb{R}^+}((x, y), (u, v))$$

*on $\mathbb{Q}^+ \times \mathbb{Q}^+$. For $n \in \mathbb{N}^+$, we obtain that the fuzzy relations $\tilde{\circ}$ and $\tilde{\odot}_n$ on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and $\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+$, defined by*

$$\mu_{\tilde{\circ}}(x, y, z) = \begin{cases} 1 & , \quad if\ z = x.y \\ \alpha.(x^\bullet \wedge y^\bullet \wedge z^\bullet) & , \quad otherwise \end{cases}$$

*and*

$$\mu_{\tilde{\odot}_n}(x, y, z) = \begin{cases} 1 & , \quad if\ z = x.y \\ \frac{\alpha}{n}.(x^\bullet \wedge y^\bullet \wedge z^\bullet) & , \quad otherwise \end{cases}$$

*are vague binary operations on $\mathbb{R}^+$ and $\mathbb{Q}^+$, respectively; furthermore, $< \mathbb{R}^+, \tilde{\circ} >$ and $< \mathbb{Q}^+, \tilde{\odot}_n >$ are vague groups from* [7, 12]. *Due to the definitions of $\mu_{\tilde{\odot}_n}(x, y, z)$ and $\mu_{\tilde{\circ}}(x, y, z)$, we have $\mu_{\tilde{\odot}_n}(x, y, z) \leq \mu_{\tilde{\circ}}(x, y, z)$ for each $x, y, z \in \mathbb{Q}^+$, i.e., $< \mathbb{Q}^+, \tilde{\odot}_n > \overset{\text{v.s}}{\leq} < \mathbb{R}^+, \tilde{\circ} >$.*

**Proposition 2.9.** [13] *Let $< G, \tilde{\circ} >$ be a vague group. If $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ and $< B, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < A, \tilde{\odot} >$, then $< B, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.*

**Proposition 2.10.** [13] *Let $< G, \tilde{\circ} >$ be a vague group, $\emptyset \neq A \subseteq G$ and let $\tilde{\odot}$ be a vague binary operation on $A$. Then*

$$< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} > \Leftrightarrow \quad \begin{array}{ll} (i) & \text{For each } x \in A\ ,\ x^{-1} \in A, \text{ and} \\ (ii) & \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)\ ,\ \forall a, b, c \in A. \end{array}$$

**Corollary 2.11.** [13] *Let $< G, \tilde{\circ} >$ be a vague group and $\tilde{\bullet}$ be a vague binary operation on $G$ such that $\mu_{\tilde{\bullet}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)$ for all $a, b, c \in G$. Let $e_{\tilde{\circ}}$ be an identity element of $G$. Then, $< \{e_{\tilde{\circ}}\}, \tilde{\circ} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ and $< G, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.*

**Corollary 2.12.** [13] *Let* $< G, \tilde{\circ} >$ *be a vague group, and let* $< A_j, \tilde{\bullet}_j > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ *for all* $j \in J$*. If* $\tilde{\star}$ *is a vague binary operation on* $\bigcap_{j\in J} A_j$ *such that*

$$\mu_{\tilde{\star}}(x,y,z) \leq \bigwedge_{j\in J} \mu_{\tilde{\bullet}_j}(x,y,z),\ \forall x,y,z \in \bigcap_{j\in J} A_j\ ,$$

*then* $< \bigcap_{j\in J} A_j, \tilde{\star} > \overset{\text{v.s}}{\leq} < A_j, \tilde{\bullet}_j >$.

## 3. Vague Rings

In a similar fashion to classical algebra, the notion of vague ring can be given in the following way:

**Definition 3.1.** *Let* $E_{\mathcal{H}\times\mathcal{H}}$ *and* $E_{\mathcal{H}}$ *be fuzzy equalities on* $\mathcal{H}\times\mathcal{H}$ *and* $\mathcal{H}$*, respectively. Let* $\tilde{\circ}$*,* $\tilde{\bullet}$ *be two vague binary operations on* $\mathcal{H}$*. Then, the 3-tuple* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *is called a vague ring w.r.t.* $E_{\mathcal{H}\times\mathcal{H}}$ *and* $E_{\mathcal{H}}$ *if the following three conditions are satisfied:*

(VR.1) $< \mathcal{H}, \tilde{\circ} >$ *is a commutative vague group,*
(VR.2) $< \mathcal{H}, \tilde{\bullet} >$ *is a vague semigroup,*
(VR.3) $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *satisfies distributive laws, i.e.,* $\forall a,b,c,d,t,x,y,z \in \mathcal{H}$*,*

$$\mu_{\tilde{\bullet}}(x,y,a) \wedge \mu_{\tilde{\bullet}}(x,z,b) \wedge \mu_{\tilde{\circ}}(a,b,c) \wedge \mu_{\tilde{\circ}}(y,z,d) \wedge \mu_{\tilde{\bullet}}(x,d,t) \leq E_{\mathcal{H}}(t,c),$$

$$\mu_{\tilde{\bullet}}(x,z,a) \wedge \mu_{\tilde{\bullet}}(y,z,b) \wedge \mu_{\tilde{\circ}}(a,b,c) \wedge \mu_{\tilde{\circ}}(x,y,d) \wedge \mu_{\tilde{\bullet}}(d,z,t) \leq E_{\mathcal{H}}(t,c).$$

