BISIMULATION FOR BL-GENERAL FUZZY AUTOMATA

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Abstract. In this note, we define bisimulation for BL-general fuzzy automata and show that if there is a bisimulation between two BL-general fuzzy automata, then they have the same behavior. For a given BL-general fuzzy automata, we obtain the greatest bisimulation for the BL-general fuzzy automata. Thereafter, if we use the greatest bisimulation, then we obtain a quotient BL-general fuzzy automata and this quotient is minimal, furthermore there is a morphism from the first one to its quotient. Also, for two given BL-general fuzzy automata we present an algorithm, which determines bisimulation between them. Finally, we present some examples to clarify these new notions.

1. Introduction

Fuzzy automata was introduced by W. G. Wee [38] in 1967 and Santos [35] in 1968. Thereafter, there were a considerable number of authors, such as Mordeson and Malik [22, 23], Topencharov and Peeva [36], and others having contributed to this field [19, 30, 31, 32, 33]. Fuzzy finite automata have many important applications such as in learning system, pattern recognition, neural networks, database theory and fuzzy discrete event systems [11, 13, 14, 22, 25, 27, 34, 39]. State reduction and equivalence of fuzzy automata were studied by [20, 28, 29, 33, 40]. A widely-used notion of "equivalence" between states of automata is that of bisimulation. Bisimulations were introduced by Milner [24] and Park [26] as a means for testing behavioral equivalence among processes, but they have also been very successfully exploited to reduce the state-space of processes. Bisimulations have been very successfully exploited to model equivalence between various systems, as well as to reduce the number of states of these systems. The most common structures on which bisimulations have been studied are labeled transition systems, tree automata, weighted automata, etc. [11, 16, 17, 24]. Roughly, at the same time, bisimulations have been discovered in some areas of mathematics, e.g., in set theory and modal logic. Recently, bisimulations have been also studied in the setting of fuzzy automata and fuzzy transition systems [5, 6, 8, 9]. The approach to bisimulations proposed in [8, 9] for fuzzy automata has been applied in [7] to ordinary nondeterministic automata and in [10] to weighted automata. In this paper, we show that this methodology can be applied in a similar form to BL-general fuzzy automata. Nowadays, they are widely employed in the computer science,

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particularly in object-oriented languages, functional languages, verification tools, data types, domains, databases, program analysis, etc. For more information of bisimulations, we refer to [3, 4, 7, 8, 21].

In 2004, M. Doostfatemeh and S.C. Kremer [12] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. Their key motivation of introducing the notion general fuzzy automata was the insufficiency of the current literature to handle the applications which rely on fuzzy automata as a modeling tool, assigning membership values to active states of a fuzzy automaton, resolve the multi -membership.

Basic logic (BL) has been introduced by Hajek [15] in order to provide a general framework for formalizing statements of fuzzy nature. Formulas of propositional BL may be interpreted by means of BL-algebras [37]. With respect to a semantics defined in this way, BL is complete: formulas proved by BL, exactly those valid in any BL-algebra. In 2012, Kh. Abolpour and M. M. Zahedi [2] extended the notion of general fuzzy automata and gave the notion of BL-general fuzzy automata (BL-GFA).

With respect to minimization, the situation for nondeterministic automata is not as satisfactory as that for deterministic automata. The content of this paper is as follows: In preliminaries Section, we define the basic concepts. In Section 3, we define bisimulation for BL-GFA and show that if there is a bisimulation between two BL-GFA, then there is a morphism between them and they have the same behavior. For two given BL-general fuzzy automata, we present an algorithm to determines bisimulation between them with time complexity $O(|X||\bar{Q}_1||\bar{Q}_2|)$. Finally for a given BL-GFA, if we use the greatest bisimulation, then we obtain a quotient BL-GFA and this quotient is minimal, furthermore there is a morphism from the first one to its quotient.

2. Preliminaries

Definition 2.1. [12] A general fuzzy automaton (GFA) $\widetilde{F}$ is an eight-tuple machine denoted by $\widetilde{F} = (Q, X, \widetilde{R}, Z, \tilde{\delta}, \omega,\ F_1, F_2)$, where

- Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$,
- X is a finite set of input symbols, $X = \{a_1, a_2, ..., a_m\}$,
- $\widetilde{R}$ is a set of fuzzy start states, $\widetilde{R} \subseteq \widetilde{P}(Q)$,
- Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$,
- $\tilde{\delta} : (Q \times [0, 1]) \times X \times Q \to [0, 1]$ is the augmented transition function,
- $\omega : Q \to Z$ is the output function,
- $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called the membership assignment function. The function $F_1(\mu, \delta)$ is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the value of a transition.

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

- $F_2 : [0, 1]^* \to [0, 1]$ is called the multi-membership resolution function.

  The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

We let the set of all transition of $\widetilde{F}$ is denoted by Δ. Now, suppose that $Q_{act}(t_i)$ be the set of all active state at time t_i, for all $i \geq 0$. We have $Q_{act}(t_0) = \widetilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) \big| \exists q' \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta\}$, for all $i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we write $q \in Domain(Q_{act}(t_i))$. Hereafter, we denote these notations by

$$q \in Q_{act}(t_i) \text{ and } T \subseteq Q_{act}(t_i).$$

Definition 2.2. [15] A BL-algebra is algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that: (i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, (ii) $(L, *, 1)$ is a commutative monoid, (iii) $*$ and $\rightarrow$ form an adjoint pair, i.e., $x \leq y \rightarrow z$ if and only if $x * y \leq z$ for all $x, y, z \in L$, (iv) $x \wedge y = x * (x \rightarrow y)$, (v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Example 2.3. Let Q be a nonempty set. Then $(P(Q), *, \rightarrow, \cap, \cup, \emptyset, Q)$ is a BL-algebra, where $P(Q)$ is a power set of Q.

Proof. First, we define $Q_1 * Q_2 = Q_1 \cap Q_2$ and

$$Q_1 \rightarrow Q_2 = \begin{cases} Q & \text{if } Q_1 \subseteq Q_2 \\ Q_2 \cup Q_1' & \text{otherwise} \end{cases},$$

for every $Q_1, Q_2 \in P(Q)$, where $Q_1' = Q - Q_1$.

Obviously, $(P(Q), \cap, \cup, \emptyset, Q)$ is a bounded lattice and $(P(Q), *, Q)$ is a commutative monoid.

(iii) Now, suppose that $Q_3 \subseteq Q_1 \rightarrow Q_2$. If $Q_1 \subseteq Q_2$, then it is clear that $Q_1 * Q_3 \subseteq Q_2$. Now, let $Q_1 \nsubseteq Q_2$. Then the claim holds by considering Example 2.7 [2]. The converse is trivial.

(iv) If $Q_1 \subseteq Q_2$, then $Q_1 \rightarrow Q_2 = Q$ and $Q_1 \cap Q = Q_1 = Q_1 \cap Q_2$. If $Q_1 \nsubseteq Q_2$, then it is clear that by considering Example 2.7 [2].