(VR.4) *A vague ring* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *is said to be a vague ring with identity if there exists* $e_{\tilde{\bullet}} \in \mathcal{H}$ *such that* $\mu_{\tilde{\bullet}}(x, e_{\tilde{\bullet}}, x) \wedge \mu_{\tilde{\bullet}}(e_{\tilde{\bullet}}, x, x) = 1$ *for each* $x \in \mathcal{H}$*.*
(VR.5) *A vague ring* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *is said to be a commutative (Abelian) if*

$$\mu_{\tilde{\bullet}}(x,y,s) \wedge \mu_{\tilde{\bullet}}(y,x,t) \leq E_{\mathcal{H}}(s,t)\ ,\ \forall x,y,s,t \in \mathcal{H}.$$

In this work, since the particular integral, commutative cqm-lattice is being studied $([0,1], \leq, \wedge)$, Definition 3.1 corresponds to a special case of Definition 7.12 in [3].

In the rest of this paper, the notation $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ always stands for the vague ring $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ w.r.t. $E_{\mathcal{H}\times\mathcal{H}}$ and $E_{\mathcal{H}}$. If $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring, then we denote the inverse of $a$ by $-a$ w.r.t. the vague group $< \mathcal{H}, \tilde{\circ} >$; additionally if $< \mathcal{H}, \tilde{\bullet} >$ is a vague group, then we denote the inverse of $a$ by $a^{-1}$ w.r.t. the vague group $< \mathcal{H}, \tilde{\bullet} >$.

**Example 3.2.** Let $< \mathcal{H}, \circ, \bullet >$ be a ring. For $x,y,a,b \in \mathcal{H}$ and $\alpha, \beta, \gamma, \nu \in \mathbb{R}$ such that $0 \leq \nu \leq \gamma \leq \beta \leq \alpha < 1$, considering the fuzzy equalities

$$E_{\mathcal{H}}(a,b) := \begin{cases} 1 & , \quad a = b \\ \alpha & , \quad otherwise \end{cases}$$

on $\mathcal{H}$ and

$$E_{\mathcal{H}\times\mathcal{H}}((a,b),(x,y)) := \begin{cases} 1 & , \quad (a,b) = (x,y) \\ \beta & , \quad otherwise \end{cases}$$

on $\mathcal{H} \times \mathcal{H}$. And, considering the vague binary operations

$$\tilde{\circ} : \mathcal{H} \times \mathcal{H} \rightsquigarrow \mathcal{H} \ , \ \mu_{\tilde{\circ}}(a,b,c) := \begin{cases} 1 & , \quad a \circ b = c \\ \gamma & , \quad otherwise \end{cases}$$

and

$$\tilde{\bullet} : \mathcal{H} \times \mathcal{H} \rightsquigarrow \mathcal{H} \ , \ \mu_{\tilde{\bullet}}(a,b,c) := \begin{cases} 1 & , \quad a \bullet b = c \\ \nu & , \quad otherwise. \end{cases}$$

In this case, it is clearly seen that $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring from the inequality in (2) and the condition $(E.3)$.

**Proposition 3.3.** [3] *If $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring, then $< \mathcal{H}, \circ, \bullet >$ is a ring.*

From classical algebra, we know that if $< \mathcal{H}, \circ, \bullet >$ is a ring, and $e_\circ$ and $e_\bullet$ are identity elements of $< H, \circ >$ and $< H, \bullet >$, respectively; then the following properties are satisfied for all $x, y \in \mathcal{H}$:

- **(1)** $x \bullet e_\circ = e_\circ \bullet x = e_\circ$
- **(2)** $(-x) \bullet (-y) = x \bullet y$
- **(3)** $x \bullet (-y) = (-x) \bullet y = -(x \bullet y)$
- **(4)** $x \bullet (y - z) = (x \bullet y) - (x \bullet z)$ and $(x - y) \bullet z = (x \bullet z) - (y \bullet z)$
- **(5)** $(-e_\bullet) \bullet x = -x$ and $(-e_\bullet) \bullet (-e_\bullet) = e_\bullet$ .

The following proposition states that these five properties are satisfied for vague rings under appropriate conditions:

**Proposition 3.4.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and $e_{\tilde{\circ}}$ an identity element of the vague group $< \mathcal{H}, \tilde{\circ} >$. Then, the following statements are satisfied for all $m, n, t, u, v, w, x, y, z \in \mathcal{H}$.*

- **(1)** $\mu_{\tilde{\bullet}}(x, e_{\tilde{\circ}}, m) \wedge \mu_{\tilde{\bullet}}(e_{\tilde{\circ}}, x, n) \leq E_{\mathcal{H}}(m, n)$.
- **(2)** $\mu_{\tilde{\bullet}}(-x, -y, m) \wedge \mu_{\tilde{\bullet}}(x, y, n) \leq E_{\mathcal{H}}(m, n)$.
- **(3)** *If the vague binary operation $\tilde{\circ}$ is transitive of the second order, then* $\mu_{\tilde{\bullet}}(x, -y, m) \wedge \mu_{\tilde{\bullet}}(x, y, n) \leq E_{\mathcal{H}}(m, -n)$ *and* $\mu_{\tilde{\bullet}}(-x, y, m) \wedge \mu_{\tilde{\bullet}}(x, y, n) \leq E_{\mathcal{H}}(m, -n)$.
- **(4)** *Let the vague binary operation $\tilde{\circ}$ be transitive of the second and third orders.*
- **(i)** *If the vague binary operation $\tilde{\bullet}$ is transitive of the second order, then*

$$\mu_{\tilde{\circ}}(y, -z, u) \wedge \mu_{\tilde{\bullet}}(x, u, m) \wedge \mu_{\tilde{\bullet}}(x, y, v) \wedge \mu_{\tilde{\bullet}}(x, z, t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \leq E_{\mathcal{H}}(m, n).$$

- **(ii)** *If the vague binary operation $\tilde{\bullet}$ is transitive of the third order, then*

$$\mu_{\tilde{\circ}}(x, -y, u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge \mu_{\tilde{\bullet}}(x, z, v) \wedge \mu_{\tilde{\bullet}}(y, z, t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \leq E_{\mathcal{H}}(m, n).$$

- **(5)** *If $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring with identity $e_{\tilde{\bullet}}$, then $\mu_{\tilde{\bullet}}(-e_{\tilde{\bullet}}, x, -x) = 1 = \mu_{\tilde{\bullet}}(-e_{\tilde{\bullet}}, -e_{\tilde{\bullet}}, e_{\tilde{\bullet}})$.*

*Proof.* **(1):** Since $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring, $< \mathcal{H}, \circ, \bullet >$ is a ring from Proposition 3.3. Thus, $e_{\tilde{\circ}} \bullet x = x \bullet e_{\tilde{\circ}} = e_{\tilde{\circ}}$ for all $x \in \mathcal{H}$. So, using the inequality in (2) and the condition $(E.3)$, we can write

$$\begin{aligned} \mu_{\tilde{\bullet}}(x, e_{\tilde{\circ}}, m) \wedge \mu_{\tilde{\bullet}}(e_{\tilde{\circ}}, x, n) &\leq E_{\mathcal{H}}(x \bullet e_{\tilde{\circ}}, m) \wedge E_{\mathcal{H}}(e_{\tilde{\circ}} \bullet x, n) \\ &\leq E_{\mathcal{H}}(e_{\tilde{\circ}}, m) \wedge E_{\mathcal{H}}(e_{\tilde{\circ}}, n) \\ &\leq E_{\mathcal{H}}(m, n) \ , \ \forall m, n \in \mathcal{H}. \end{aligned}$$

**(2):** By using the inequality in (2) and the condition $(E.3)$, we get the following inequalities

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(-x,-y,m) \wedge \mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}((-x) \bullet (-y), m) \wedge E_{\mathcal{H}}(x \bullet y, n) \\ & = & E_{\mathcal{H}}(x \bullet y, m) \wedge E_{\mathcal{H}}(x \bullet y, n) \\ & \leq & E_{\mathcal{H}}(m,n) \ , \ \forall x,y,m,n \in \mathcal{H}. \end{array}$$

**(3):** We suppose that the vague binary operation $\tilde{\circ}$ is transitive of the second order. In this case, we have $E_{\mathcal{H}}(x \bullet y, n) = E_{\mathcal{H}}(-(x \bullet y), -n)$ from Theorem 2.6, so we can write the following inequalities from the inequality in (2) and the condition $(E.3)$,

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(x,-y,m) \wedge \mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}(x \bullet (-y), m) \wedge E_{\mathcal{H}}(x \bullet y, n) \\ & = & E_{\mathcal{H}}(-(x \bullet y), m) \wedge E_{\mathcal{H}}(-(x \bullet y), -n) \\ & \leq & E_{\mathcal{H}}(m,-n) \end{array}$$

and

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(-x,y,m) \wedge \mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}((-x) \bullet y, m) \wedge E_{\mathcal{H}}(x \bullet y, n) \\ & = & E_{\mathcal{H}}(-(x \bullet y), m) \wedge E_{\mathcal{H}}(-(x \bullet y), -n) \\ & \leq & E_{\mathcal{H}}(m,-n). \end{array}$$

**(4): (i)** From classical algebra, we know that $E_{\mathcal{H}}(x \bullet (y \circ (-z)), m) = E_{\mathcal{H}}((x \bullet y) \circ (-(x \bullet z)), m)$ for all $x,y,z,m \in \mathcal{H}$. Since the vague binary operation $\tilde{\circ}$ is transitive of the second order, by making use of Theorem 2.6, we get $E_{\mathcal{H}}(x \bullet z, t) = E_{\mathcal{H}}(-(x \bullet z), -t)$ for all $x,z,t \in \mathcal{H}$.