(v) If $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$, then $Q_1 \rightarrow Q_2 \cup Q_2 \rightarrow Q_1 = Q$. Now, let $Q_1 \nsubseteq Q_2$ and $Q_2 \nsubseteq Q_1$. Then $Q_1 \rightarrow Q_2 \cup Q_2 \rightarrow Q_1 = (Q_2 \cup Q_1') \cup (Q_1 \cup Q_2') = Q$. Hence $(P(Q), *, \rightarrow, \cap, \cup, \emptyset, Q)$ is a BL-algebra. $\square$

Let $L = (L, \vee, \wedge, 0, 1)$ be a bounded complete lattice. Now, by considering bounded lattice L, Example 2.3 and Definition 3.1 of [2] we give the following definition:

Definition 2.4. Let $L = (L, \vee, \wedge, 0, 1)$ be a bounded complete lattice and let $\widetilde{F} = (Q, X, \widetilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton and $\bar{Q} = (P(Q), \subseteq, \cap, \cup, \emptyset, Q)$ be a BL-algebra in Example 2.3. We define the BL-general fuzzy automaton (BL-GFA) as a ten-tuple machine denoted by

$$\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2),$$

where

(i) $\bar{Q} = P(Q)$, where Q is a finite set and $\bar{Q}$ is the power set of Q,

(ii) X is a finite set of input symbols,

(iii) $\widetilde{R}$ is the set of fuzzy start states,

(iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z,
(v) $\omega_l : \bar{Q} \to \bar{Z}$ is the output function defined by: $\omega_l(Q_i) = \{\omega(q) \big| q \in Q_i\}$,
(vi) $\delta_l : \bar{Q} \times X \times \bar{Q} \to L$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \vee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$, for all $Q_i, Q_j \in P(Q)$ and $a \in X$,
(vii) $f_l : \bar{Q} \times X \to \bar{Q}$ is the next state map defined by:

$$f_l(Q_i, a) = \cup_{q_i \in Q_i} \{q_j \big| \delta(q_i, a, q_j) \in \Delta\},$$

(viii) $\tilde{\delta}_l : (\bar{Q} \times L) \times X \times \bar{Q} \to L$ is the augmented transition function defined $\tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$,
(ix) $F_1 : L \times L \to L$ is called membership assignment function,
(x) $F_2 : L^* \to L$ is called multi-membership resolution function.

In the rest of this note, L always denotes a bounded complete lattice.

Definition 2.5. [2] Let $\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA. The run map of the BL-GFA $\widetilde{F}_l$ is the map $\rho : X^* \to \bar{Q}$ defined by the following induction: $\rho(\Lambda) = \{q_0\}$ and $\rho(a_1a_2...a_n) = Q_{i_n}, \rho(a_1a_2...a_na_{n+1}) = f_l(Q_{i_n}, a_{n+1})$, where $(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in \bar{Q}_{lact}(a_1a_2...a_n)$, for every $a_1, ..., a_n \in X$.

The behavior of $\widetilde{F}_l$ is the map $\beta = \omega_l \circ \rho : X^* \to \bar{Z}$.

Definition 2.6. [2] Given $(\bar{Q}, f_l, \delta_l)$ and $(\bar{Q}', f_l', \delta_l')$, we say that $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is a homomorphism with threshold $\frac{\tau_1}{\tau_2}$ if there is a map of $\bar{Q}$ into $\bar{Q}'$ such that for every $Q_i, Q_j \in \bar{Q}$ the following hold:

(i) $g \circ f_l = f_l' \circ (g \times id_X)$,
(ii) $\tau_1 \leq \delta_l(f_l(Q_i, a_1), a_2, Q_j) \leq \tau_2$ if and only if $\tau_1 \leq \delta_l'(g(f_l(Q_i, a_1)), a_2, g(Q_j)) \leq \tau_2$.

We say that $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is homomorphism if and only if $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is homomorphism with threshold $\frac{0}{1}$.

Definition 2.7. [2] Let $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ be two BL- GFA. We say that $(g, g_{out}) : \widetilde{F}_l \to \widetilde{F}_l'$ is a morphism with threshold $\frac{\tau_1}{\tau_2}$ if and only if the following hold:

(i) $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is a homomorphism with threshold $\frac{\tau_1}{\tau_2}$,
(ii) $g_{out} \circ \omega_l = \omega_l' \circ g$,
(iii) $g(\{q_0\}) = \{q_0'\}$.

We say that $(g, g_{out}) : \widetilde{F}_l \to \widetilde{F}_l'$ is morphism if and only if $(g, g_{out}) : \widetilde{F}_l \to \widetilde{F}_l'$ is morphism with threshold $\frac{0}{1}$.

3. Bisimulation for BL-general Fuzzy Automata

Definition 3.1. Let $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2)$, $i = 1, 2$ be two BL- GFA. Then the relation $\approx$ between $\bar{Q}_1$ and $\bar{Q}_2$ is called a bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$ if the following hold:

(1) $\{q_{01}\} \approx \{q_{02}\}$,

(2) $Q' \approx Q''$ implies that

$$(\forall \alpha \in L)(Q'_1 \in \bar{Q}_1)(a \in X)(\delta_{l1}(Q', a, Q'_1) = \alpha$$
$$\Longrightarrow (\exists Q'_2 \in \bar{Q}_2)\delta_{l2}(Q'', a, Q'_2) \geq \alpha, Q'_1 \approx Q'_2) \text{ and vice versa,}$$

(3) $Q' \approx Q''$ implies that $\omega_{l1}(Q') = \omega_{l2}(Q'')$,

where $Q' \in \bar{Q}_1$ and $Q'' \in \bar{Q}_2$

Definition 3.2. Let $\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA. If for every BL-GFA $\widetilde{F}'_1$, which $\widetilde{F}_l$ is bisimilar with $\widetilde{F}'_l$, $|\widetilde{F}_l| \leq |\widetilde{F}'_l|$, then $\widetilde{F}_l$ is a minimal BL-GFA.

We will show that if two BL-GFA are bisimilar, then they have the same behavior.

Remark 3.3. Let $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2)$, $i = 1, 2, 3$ be three BL-GFA.
(i) It is easy to show that, if $\approx$ is a bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$, then its reverse is a bisimulation between $\widetilde{F}_{l2}$ and $\widetilde{F}_{l1}$. Therefore, the relation $\approx$ is a symmetric relation between $\bar{Q}_1$ and $\bar{Q}_2$.
(ii) If $\approx_1$ is a bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$ and $\approx_2$ is a bisimulation between $\widetilde{F}_{l2}$ and $\widetilde{F}_{l3}$, then their composition

$$\approx = \approx_1 \circ \approx_2 = \{(P, R) \big| \exists Q', P \approx_1 Q' and Q' \approx_2 R\},$$

is a bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l3}$.

(1) $\{q_{01}\} \approx_1 \{q_{02}\}$ and $\{q_{02}\} \approx_2 \{q_{03}\}$ imply that $\{q_{01}\} \approx \{q_{03}\}$.