If we denote

$$\alpha = \mu_{\tilde{\circ}}(y,-z,u) \wedge \mu_{\tilde{\bullet}}(x,u,m) \wedge \mu_{\tilde{\bullet}}(x,y,v) \wedge \mu_{\tilde{\bullet}}(x,z,t) \wedge \mu_{\tilde{\circ}}(v,-t,n),$$

then we get the following inequalities by using the vague binary operation $\tilde{\circ}$ is transitive of the second and third orders, the vague binary operation $\tilde{\bullet}$ is transitive of the second order, the inequality in (2) and the condition $(E.3)$.

$$\begin{array}{rl} \alpha & \leq \ E_{\mathcal{H}}(y \circ (-z), u) \wedge \mu_{\tilde{\bullet}}(x,u,m) \wedge E_{\mathcal{H}}(x \bullet y, v) \wedge E_{\mathcal{H}}(x \bullet z, t) \wedge \mu_{\tilde{\circ}}(v,-t,n) \\ & \leq \ \mu_{\tilde{\bullet}}(x, y \circ (-z), m) \wedge E_{\mathcal{H}}(x \bullet y, v) \wedge E_{\mathcal{H}}(x \bullet z, t) \wedge \mu_{\tilde{\circ}}(v,-t,n) \\ & \leq \ \mu_{\tilde{\bullet}}(x, y \circ (-z), m) \wedge E_{\mathcal{H}}(x \bullet y, v) \wedge E_{\mathcal{H}}(-(x \bullet z), -t) \wedge \mu_{\tilde{\circ}}(v,-t,n) \\ & \leq \ E_{\mathcal{H}}(x \bullet (y \circ (-z)), m) \wedge E_{\mathcal{H}}(x \bullet y, v) \wedge \mu_{\tilde{\circ}}(v, -(x \bullet z), n) \\ & \leq \ E_{\mathcal{H}}(x \bullet (y \circ (-z)), m) \wedge \mu_{\tilde{\circ}}(x \bullet y, -(x \bullet z), n) \\ & \leq \ E_{\mathcal{H}}(x \bullet (y \circ (-z)), m) \wedge E_{\mathcal{H}}((x \bullet y) \circ (-(x \bullet z)), n) \\ & \leq \ E_{\mathcal{H}}(m,n). \end{array}$$

**(ii)** Since the vague binary operation $\tilde{\circ}$ is transitive of the second order, we have $E_{\mathcal{H}}(y \bullet z, t) = E_{\mathcal{H}}(-(y \bullet z), -t)$ for all $x,y,z,t \in \mathcal{H}$ from Theorem 2.6. Furthermore, because $< \mathcal{H}, \circ, \bullet >$ is a ring, we can write that $E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) = E_{\mathcal{H}}((x \bullet z) \circ (-(y \bullet z)), m)$ for all $x,y,z,m \in \mathcal{H}$. If we denote

$$\beta = \mu_{\tilde{\circ}}(x,-y,u) \wedge \mu_{\tilde{\bullet}}(u,z,m) \wedge \mu_{\tilde{\bullet}}(x,z,v) \wedge \mu_{\tilde{\bullet}}(y,z,t) \wedge \mu_{\tilde{\circ}}(v,-t,n),$$

then we can write the following inequalities from the hypothesis, the inequality in (2) and the condition $(E.3)$.

$$\begin{array}{rl} \beta & \leq E_{\mathcal{H}}(x \circ (-y), u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(y \bullet z, t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\ & \leq E_{\mathcal{H}}(x \circ (-y), u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(-(y \bullet z), -t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\ & \leq \mu_{\tilde{\bullet}}((x \circ (-y)), z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(-(y \bullet z), -t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\ & \leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge \mu_{\tilde{\circ}}(v, -(y \bullet z), n) \\ & \leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge \mu_{\tilde{\circ}}(x \bullet z, -(y \bullet z), n) \\ & \leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge E_{\mathcal{H}}((x \bullet z) \circ (-(y \bullet z)), n) \\ & \leq E_{\mathcal{H}}(m, n). \end{array}$$

**(5):** It is clear that, if $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ is a vague ring with identity $e_{\tilde{\bullet}}$, then $< \mathcal{H}, \circ, \bullet >$ is a ring with identity $e_{\tilde{\bullet}}$. Thus, we get $-e_{\tilde{\bullet}} \bullet x = -x$ for all $x \in \mathcal{H}$, i.e., $\mu_{\tilde{\bullet}}(-e_{\tilde{\bullet}}, x, -x) = 1$. The result then follows immediately. $\square$

Now, we can define the concept of vague subring which corresponds to the concept of subring in classical algebra as follows:

**Definition 3.5.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and $A$ be a nonempty, crisp subset of $\mathcal{H}$. Let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on $A$ such that*

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \ , \quad \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \quad \forall a, b, c \in A.$$

*If $< A, \tilde{\oplus}, \tilde{\odot} >$ is itself a vague ring w.r.t. $E_{A \times A}$ and $E_A$, then $< A, \tilde{\oplus}, \tilde{\odot} >$ is said to be a vague subring of $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$, denoted by $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$.*

The following propositions and corollaries state that some results of classical algebra are also valid for vague algebra.