(2) Let $Q'_1 \approx Q'_3$. Then there exists $Q'_2 \in \bar{Q}_2$, where $Q'_1 \approx_1 Q'_2$ and $Q'_2 \approx_2 Q'_3$. Thus by Definition 3.1, it is obvious that:

$$(\forall \alpha \in L)(Q''_1 \in \bar{Q}_1)(a \in X)(\delta_{l1}(Q'_1, a, Q''_1) = \alpha \Longrightarrow$$
$$(\exists Q''_3 \in \bar{Q}_3)\delta_{l3}(Q'_3, a, Q''_3) \geq \alpha, Q''_1 \approx Q''_3) \text{ and vice versa.}$$

(3) Let $Q'_1 \approx Q'_3$. Then there exists $Q'_2 \in \bar{Q}_2$ such that $Q'_1 \approx_1 Q'_2$ and $Q'_2 \approx_2 Q'_3$. Thus $\omega_{l1}(Q'_1) = \omega_{l3}(Q'_3)$. These imply that the relation $\approx$ is a transitive relation between $\bar{Q}_1$ and $\bar{Q}_3$.

Lemma 3.4. *Let* $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ *be two BL-GFAs. Then the union of any nonempty family of bisimulations between* $\widetilde{F}_{l1}$ *and* $\widetilde{F}_{l2}$ *is also a bisimulation between them.*

Proof. Let $\{\approx_i \big| i \in I\}$ be a nonempty set of bisimulations between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$. Define $\approx = \cup_{i \in I} \approx_i$. Then $Q' \approx Q''$ if and only if there exists $i \in I$ such that $Q' \approx_i$

Q''. Since I is nonempty, then $\{q_{01}\} \approx_i \{q_{02}\}$ for every $i \in I$. So $\{q_{01}\} \approx \{q_{02}\}$. Now, let $Q'_1 \approx Q'_2$. Then for some $i \in I$, $Q'_1 \approx_i Q'_2$. Therefore

$$(\forall \alpha \in L)(Q''_1 \in \bar{Q}_1)(a \in X)(\delta_{l1}(Q'_1, a, Q''_1) = \alpha \\ \Longrightarrow (\exists Q''_2 \in \bar{Q}_2)\delta_{l2}(Q'_2, a, Q''_2) \geq \alpha, Q''_1 \approx_i Q''_2) \text{ and vice versa.}$$

So

$$(\forall \alpha \in L)(Q''_1 \in \bar{Q}_1)(a \in X)(\delta_{l1}(Q'_1, a, Q''_1) = \alpha \\ \Longrightarrow (\exists Q''_2 \in \bar{Q}_2)\delta_{l2}(Q'_2, a, Q''_2) \geq \alpha, Q''_1 \approx Q''_2) \text{ and vice versa.}$$

Now, suppose that $Q_1 \approx Q_2$. Then there exists $i \in I$, where $Q_1 \approx_i Q_2$. So $\omega_{l1}(Q_1) = \omega_{l2}(Q_2)$. Hence, the claim holds. □

Theorem 3.5. *Let* $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2)$, $i = 1, 2$ *be two BL-GFAs and let* $\approx$ *be a bisimulation between* $\widetilde{F}_{l1}$ *and* $\widetilde{F}_{l2}$. *Then* $\beta_{\tilde{F}_{l1}} = \beta_{\tilde{F}_{l2}}$.

Proof. Let $\approx$ be a bisimulation and let ρ_1, ρ_2 be the run relations of $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$, respectively. First, we prove that for every $a_1...a_n = x \in X^*$, there exist $Q'_1 \in \bar{Q}_1$ and $Q'_2 \in \bar{Q}_2$ such that $\rho_1(x) \approx Q'_2 \subseteq f_{l2}(\rho_2(a_1a_2...a_{n-1}), a_n)$ and $\rho_2(x) \approx Q'_1 \subseteq f_{l1}(\rho_1(a_1a_2...a_{n-1}), a_n)$. Now, if $x = \Lambda$, then $\rho_1(\Lambda) = \{q_{01}\} \approx \{q_{02}\} = \rho_2(\Lambda)$. Let $x = a \in X$. Then $\rho_1(a) = f_{l1}(\{q_{01}\}, a)$. Let $\alpha \in L$ be such that $\delta_{l1}(\{q_{01}\}, a, f_{l1}(\{q_{01}\}, a)) = \alpha$. Then by Definition 3.1, there exists $Q'_2 \in \bar{Q}_2$ such that $\delta_{l2}(\{q_{02}\}, a, Q'_2) \geq \alpha$ and $\rho_1(a) \approx Q'_2 \subseteq f_{l2}(\{q_{02}\}, a)$.

Now, suppose that $x = a_1a_2 \in X^*$ and $\alpha \in L$ be such that $\delta_{l1}(\rho_1(a_1), a_2, \rho_1(a_1a_2)) = \alpha$. Then there is $Q''_2 \in \bar{Q}_2$ such that $\delta_{l2}(Q'_2, a_2, Q''_2) \geq \alpha$ and $\rho_1(a_1a_2) \approx Q''_2 \subseteq f_{l2}(Q'_2, a_2) \subseteq f_{l2}(\rho_2(a_1), a_2)$. The rest of the proof by inductively. By similar way, we have $\rho_2(x) \approx Q'_1 \subseteq f_{l1}(\rho_1(a_1...a_{n-1}), a_n)$.

Then for every $a_1a_2...a_n = x \in X^*$

$$\beta_{\tilde{F}_{l1}}(x) = \omega_{l1}(\rho_1(x)) = \omega_{l2}(Q'_2) \subseteq \omega_{l2}(f_{l2}(\rho_2(a_1...a_{n-1}), a_n)) = \omega_{l2}(\rho_2(x)) = \beta_{\tilde{F}_{l2}}(x),$$

for some $Q'_2 \in \bar{Q}_2$, where $\rho_1(x) \approx Q'_2$. Also,

$$\beta_{\tilde{F}_{l2}}(x) = \omega_{l2}(\rho_2(x)) = \omega_{l1}(Q'_1) \subseteq \omega_{l1}(f_{l1}(\rho_1(a_1...a_{n-1}), a_n)) = \omega_{l1}(\rho_1(x)) = \beta_{\tilde{F}_{l1}}(x),$$

for some $Q'_1 \in \bar{Q}_1$, where $\rho_2(x) \approx Q'_1$. Hence $\beta_{\tilde{F}_{l1}}(x) = \beta_{\tilde{F}_{l2}}(x)$ □

Definition 3.6. Let $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2)$, $i = 1, 2$ be two BL-GFAs and let $\approx$ be a bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$. The support of $\approx$ in $\widetilde{F}_{l1}$ is the set $C_{\approx}(\bar{Q}_2)$, the set of states of $\widetilde{F}_{l1}$ that are related by $\approx$ to some states of $\widetilde{F}_{l2}$.

Definition 3.7. Let $\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA. Then $\emptyset \neq Q' \in \bar{Q}$ is an accessible state if there exists $x \in X^*$ such that $f_l(\{q_0\}, x) = Q'$.

Note that a bisimulation between a BL-GFA and itself is called a bisimulation on BL-GFA.

Theorem 3.8. *Let $\widetilde{F}_l$ be a BL-GFA and let B be the set of all bisimulations on $\widetilde{F}_l$. Then union of all the relations in B is a bisimulation on $\widetilde{F}_l$ and it is also an equivalence relation on $\bar{Q}$.*

Proof. Let $\equiv$ be the union of all the relations in B. By Lemma 3.4, $\equiv$ is a bisimulation on $\widetilde{F}_l$. Also the relation $\equiv$ is reflexive, since the identity relation is in B, and by Note 3.3 it is symmetric and transitive. $\square$

Let $\equiv$ be the union of all bisimulations on $\widetilde{F}_l$. We define

$$[P] = \{Q' \big| P \equiv Q'\},\ \simeq= \{(P, [P]) \big| P \in \bar{Q}\}.$$

For any $A \subseteq \bar{Q}$, define

$$A' = \{[P] \big| P \in A\}.$$

Lemma 3.9. *For all $A, B \subseteq \bar{Q}$:*

(i) *$A \subseteq C_{\equiv}(B)$ if and only if $A' \subseteq B'$,*

(ii) *$A \equiv B$ if and only if $A' = B'$,*

(iii) *$A \simeq A'$.*

Proof. (*i*) Let $A \subseteq C_{\equiv}(B)$. Let $[P] \in A'$. Then $P \in A$. It implies that $P \in C_{\equiv}(B)$. Therefore there exists $P' \in B$ such that $P \equiv P'$. Hence $[P] = [P'] \in B'$. Now, let $A' \subseteq B'$. Suppose that $P \in A$. Then $[P] \in A'$. By considering hypothesis $[P] \in B'$. So, $P \in B$. Thus $P \in C_{\equiv}(B)$.

(*ii*) Let $A \equiv B$. Suppose that $[P] \in A'$. Then $P \in A$. Therefore there exists $Q' \in B$ such that $P \equiv Q'$. Thus $[P] = [Q'] \in B'$. Therefore $A' \subseteq B'$. By a similar way $B' \subseteq A'$. Now, suppose that $A' = B'$. Let $P \in A$. Then $[P] \in A' = B'$. Therefore $[P] \in B'$. Hence $P \in B$.

(*iii*) Clearly it holds. $\square$

Let $\widetilde{F}_l = (\bar{Q}_l, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA and $\equiv$ be the union of all bisimulations on $\widetilde{F}_l$. Now, we define the quotient BL-GFA $\widetilde{F'}_l = (\bar{Q}'_l, X, \widetilde{R'}, \bar{Z}, \omega'_l, \delta'_l, f'_l, \tilde{\delta}'_l, F_1, F_2)$, where $\bar{Q}'_l = \{[Q'] \big| Q' \in \bar{Q}_l\}, [Q'] = \{P \big| Q' \equiv P\}, \widetilde{R'} = [\{q_0\}]$. Also, we define $f'_l : \bar{Q}'_l \times X \to \bar{Q}'_l$ by $f'_l([Q'], a) = [f_l(Q', a)]$, $\delta'_l : \bar{Q}'_l \times X \times \bar{Q}'_l \to L$ by

(1)

$$\delta'_l([Q'], a, [P]) = \vee\{\delta_l(Q'', a, P') \big| Q'' \equiv Q', P' \equiv P\} = \vee\{\delta_l(Q', a, P') \big| P' \equiv P\},$$

and $\omega'_l : \bar{Q}'_l \to Z$ by $\omega'_l([Q']) = \omega_l(Q')$.

It is clear that $\tilde{\delta}_l$ is well-defined. If $[P] = [Q']$, then $P \equiv Q'$ and $\omega_l(P) = \omega_l(Q')$, for every $[P], [Q'] \in \bar{Q}'_l$. Therefore $\omega'_l(P) = \omega'_l(Q')$. Hence ω'_l is well-defined.

Theorem 3.10. *Let $\widetilde{F}_l$ be a BL-GFA with no inaccessible states and let $\equiv$ be the greatest bisimulation on $\bar{Q}_l$. The quotient BL-GFA $\widetilde{F'}_l$ on $\widetilde{F}_l$, under bisimulation $\equiv$, is a morphism to $\widetilde{F}_l$.*

Proof. First, we show that $\widetilde{F}_l$ and $\widetilde{F}'_l$ are homomorphic. We define $g : \bar{Q} \to \bar{Q}'$ by $g(Q') = [Q']$. For every $Q' \in \bar{Q}$ and $a \in X$, we have

$$g \circ f_l(Q', a) = g(f_l(Q', a)) = [f_l(Q', a)],$$

and

$$f'_l \circ (g \times id_X)(Q', a) = f'_l(g(Q'), a) = f'_l([Q'], a) = [f_l(Q', a)].$$

Then $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f'_l, \delta'_l)$ is a homomorphism.

Now, let $g_{out} : \bar{Z} \to \bar{Z}$ be the identity map. Then for every $Q' \in \bar{Q}$

$$g_{out} \circ \omega_l(Q') = g_{out}(\omega_l(Q')) = \omega_l(Q'),$$

and

$$\omega'_l \circ g(Q') = \omega'_l(g(Q')) = \omega'_l([Q']) = \omega_l(Q').$$

Also, we have $g(\{q_0\}) = [\{q_0\}]$. Then $\widetilde{F}_l$ and $\widetilde{F}'_l$ are morphic. □

Theorem 3.11. *Let $\widetilde{F}_l$ be a BL-GFA with no inaccessible states and $\widetilde{F}'_l$ be the quotient BL-GFA of $\widetilde{F}_l$. Then $\widetilde{F}_l$ and $\widetilde{F}'_l$ have the same behavior.*

Proof. The proof follows from Theorem 3.10, and Corollary 2.18 [1]. □

Theorem 3.12. *The relation $\simeq = \{(P, [P]) \big| P \in \bar{Q}_l\}$, where $[P] = \{Q' \big| P \equiv Q'\}$ is a bisimulation between BL-GFA $\widetilde{F}_l$ and quotient BL-GFA $\widetilde{F'}_l$. Also, $\widetilde{F}_l$ and $\widetilde{F'}_l$ have the same behavior.*

Proof. It is clear that $\{q_0\} \simeq [\{q_0\}]$. Now, let $P \simeq [Q']$. Then $P \equiv Q'$. Suppose that there exist $P' \in \bar{Q}_l$ and $a \in X$ such that $\delta_l(P, a, P') = \alpha$. Then there is $Q'' \in \bar{Q}'_l$ such that $\delta_l(Q', a, Q'') \geq \alpha$ and $P' \equiv Q''$. Therefore $\delta'_l([Q'], a, [Q'']) \geq \alpha$ and $P' \simeq [Q'']$. Now, let there exist $[Q''] \in \bar{Q}'_l$ and $a \in X$ such that $\delta'_l([Q'], a, [Q'']) = \alpha$. By (1) there exists $R \equiv Q''$, where

$$\delta'_l([Q'], a, [Q'']) = \delta_l(Q', a, R) = \alpha.$$

Then there is $P' \in \bar{Q}_l$ such that $\delta_l(P, a, P') \geq \alpha$ and $P' \equiv R, P' \simeq [Q'']$.