**Proposition 3.6.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and $A \subseteq \mathcal{H}$. Let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on $A$. Then the following equivalence is satisfied:*

$$< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} > \ \Leftrightarrow \ \begin{array}{ll} (i) & < A, \tilde{\oplus} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} > \\ (ii) & \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \forall a, b, c \in A. \end{array}$$

*Proof.* $(\Rightarrow)$: Obvious from Definition 3.5.

$(\Leftarrow)$**:** By making use of $(i)$ and $(ii)$, we can write

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ and } \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \ \forall a, b, c \in A.$$

Therefore, it is sufficient to show that $< A, \tilde{\oplus}, \tilde{\odot} >$ is a vague ring. The conditions $(VR.1)$ and $(VR.2)$ are satisfied for $< A, \tilde{\oplus}, \tilde{\odot} >$ under the assumptions $(i)$ and $(ii)$. On the other hand, since $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ satisfies distributive laws, the condition $(VR.3)$ is also obtained for $< A, \tilde{\oplus}, \tilde{\odot} >$. Hence, $< A, \tilde{\oplus}, \tilde{\odot} >$ must be a vague ring, i.e., $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. $\square$

The following corollary explains that the intersection of vague subrings is also a vague subring.

**Corollary 3.7.** *Let* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *be a vague ring and* $<A_j,\tilde{\oplus}_j,\tilde{\odot}_j> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *for all* $j \in J = \{1,2,...,n\}$. *Let* $A=\bigcap_{j\in J} A_j$, *and let* $\tilde{\oplus}$ , $\tilde{\odot}$ *be two vague binary operations on* $A$ *such that*

$$\mu_{\tilde{\oplus}}(a,b,c) \leq \bigwedge_{j\in J} \mu_{\tilde{\oplus}_j}(a,b,c) \quad \text{and} \quad \mu_{\tilde{\odot}}(a,b,c) \leq \bigwedge_{j\in J} \mu_{\tilde{\odot}_j}(a,b,c) \quad \forall a,b,c \in A.$$

*Then,* $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$.

*Proof.* Because of $<A_j,\tilde{\oplus}_j,\tilde{\odot}_j> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$, we have $<A_j,\tilde{\oplus}_j> \overset{\text{v.s}}{\leq} <\mathcal{H},\tilde{\circ}>$ for all $j\in J$. Thus, it is clearly seen that, $<A,\tilde{\oplus}> \overset{\text{v.s}}{\leq} <\mathcal{H},\tilde{\circ}>$ from Corollary 2.12 and Proposition 2.9. On the other hand, since $\mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c)$ for all $a,b,c\in A$, we obtain $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ from Proposition 3.6. $\square$

**Corollary 3.8.** *Let* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *be a vague ring and* $e_{\tilde{\circ}}$ *be an identity element of* $<\mathcal{H},\tilde{\circ}>$. *Let* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *be two vague binary operations on* $\mathcal{H}$ *such that* $\mu_{\tilde{\oplus}}(x,y,z) \leq \mu_{\tilde{\circ}}(x,y,z)$, $\mu_{\tilde{\odot}}(x,y,z) \leq \mu_{\tilde{\bullet}}(x,y,z)$ *for all* $x,y,z\in\mathcal{H}$. *Then the following properties are satisfied:*

*(a)* $<\{e_{\tilde{\circ}}\},\tilde{\circ},\tilde{\bullet}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *(b)* $<\mathcal{H},\tilde{\oplus},\tilde{\odot}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$

*Proof.* We know that $<\{e_{\tilde{\circ}}\},\tilde{\circ}> \overset{\text{v.s}}{\leq} <\mathcal{H},\tilde{\circ}>$ and $<\mathcal{H},\tilde{\oplus}> \overset{\text{v.s}}{\leq} <\mathcal{H},\tilde{\circ}>$ from Corollary 2.11. Using the inequalities in hypothesis and Proposion 3.6, we have $<\{e_{\tilde{\circ}}\},\tilde{\circ},\tilde{\bullet}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ and $<\mathcal{H},\tilde{\oplus},\tilde{\odot}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$. This completes the proof. $\square$

## 4. **Vague Ideals**

In this section we will define the concept of vague ideal, which is one of the basic concepts of this work, and we will obtain some fundamental properties of this concept.

**Definition 4.1.** *Let* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *be a vague ring and* $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.r}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$. *If for all* $a\in A$ *and for all* $h,t,s\in\mathcal{H}$

$$\mu_{\tilde{\bullet}}(a,h,t)=1 \Longrightarrow t\in A \quad \text{and} \quad \mu_{\tilde{\bullet}}(h,a,s)=1 \Longrightarrow s\in A,$$

*then* $<A,\tilde{\oplus},\tilde{\odot}>$ *is said to be a vague ideal of* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$, *it is denoted by* $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.i}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$.