Also, if $P \simeq [Q']$, then $P \equiv Q'$ and $\omega_l(P) = \omega_l(Q') = \omega'_l([Q'])$. By Theorem 3.5, it is clear that $\widetilde{F}_l$ and $\widetilde{F'}_l$ have the same behavior. □

Lemma 3.13. *The only bisimulation on the quotient BL-GFA $\widetilde{F'}_l$ is the identity relation.*

Proof. Let $\sim$ be a bisimulation on $\widetilde{F'}_l$ and let $[P] \neq [Q']$ and $[P] \sim [Q']$. Now, consider the composition $\simeq \circ \sim \circ (\simeq)^{-1}$, where $(\simeq)^{-1}$ is the inverse of $\simeq$. Then we have $P \simeq [P] \sim [Q'] \simeq Q'$. Therefore $P \equiv Q'$. So it is a contradiction. Thus $\sim$ is the identity relation. □

We say that two BL-GFA $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$ are bisimilar if there exists a bisimulation between them.

Theorem 3.14. *Let $\widetilde{F}_{l1}$ be a BL-GFA with no inaccessible states and let $\equiv$ be the greatest bisimulation on $\bar{Q}_{l1}$. Then the quotient BL-GFA $\widetilde{F'}_{l1}$ is the minimal BL-GFA bisimilar to $\widetilde{F}_{l1}$.*

Proof. To show this, it will suffice to show that for any BL-GFA $\widetilde{F}_{l2}$ with no inaccessible states bisimilar to $\widetilde{F}_{l1}$ we should have for any bisimulation between $\widetilde{F}'_{l1}$ and $\widetilde{F}'_{l2}$ gives a one-to-one correspondence between the states of $\widetilde{F}'_{l1}$ and $\widetilde{F}'_{l2}$, where $\widetilde{F}'_{li}$ is the quotient BL-GFA of $\widetilde{F}_{li}, i = 1, 2$ according to greatest bisimulation. Now,

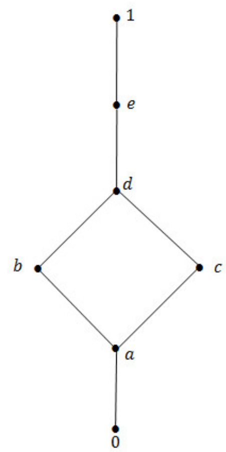

FIGURE 1. The Complete Lattice L of Example 3.15

let $\approx$ be a bisimulation between $\widetilde{F}'_{l1}$ and $\widetilde{F}'_{l2}$ and let under this relation every state of $\widetilde{F}'_{l2}$ is related to at least one state of $\widetilde{F}'_{l1}$ and every state of $\widetilde{F}'_{l1}$ is related to at most one state of $\widetilde{F}'_{l2}$. Then the composition of $\approx$ with its inverse would not be the identity on $\widetilde{F}'_{l1}$, which contradicts Lemma 3.13. Thus $\approx$ gives a one-to-one correspondence between the states of $\widetilde{F}'_{l1}$ and $\widetilde{F}'_{l2}$. □

Example 3.15. Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice as in Figure 1. Now, consider the general fuzzy automaton $\widetilde{F} = (Q, X, \tilde{\delta}, \widetilde{R}, Z, \omega, F_1, F_2)$ as in Figure 2, where $Q = \{q_0, q_1\}$, $\widetilde{R} = \{(q_0, 1)\}, X = \{\sigma\}, Z = \{z_1, z_2\}, \omega(q_0) = z_1 = \omega(q_1)$ and

$$\delta(q_0, \sigma, q_0) = a,\ \delta(q_0, \sigma, q_1) = b,$$
$$\delta(q_1, \sigma, q_0) = c,\ \delta(q_1, \sigma, q_1) = e.$$

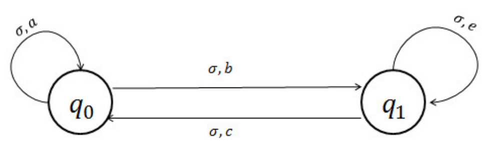

FIGURE 2. General Fuzzy Automaton $\widetilde{F}$ of Example 3.15

Also, by considering Definition 2.4, we have BL- general fuzzy automaton $\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z},\ \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ as Figure 3, where $\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$, $\bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}, \omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0, q_1\}) = \{z_1\}, f_l(\{q_0\}, \sigma) = f_l(\{q_1\}, \sigma) = f_l(\{q_0, q_1\}, \sigma) = \{q_0, q_1\}$ and

$$\delta_l(\{q_0\}, \sigma, \{q_0\}) = a,\ \delta_l(\{q_0\}, \sigma, \{q_1\}) = b,$$
$$\delta_l(\{q_0\}, \sigma, \{q_0, q_1\}) = b,\ \delta_l(\{q_1\}, \sigma, \{q_0\}) = c,$$
$$\delta_l(\{q_1\}, \sigma, \{q_1\}) = e,\ \delta_l(\{q_1\}, \sigma, \{q_0, q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma, \{q_0\}) = c,\ \delta_l(\{q_0, q_1\}, \sigma, \{q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = e.$$

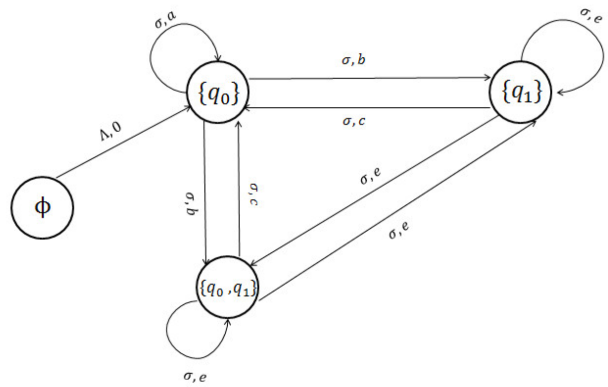

FIGURE 3. The BL-general Fuzzy Automaton $\widetilde{F}_l$ of Example 3.15

By considering Definition 3.1, we have $[\{q_1\}] = [\{q_0, q_1\}]$. Then we obtain the quotient BL-general fuzzy automaton of $\widetilde{F}_l$, which is called $\widetilde{F}_l'$, as Figure 4, where $\bar{Q}' = \{[\emptyset], [\{q_0\}], [\{q_1\}]\}, \bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\},\ \omega_l'([\{q_0\}]) = \omega_l'([\{q_1\}]) = \{z_1\}, f_l'([\{q_0\}], \sigma) = f_l([\{q_1\}], \sigma) = [\{q_1\}]$ and

$$\bar{\delta}'([\{q_0\}], \sigma, [\{q_0\}]) = a,\ \bar{\delta}'([\{q_0\}], \sigma, [\{q_1\}]) = b,$$
$$\bar{\delta}'([\{q_1\}], \sigma, [\{q_0\}]) = c,\ \bar{\delta}'([\{q_1\}], \sigma, [\{q_1\}]) = e.$$

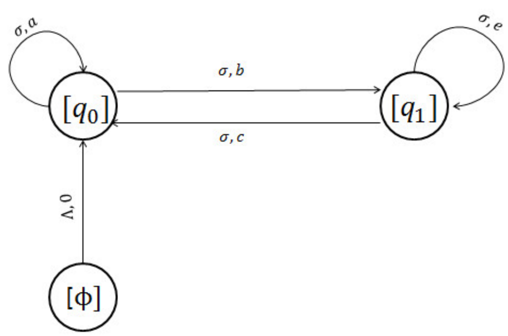

FIGURE 4. The Quotient BL-general Fuzzy Automata $\widetilde{F}_l'$ of Example 3.15