It is clear from Definition 4.1 that if $E_{\mathcal{H}}=E^c_{\mathcal{H}}$, $E_{\mathcal{H}\times\mathcal{H}}=E^c_{\mathcal{H}\times\mathcal{H}}$, $\mu_{\tilde{\circ}}(\mathcal{H}\times\mathcal{H}\times\mathcal{H})\in\{0,1\}$ and $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.i}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$, then $<A,\oplus,\odot>$ is an ideal of $<\mathcal{H},\circ,\bullet>$. Therefore, in this case, a vague ideal $<A,\tilde{\oplus},\tilde{\odot}>$ of $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is nothing but an ideal of the classical ring $<\mathcal{H},\circ,\bullet>$ in the classical sense.

**Proposition 4.2.** *Let* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *be a vague ring. If* $<A,\tilde{\oplus},\tilde{\odot}> \overset{v.i}{\leq} <\mathcal{H},\tilde{\circ},\tilde{\bullet}>$, *then* $<A,\oplus,\odot>$ *is an ideal of* $<\mathcal{H},\circ,\bullet>$.

*Proof.* Assume that $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. Then, due to Definition 4.1 and Proposition 3.3, we get $< A, \oplus, \odot >$ is a (classical) subring of $< \mathcal{H}, \circ, \bullet >$. On the other hand, since

$$\mu_{\tilde{\circ}}(h, a, h \circ a) = 1 = \mu_{\tilde{\circ}}(a, h, a \circ h), \forall a \in A \ , \ \forall h \in \mathcal{H},$$

we can write $a \circ h$, $h \circ a \in A$ from Definition 4.1. Hence, we obtain $< A, \oplus, \odot >$ is an ideal of $< \mathcal{H}, \circ, \bullet >$. This completes the proof. □

**Proposition 4.3.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring. Then*

$$< \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} > \ \text{ and } \ < \mathcal{H}, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >,$$

*where $\tilde{\oplus}$ and $\tilde{\odot}$ are vague binary operations on $\mathcal{H}$ such that $\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)$ and $\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c)$ for all $a, b, c \in \mathcal{H}$.*

*Proof.* From Corollary 3.8, we can write $< \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. Additionally, for all $h, s, t \in \mathcal{H}$ we get $\mu_{\tilde{\bullet}}(e_{\tilde{\circ}}, h, s) = 1 = \mu_{\tilde{\bullet}}(h, e_{\tilde{\circ}}, t)$ implies $s = e_{\tilde{\circ}} \bullet h = e_{\tilde{\circ}} \in \{e_{\tilde{\circ}}\}$ and $t = h \bullet e_{\tilde{\circ}} = e_{\tilde{\circ}} \in \{e_{\tilde{\circ}}\}$ from classical algebra, i.e., $< \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$.

The same statement for $< \mathcal{H}, \tilde{\oplus}, \tilde{\odot} >$ can be obtained in a similar way. □

**Proposition 4.4.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring, $A \subseteq \mathcal{H}$, let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on $A$. Thus, the following equivalence is satisfied:*

$$< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} > \quad \Leftrightarrow \quad \begin{array}{l} (i) \ < A, \tilde{\oplus} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} > \\ (ii) \ \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \ \forall a, b, c \in A \\ (iii) \ \mu_{\tilde{\bullet}}(a, h, t) = 1 \Rightarrow t \in A, \ \forall a \in A, \forall h, t \in \mathcal{H} \\ (iv) \ \mu_{\tilde{\bullet}}(h, a, s) = 1 \Rightarrow s \in A, \ \forall a \in A, \forall h, s \in \mathcal{H}. \end{array}$$

*Proof.* ($\Rightarrow$): Obvious from Definition 4.1.

($\Leftarrow$): We have $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ from Proposition 3.6 under the assumptions $(i)$ and $(ii)$. Then, using this fact, and utilizing the conditions $(iii)$-$(iv)$, we can easily observe that $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. □

The following proposition explains that the intersection of vague ideals is also a vague ideal.

**Proposition 4.5.** *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and $< A_j, \tilde{\oplus}_j, \tilde{\odot}_j > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ for each $j \in J = \{1, 2, ..., n\}$. If $\tilde{\oplus}$ and $\tilde{\odot}$ are vague binary operations on $\bigcap_{j \in J} A_j$ such that $\mu_{\tilde{\oplus}}(a, b, c) \leq \bigwedge_{j \in J} \mu_{\tilde{\circ}_j}(a, b, c)$ and $\mu_{\tilde{\odot}}(a, b, c) \leq \bigwedge_{j \in J} \mu_{\tilde{\bullet}_j}(a, b, c)$, then $< \bigcap_{j \in J} A_j, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ for all $a, b, c \in \bigcap_{j \in J} A_j$.*