It is clear that, $\simeq = \{(P, [P]) \big| P \in \bar{Q}\}$ is a bisimulation between $\widetilde{F}_l$ and $\widetilde{F}_l'$, where $\{q_0\} \simeq [\{q_0\}], \{q_1\} \simeq [\{q_1\}]$ and $\{q_0, q_1\} \simeq [\{q_1\}]$. Now, define $g : \bar{Q} \to \bar{Q}'$ by $g(\{P\}) = [P]$. Then

$$g(f_l(\{q_0\}, \sigma)) = g(\{q_0, q_1\}) = [\{q_1\}] = f_l'((g \times id_X)(\{q_0\}, \sigma)) = f_l'([\{q_0\}], \sigma),$$
$$g(f_l(\{q_1\}, \sigma)) = g(\{q_0, q_1\}) = [\{q_1\}] = f_l'((g \times id_X)(\{q_1\}, \sigma)) = f_l'([\{q_1\}], \sigma),$$
$$g(f_l(\{q_0, q_1\}, \sigma)) = g(\{q_0, q_1\}) = [\{q_1\}] = f_l'((g \times id_X)(\{q_0, q_1\}, \sigma)) = f_l'([\{q_1\}], \sigma).$$

Also, we have

$$a \leq \delta_l(\bar{f}_l(Q_i, \sigma_1), \sigma_2, Q_j) \leq e \Longleftrightarrow a \leq \delta'_l(g(\bar{f}_l(Q_i, \sigma_1)), \sigma_2, g(Q_j)) \leq e.$$

Then $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f'_l, \delta'_l)$ is a homomorphism with threshold $\frac{a}{e}$. Let g_{out} : $\bar{Z} \to \bar{Z}$ be an identity map. Then it is clear that $g_{out} \circ \omega_l = \omega'_l \circ g$. Also, we have $g(\{q_0\}) = [\{q_0\}]$. So, $\widetilde{F}_l$ and $\widetilde{F}'_l$ are morphic with threshold $\frac{a}{e}$. Hence, by Theorems 3.5 and 3.11, $\widetilde{F}_l$ and $\widetilde{F}'_l$ have the same behavior.

Example 3.16. Let $L = [0, 1]$ and the general fuzzy automata $\widetilde{F} = (Q, X, \tilde{\delta}, \widetilde{R}, Z, \omega, F_1, F_2)$ as in Figure 5, where $Q = \{q_0, q_1\}$, $\widetilde{R} = \{(q_0, 1)\}$, $X = \{\sigma\}$, $Z = \{z_1, z_2\}$, $\omega(q_0) = z_1 = \omega(q_1)$ and

$$\delta(q_0, \sigma, q_0) = 0.1,\ \delta(q_0, \sigma, q_1) = 0.2,$$
$$\delta(q_1, \sigma, q_0) = 0.2,\ \delta(q_1, \sigma, q_1) = 0.4.$$

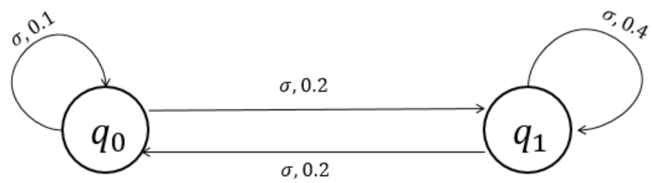

FIGURE 5. General Fuzzy Automata $\widetilde{F}$ of Example 3.16

So, we have BL- general fuzzy automata $\widetilde{F}_l = (\bar{Q}, X, \widetilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ as Figure 6, where $\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$, $\bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$, $\omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0, q_1\}) = \{z_1\}$, $f_l(\{q_0\}, \sigma) = f_l(\{q_1\}, \sigma) = f_l(\{q_0, q_1\}, \sigma) = \{q_0, q_1\}$ and

$$\delta_l(\{q_0\}, \sigma, \{q_0\}) = 0.1,\ \delta_l(\{q_0\}, \sigma, \{q_1\}) = 0.2,$$
$$\delta_l(\{q_0\}, \sigma, \{q_0, q_1\}) = 0.2,\ \delta_l(\{q_1\}, \sigma, \{q_0\}) = 0.2,$$
$$\delta_l(\{q_1\}, \sigma, \{q_1\}) = 0.4,\ \delta_l(\{q_1\}, \sigma, \{q_0, q_1\}) = 0.4,$$
$$\delta_l(\{q_0, q_1\}, \sigma, \{q_0\}) = 0.2,\ \delta_l(\{q_0, q_1\}, \sigma, \{q_1\}) = 0.4,$$
$$\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = 0.4.$$

By considering Definition 3.1 we have $[\{q_1\}] = [\{q_0, q_1\}]$. The rest of the proof is similar to the proof of Example 3.15.

The following algorithm determines a bisimulation between two given BL-GFA $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$. There may exists no bisimulation between $\widetilde{F}_{l1}$ and $\widetilde{F}_{l2}$, in which case the algorithm stops and reports failure.

1. Algorithm for Computing Bisimulation

Step 1: input: Two BL-GFA $\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2, Q' \in \bar{Q}_1, Q'' \in \bar{Q}_2, X = \{a_1, a_2, ..., a_n\}, J = 1,$

Step 2: $Q' \approx_j Q''$ if and only if $\omega_{l1}(Q') = \omega_{l2}(Q'')$,

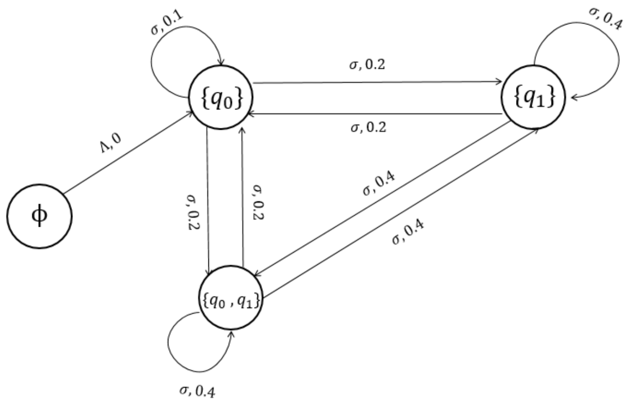

FIGURE 6. The BL-general Fuzzy Automata $\widetilde{F}_l$ of Example 3.16

Step 3: $k = 1$, $j = j + 1$
Step 4: $Q' \approx_j Q''$ if and only if $Q' \approx_{j-1} Q''$ and

$$(\forall \alpha \in L)(Q'_1 \in \bar{Q}_1)(a_k \in X)(\delta_{l1}(Q', a_k, Q'_1) = \alpha)$$
$$\Longrightarrow (\exists Q'_2 \in \bar{Q}_2)(\delta_{l2}(Q'', a_k, Q'_2) \geq \alpha, Q'_1 \approx_{j-1} Q'_2) \text{ and vice versa,}$$

Step 5: $k = k + 1$, if $k > n$, then go to next step, else go to Step 4,
Step 6: if $\approx_j = \approx_{j-1}$, then go to next step, else go to Step 3,
Step 7: if $\{q_{01}\} \approx_j \{q_{02}\}$, then go to next step, else go to Step 9,
Step 8: **output:** $\approx = \approx_j$,
Step 9: **output:** fail.