*Proof.* From Corollary 3.7, we have $< \bigcap\limits_{j \in J} A_j, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. For all $a \in \bigcap\limits_{j \in J} A_j$ and $h, s, t \in \mathcal{H}$, we may write

$$\mu_{\tilde{\bullet}}(a,h,t)=1 \Longrightarrow t\in A_j\ ,\quad j\in J \Longrightarrow t\in \bigcap_{j\in J} A_j,$$

and

$$\mu_{\tilde{\bullet}}(h,a,s)=1 \Longrightarrow s\in A_j\ ,\quad j\in J \Longrightarrow s\in \bigcap_{j\in J} A_j.$$

Thus, it must be $< \bigcap_{j\in J} A_j, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. This completes the proof. □

**Definition 4.6.** *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring,* $A \subsetneqq \mathcal{H}$ *and* $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*. If* $\tilde{\bullet}$ *is a vague binary operation on* $\mathcal{H} \setminus A$*, then* $< A, \tilde{\oplus}, \tilde{\odot} >$ *is said to be a vague prime ideal of* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*.*

It is clear that, if $< A, \tilde{\oplus}, \tilde{\odot} >$ is a vague prime ideal of $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$, then $< A, \oplus, \odot >$ is a prime ideal of $< \mathcal{H}, \circ, \bullet >$.

**Proposition 4.7.** *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring and* $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*. Then the following two statements are equivalent.*
*(i)* $< A, \tilde{\oplus}, \tilde{\odot} >$ *is a vague prime ideal of* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*.*
*(ii)* $\mu_{\tilde{\bullet}}(x,y,z) < 1$ *for each* $z \in A$ *and for each* $x, y \in \mathcal{H} \setminus A$*.*

*Proof.* $(i) \Rightarrow (ii)$ : We assume that there exists $z \in A$ and $x, y \in \mathcal{H} \setminus A$ such that $\mu_{\tilde{\bullet}}(x,y,z) = 1$. In this case, from Definition 4.6, there exists $t \in \mathcal{H} \setminus A$ such that $\mu_{\tilde{\bullet}}(x,y,t) = 1$. Utilizing the condition $(F.2)$, we get $z = t \in \mathcal{H} \setminus A$, this contradicts with $z \in A$. Therefore $\mu_{\tilde{\bullet}}(x,y,z) < 1$ for each $z \in A$ and for each $x, y \in \mathcal{H} \setminus A$.

$(ii) \Rightarrow (i)$ : Since $\tilde{\bullet}$ is a vague binary operation on $\mathcal{H}$, for each $a, b \in \mathcal{H} \setminus A$ there exists $c \in \mathcal{H}$ such that $\mu_{\tilde{\bullet}}(a,b,c) = 1$. In this case, it must be $c \in \mathcal{H} \setminus A$ from the statement $(ii)$. This completes the proof. □

**Definition 4.8.** *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring,* $\emptyset \neq M \subsetneqq \mathcal{H}$ *and* $< M, \tilde{\oplus}, \tilde{\odot} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*. If there is no* $< N, \tilde{\ominus}, \tilde{\star} >$ *vague ideal of* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *such that*

$$< M, \tilde{\oplus}, \tilde{\odot} > \subset < N, \tilde{\ominus}, \tilde{\star} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >,$$

*then* $< M, \tilde{\oplus}, \tilde{\odot} >$ *is said to be a maximal vague ideal of* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *(*$< M, \tilde{\oplus}, \tilde{\odot} > \subset < N, \tilde{\ominus}, \tilde{\star} >$ *shows that at least one of the following statements is satisfied:* $M \subsetneqq N \subsetneqq \mathcal{H}$*,* $\mu_{\tilde{\oplus}}(a,b,c) < \mu_{\ominus}(a,b,c)$*,* $\mu_{\tilde{\odot}}(a,b,c) < \mu_{\tilde{\star}}(a,b,c)$ *for some* $a, b, c \in M$*).*

**Proposition 4.9.** *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring. If* $< M, \tilde{\oplus}, \tilde{\odot} >$ *is a maximal vague ideal of* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$*, then* $< M, \oplus, \odot >$ *is a maximal ideal of* $< \mathcal{H}, \circ, \bullet >$*.*

*Proof.* Utilizing Proposition 4.2, we get $< M, \oplus, \odot >$ is an ideal of $< \mathcal{H}, \circ, \bullet >$. We assume that $< M, \oplus, \odot >$ is not a maximal ideal of $< \mathcal{H}, \circ, \bullet >$. In this case, there exists a maximal ideal $N$ of $< \mathcal{H}, \circ, \bullet >$ such that $M \subset N \subset \mathcal{H}$. This implies

$$< M, \tilde{\oplus}, \tilde{\odot} > \subset < N, \tilde{\circ}, \tilde{\bullet} > \overset{v.i}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >,$$

but this fact contradicts $< M, \tilde{\oplus}, \tilde{\odot} >$ is a maximal vague ideal of $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. Therefore $< M, \oplus, \odot >$ must be a maximal ideal of $< \mathcal{H}, \circ, \bullet >$. □

**Proposition 4.10.** *Let* $<\mathcal{H}, \tilde{\circ}, \tilde{\bullet}>$ *be a vague ring. Then, maximal vague ideals of* $<\mathcal{H}, \tilde{\circ}, \tilde{\bullet}>$ *are* $<M, \tilde{\circ}, \tilde{\bullet}>$ *where* $M$ *is one of the maximal ideals of* $<\mathcal{H}, \circ, \bullet>$.