Steps 3 to 5 of Algorithm 1, are a loop. The loop must be repeated at most $|\bar{Q}_1| \times |\bar{Q}_2|$ times. By considering $|X|$ and Steps 3 to 5, the order of time complexity is at most $O(|X||\bar{Q}_1||\bar{Q}_2|)$.

Example 3.17. Let $L = [0, 1]$. Now, consider the GFAs $\widetilde{F}_i = (Q_i, X, \tilde{\delta}_i, \widetilde{R}_i, Z, \omega_i, F_1, F_2), i = 1, 2$, where $Q_1 = \{q_1, q_2\}, Q_2 = \{p_1, p_2\}$, $\widetilde{R}_1 = \{(q_1, 1)\}, \widetilde{R}_2 = \{(p_1, 1)\}$, $X = \{a\}, Z = \{z_1, z_2\}, \omega_1(q_1) = \omega_1(q_2) = z_1 = \omega_2(p_1) = \omega_2(p_2)$ and

$$\delta_1(q_1, a, q_1) = 0.1,\ \ \delta_1(q_1, a, q_2) = 0.2,$$
$$\delta_1(q_2, a, q_1) = 0.2,\ \ \delta_1(q_2, a, q_2) = 0.1,$$
$$\delta_2(p_1, a, p_1) = 0.1,\ \ \delta_2(p_1, a, p_2) = 0.2,$$
$$\delta_2(p_2, a, p_1) = 0.1,\ \ \delta_2(p_2, a, p_2) = 0.2.$$

Then by considering Definition 2.4 we have BL- GFAs

$$\widetilde{F}_{li} = (\bar{Q}_i, X, \widetilde{R}_i, \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2,$$

as follow: $\bar{Q}_1 = \{\emptyset, \{q_1\}, \{q_2\}, \{q_1, q_2\}\}, \bar{Q}_2 = \{\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}, \bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$, $\omega_{l1}(\{q_1\}) = \omega_{l1}(\{q_2\}) = \omega_{l1}(\{q_1, q_2\}) = \{z_1\} = \omega_{l2}(\{p_1\}) = \omega_{l2}(\{p_2\}) = \omega_{l2}(\{p_1, p_2\}), f_{l1}(\{q_1\}, a) = f_{l1}(\{q_2\}, a) = f_{l1}(\{q_1, q_2\}, a) = \{q_1, q_2\}$,

$f_{l2}(\{p_1\},a) = f_{l2}(\{p_2\},a) = f_{l2}(\{p_1,p_2\},a) = \{p_1,p_2\}$ and

$$\begin{aligned}
&\delta_{l1}(\{q_1\},a,\{q_1\}) = 0.1,\ \delta_{l1}(\{q_1\},a,\{q_2\}) = 0.2,\\
&\delta_{l1}(\{q_1\},a,\{q_1,q_2\}) = 0.2,\ \delta_{l1}(\{q_2\},a,\{q_1\}) = 0.2,\\
&\delta_{l1}(\{q_2\},a,\{q_2\}) = 0.1,\ \delta_{l1}(\{q_2\},a,\{q_1,q_2\}) = 0.2,\\
&\delta_{l1}(\{q_1,q_2\},a,\{q_1\}) = 0.2,\ \delta_{l1}(\{q_1,q_2\},a,\{q_2\}) = 0.2,\\
&\delta_{l1}(\{q_1,q_2\},a,\{q_1,q_2\}) = 0.2,\\
&\delta_{l2}(\{p_1\},a,\{p_1\}) = 0.1,\ \delta_{l2}(\{p_1\},a,\{p_2\}) = 0.2,\\
&\delta_{l2}(\{p_1\},a,\{p_1,p_2\}) = 0.2,\ \delta_{l2}(\{p_2\},a,\{p_1\}) = 0.1,\\
&\delta_{l1}(\{p_2\},a,\{p_2\}) = 0.2,\ \delta_{l2}(\{p_2\},a,\{p_1,p_2\}) = 0.2,\\
&\delta_{l2}(\{p_1,p_2\},a,\{p_1\}) = 0.1,\ \delta_{l2}(\{p_1,p_2\},a,\{p_2\}) = 0.2,\\
&\delta_{l2}(\{p_1,p_2\},a,\{p_1,p_2\}) = 0.2,
\end{aligned}$$

By considering Algorithm for computing bisimulation we have:
Stage 1.
1. $i = 1, X = \{a_1\}$,
2. $\{q_1\} \approx_1 \{p_1\} \approx_1 \{q_2\} \approx_1 \{p_2\} \approx_1 \{q_1,q_2\} \approx_1 \{p_1,p_2\}$,
3. $k = 1, i = 2$,
4. $\{q_1\} \approx_2 \{p_1\} \approx_2 \{q_2\} \approx_2 \{p_2\} \approx_2 \{q_1,q_2\} \approx_2 \{p_1,p_2\}$,
5. $k = 2$,
6. $\approx_2 = \approx_1$,
8. output : $\approx$.

If in the BL-GFA $\widetilde{F}_{l2}$, we put $\delta_2(p_2,a,p_2) = 0.1$ instead of $\delta_2(p_2,a,p_2) = 0.2$. Then we have
Stage 1.
1. $i = 1, X = \{a_1\}$,
2. $\{q_1\} \approx_1 \{p_1\} \approx_1 \{q_2\} \approx_1 \{p_2\} \approx_1 \{q_1,q_2\} \approx_1 \{p_1,p_2\}$,
3. $k = 1, i = 2$,
4. $\{p_1\} \approx_2 \{q_1\} \approx_2 \{p_1,p_2\}$, $\{p_1,p_2\} \approx_2 \{q_2\} \approx_2 \{p_1\}$, $\{p_1\} \approx_2 \{q_1,q_2\} \approx_2 \{p_1,p_2\}$,
5. $k = 2$,
6. $\approx_2 \neq \approx_1$.

Stage 2.
3. $k = 1, i = 3$,
4. $\{p_1\} \not\approx_3 \{q_1\} \not\approx_3 \{p_1,p_2\}$, $\{p_1,p_2\} \not\approx_3 \{q_2\} \not\approx_3 \{p_1\}$, $\{p_1\} \not\approx_3 \{q_1,q_2\} \not\approx_3 \{p_1,p_2\}$,
5. $k = 2$,
6. $\approx_3 \neq \approx_2$.

Stage 3.
3. $k = 1, i = 4$,
4. $\{p_1\} \not\approx_4 \{q_1\} \not\approx_4 \{p_1,p_2\}$, $\{p_1,p_2\} \not\approx_4 \{q_2\} \not\approx_4 \{p_1\}$, $\{p_1\} \not\approx_4 \{q_1,q_2\} \not\approx_4 \{p_1,p_2\}$,
5. $k = 2$,
6. $\approx_4 = \approx_3$,
7. $\{p_1\} \not\approx_4 \{q_1\}$,
8. fail.