*Proof.* It is easily seen that, if $M$ is a maximal ideal of $<\mathcal{H}, \circ, \bullet>$, then $<M, \tilde{\circ}, \tilde{\bullet}>$ is a maximal vague ideal of $<\mathcal{H}, \tilde{\circ}, \tilde{\bullet}>$ from Definition 4.8. On the other hand, we know that if $<N, \tilde{\ominus}, \tilde{\star}>$ is a maximal vague ideal of $<\mathcal{H}, \tilde{\circ}, \tilde{\bullet}>$, then $N$ is a maximal ideal of $<\mathcal{H}, \circ, \bullet>$ from Proposition 4.9. This completes the proof. $\square$

**Example 4.11.** *Let* $G = \mathbb{Z}$, $A = 2\mathbb{Z}$ , $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $0 \leq \gamma \leq \beta \leq \alpha < 1$. *We define*

$$E_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to [0,1] \ , \ E_{\mathbb{Z}}(u,v) = \begin{cases} 1 & , \ if \ u = v \\ \alpha & , \ otherwise \end{cases}$$

$E_{2\mathbb{Z}} : 2\mathbb{Z} \times 2\mathbb{Z} \to [0,1]$ , $E_{2\mathbb{Z}}(m,n) = E_{\mathbb{Z}}(m,n)$, $E_{\mathbb{Z}\times\mathbb{Z}} = E^c_{\mathbb{Z}\times\mathbb{Z}}$, $E_{2\mathbb{Z}\times 2\mathbb{Z}} = E^c_{2\mathbb{Z}\times 2\mathbb{Z}}$ ,

$$\tilde{\circ} : \mathbb{Z} \times \mathbb{Z} \rightsquigarrow \mathbb{Z} \ , \ \mu_{\tilde{\circ}}(x,y,z) = \begin{cases} 1 & , \ if \ x + y = z \\ \beta & , \ otherwise \end{cases}$$

*and*

$$\tilde{\oplus} : 2\mathbb{Z} \times 2\mathbb{Z} \rightsquigarrow 2\mathbb{Z} \ , \ \mu_{\tilde{\oplus}}(a,b,c) = \begin{cases} 1 & , \ if \ a + b = c \\ \gamma & , \ otherwise. \end{cases}$$

*We get* $<2\mathbb{Z}, \tilde{\oplus}> \overset{\text{v.s}}{\leq} <\mathbb{Z}, \tilde{\circ}>$ *from* [13]. *Let* $\nu$, $\eta \in \mathbb{R}$ *such that* $0 \leq \nu \leq \eta < 1$, *and we define*

$$\tilde{\bullet} : \mathbb{Z} \times \mathbb{Z} \rightsquigarrow \mathbb{Z} \ , \ \mu_{\tilde{\bullet}}(x,y,z) = \begin{cases} 1 & , \ if \ x.y = z \\ \eta & , \ otherwise \end{cases}$$

*and*

$$\tilde{\odot} : 2\mathbb{Z} \times 2\mathbb{Z} \rightsquigarrow 2\mathbb{Z} \ , \ \mu_{\tilde{\odot}}(a,b,c) = \begin{cases} 1 & , \ if \ a.b = c \\ \nu & , \ otherwise. \end{cases}$$

*In this case, it is clearly seen that* $<\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}>$ *is a vague ring and* $\tilde{\odot}$ *is a vague binary operation on* $2\mathbb{Z}$. *Therefore, by using Proposition 4.4, we get* $<2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot}> \overset{v.i}{\leq} <\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}>$. *On the other hand, since* $\mu_{\tilde{\odot}}(a,b,c) = \nu < 1$ *for each* $a, b \in \mathbb{Z} \setminus 2\mathbb{Z}$ *and for each* $c \in 2\mathbb{Z}$, $<2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot}>$ *is a vague prime ideal of* $<\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}>$ *from Proposition 4.7. For* $\nu < \eta$, *we get* $<2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot}>$ *is not a vague maximal ideal of* $<\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}>$ *since*

$$<2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot}> \subset <2\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}> \overset{v.i}{\leq} <\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}> .$$

*For* $\nu = \eta$, *we obtain* $<2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot}>$ *is a vague maximal ideal of* $<\mathbb{Z}, \tilde{\circ}, \tilde{\bullet}>$ *from Proposition 4.10.*

## 5. Conclusion

In the present paper, the concepts of vague ring, vague ideal, vague prime ideal and vague maximal ideal are introduced, and various elementary properties of these concepts are investigated. Furthermore, these concepts and their properties are explained with some examples.

Although the results in this paper are formulated on $([0,1],\leq,\wedge)$, it seems that most of them can be restated for any t-norm instead of the minimum t-norm. This topic is left to the readers for future investigations.

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