4. **Conclusion**

In this note, we show that if there is a bisimulation between two BL-general fuzzy automata, then there is a morphism between them and they have the same behavior. Also for a given BL-general fuzzy automata, if we use the greatest bisimulation, then we obtain a quotient BL-general fuzzy automata and this quotient is minimal, furthermore, there is a morphism from the first one to its quotient.

Now, there is an important question: Suppose that there are two BL-general fuzzy automata. Is there a weak bisimulation between these two BL-general fuzzy automata, and also for a given bisimulation between these two BL-general fuzzy automata is there a weak bisimulation between them?

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References

[1] K. Abolpour and M. M. Zahedi, *BL-general fuzzy automata and accept behavior*, Journal Applied Mathematics and Computing, **38** (2012), 103-118.

[2] K. Abolpour and M. M. Zahedi, *Isomorphism between two BL-general fuzzy automata*, Soft Computing, **16** (2012), 729-736.

[3] C. Baier, B. Engelen and M. Majster Cederbaum, *Deciding bisimilarity and similarity for probabilistic processes*, Journal of Computer and System Sciences, **60** (2000), 187-231.

[4] P. Buchholz, *Bisimulation relations for weighted automata*, Theoretical Computer Science, **393** (2008), 109-123.

[5] Y. Cao, G. Chen and E. Kerre, *Bisimulations for fuzzy transition systems*, IEEE Transactions on Fuzzy Systems, **19** (2011), 540-552.

[6] Y. Cao, H. Wang, S. X. Sun and G. Chen, *A behavioral distance for fuzzy-transition systems*, IEEE Transactions on Fuzzy Systems, **21** (2012), 735-747.

[7] M. Ćirić, J. Ignjatović, M. Bašić and I. Jančić, *Nondeterministic automata: equivalence, bisimulations, and uniform relations*, Information Sciences, **261** (2013), 185-218.

[8] M. Ćirić, J. Ignjatović, N. Damljanović and M. Bašić, *Bisimulations for fuzzy automata*, Fuzzy Sets and Systems, **186** (2012) 100-139.

[9] M. Ćirić, J. Ignjatović, I. Jančić and N. Damljanović, *Computation of the greatest simulations and bisimulations between fuzzy automata*, Fuzzy Sets and Systems, **208** (2012), 22-42.

[10] N. Damljanović, M. Ćirić and J. Ignjatović, *Bisimulations for weighted automata over an additively idempotent semiring*, Theoretical Computer Science, **534** (2014), 86-100.

[11] W. Deng and D. W. Qiu, *Supervisory control of fuzzy discrete event systems for simulation equivalence*, IEEE Transactions on Fuzzy Systems, **23** (2015), 178-192.

[12] M. Doostfatemeh and S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175-214.

[13] C. L. Giles, C. W. Omlin and K. K. Thornber, *Equivalence in knowledge representation: automata, recurrent neural networks, and dynamical fuzzy systems*, Proceedings of IEEE, **87** (1999), 1623-1640.

[14] M. M. Gupta, G. N. Saridis and B. R. Gaines, *Fuzzy Automata and Decision Processes*, North Holland, New York, (1977), 111-175.

[15] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic, Kluwer, Dordercht, **4** (1998).

[16] J. Hgberg, A. Maletti and J. May, *Backward and forward bisimulation minimisation of tree automata*, In: J. Holub, J. drek (Eds.), IAA07, in: Lecture Notes in Computer Science, **4783** (2007), 109-121.

[17] J. Hgberg, A. Maletti and J. May, *Backward and forward bisimulation minimisation of tree automata*, Theoretical Computer Science, **410** (2009) 3539-3552.
[18] D. C. Kozen, *Automata and computability*, Springer, USA, 1997.
[19] E. T. Lee and L. A. Zadeh, *Note on fuzzy languages*, Information Sciences, **1** (1969), 421-434.
[20] L. Li and D. Qiu, *On the state minimization of fuzzy automata*, IEEE Transactions on Fuzzy Systems, **23** (2015), 434-443.
[21] N. Lynch and F. Vaandrager, *Forward and backward simulations*, Information and Computation, **121** (1995), 214-233.
[22] D. S. Malik and J. N. Mordeson, *Fuzzy Automata and Languages: Theory and Applications*, Chapman Hall, CRC Boca Raton, London, New York, Washington DC, 2002.
[23] D. S. Malik and J. N. Mordeson, *Fuzzy discrete structures*, Physica-Verlag, New York, (2000), London, 2002.
[24] R. Milner, *Acalculus of communicating systems*, In: G. Goos, J. Hartmanis (Eds.), Lecture Notes in Computer Science, Springer, **92** (1980).
[25] C. W. Omlin, K. K. Thornber and C. L. Giles, *Fuzzy finite-state automata can be deterministically encoded in recurrent neural networks*, IEEE Transactions on Fuzzy Systems, **5** (1998), 76-89.
[26] D. Park, *Concurrency and automata on infinite sequences*, In: P.Deussen(Ed.), Proceedings of the 5th GI Conference, Karlsruhe, Germany, Lecture Notesin Computer Science, Springer-Verlag, **104**, (1981), 167-183.
[27] W. Pedrycz and A. Gacek, *Learning of fuzzy automata*, International Journal of Computational Intelligence and Applications, **1** (2001), 19-33.
[28] K. Peeva, *Behavior, reduction and minimization of finite L-automata*, Fuzzy Sets and Systems, **28** (1988), 171-181.
[29] K. Peeva, *Equivalence, reduction and minimization of finite automata over semirings*, Theoretical Computer Science, **88** (1991), 269-285.
[30] D. Qiu, *Automata theory based on complete residuated lattice-valued logic*, Science in China Series: Information Sciences, **44** (2001), 419-429.
[31] D. Qiu, *Automata theory based on complete residuated lattice-valued logic (II)*, Science in China Series F: Information Sciences, **45** (2002), 442-452.
[32] D. Qiu, *Characterizations of fuzzy finite automata*, Fuzzy Sets and Systems, **141** (2004), 391-414.
[33] D. Qiu, *Pumping lemma in automata theory based on complete residuated lattice-valued logic: A not*, Fuzzy Sets and Systems, **157** (2006), 2128-2138.
[34] D. Qiu, *Supervisory control of fuzzy discrete event systems: a formal approach*, IEEE Transactions on Systems, Man and CyberneticsPart B, **35** (2005), 72-88.
[35] E. S. Santos, *Maxmin automata*, Information Control, **13** (1968), 363-377.
[36] V. Topencharov and K. Peeva, *Equivalence, reduction and minimization of finite fuzzy automata*, Journal of Mathematical Analysis and Applications, **84** (1981), 270-281.
[37] E. Turunen, *Boolean deductive systems of BL-algebras*, Archive for Mathematical Logic, **40** (2001), 467-473.
[38] W. G. Wee, *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification*, Ph.D. Thesis, Purdue University, Lafayette, IN, 1967.
[39] W. G. Wee and K. S. Fu, *A formulation of fuzzy automata and its application as a model of learning systems*, IEEE Transactions on Systems, Man and Cybernetics, **5** (1969), 215-223.
[40] L. Wu and D. Qiu, *Automata theory based on complete residuated lattice-valued logic: Reduction and minimization*, Fuzzy Sets and Systems, **161** (2010), 1635-1656.
[41] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965,) 338-353.

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