CHARACTERIZATIONS OF L-CONVEX SPACES

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Abstract. In this paper, the concepts of L-concave structures, concave L-interior operators and concave L-neighborhood systems are introduced. It is shown that the category of L-concave spaces and the category of concave L-interior spaces are isomorphic, and they are both isomorphic to the category of concave L-neighborhood systems whenever L is a completely distributive lattice. Also, it is proved that these categories are all isomorphic to the category of L-convex spaces whenever L is a completely distributive lattice with an order-reversing involution operator.

1. **Introduction**

Abstract convexity theory [20, 23] is an important branch of mathematics. As in the situation of topology theory, convexity theory usually deals with set-theoretic structures satisfying axioms similar to that usual convex sets fulfill. Convexities exist in many different mathematical research areas, such as convexities in lattices and in Boolean algebras [22, 24], convexities in metric spaces and graphs [8, 11, 19]. Especially, convexities appear naturally in topology and possess many topological properties, such as product spaces [4, 5], convex invariants [9, 16, 17, 18, 21] and separation [2, 3, 6, 7]. Like closure operators in topology, hull operators (also called algebraic closure operators) can be used to characterize convex structures [23].

With the development of fuzzy mathematics, convex structures have been endowed with fuzzy set theory. In [10] and [14], the authors independently introduced the concept of L-convex structures. Based on L-convex structures, Pang and Shi [12] proposed several types of L-convex structures and investigated their categorical relations. Since algebraic closure operators cannot be directly generalized to L-convex spaces, there have not been algebraic L-closure operators. That is to say, there have not been suitable axiomatic definition for algebraic L-closure operators in the framework of L-convex spaces. Recently, Shi and Xiu [15] provided a new approach to fuzzification of convex structures and proposed the notion of M-fuzzifying convex structures. In this environment, algebraic M-fuzzifying closure operators are introduced and they are shown to be equivalent to M-fuzzifying convex structures. Inspired by this, Pang and Shi [13] introduced axiomatic L-hull operators to characterize L-convex structures.

In the theory of L-topology, there are several characterizations of L-topologies, including L-closure operators, L-interior operators and L-neighborhood systems.

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As we all know, L-convex structures possess so many characters of L-topologies. This motivates us to consider the corresponding characterizations of L-convex structures. In this paper, we will firstly introduce the dual concept of L-convex structures, which is called L-concave structures. And then we will consider the corresponding fuzzy interior operators and fuzzy neighborhood systems.

2. **Preliminaries**

Throughout this paper, let L denote a complete lattice. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. For $a, b \in L$, we say that a is wedge below b in L, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geqslant b$ implies $d \geqslant a$ for some $d \in D$. An element a in L is called coprime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of nonzero coprime elements in L is denoted by $J(L)$. Let $\beta(a) = \{b \in L \mid b \prec a\}$ and $\beta^*(a) = \beta(a) \cap J(L)$. A complete lattice L is completely distributive if and only if $a = \bigvee \beta(a)$ (or $a = \bigvee \beta^*(a)$) for each $a \in L$. Also, we adopt the convention that $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. L^X is also a complete lattice when it inherits the structure of the lattice L in a natural way, by defining $\bigvee$, $\bigwedge$ and $\leqslant$ pointwisely. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively. We say $\{A_j\}_{j\in J}$ is a directed (resp. co-directed) subset of L^X, in symbols $\{A_j\}_{j\in J} \overset{dir}{\subseteq} L^X$ (resp. $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} L^X$), if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_1}, A_{j_2} \leqslant A_{j_3}$ (resp. $A_{j_1}, A_{j_2} \geqslant A_{j_3}$).

Let X, Y be two nonempty sets and $\varphi : X \longrightarrow Y$ be a mapping. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B \circ \varphi$ for $B \in L^Y$, respectively.

Definition 2.1. [10, 14] An L-convex structure $\mathcal{E}$ on X is a subset of L^X which satisfies:

(LC1) $\underline{\perp}, \underline{\top} \in \mathcal{E}$;

(LC2) $\{A_k\}_{k\in K} \subseteq \mathcal{E}$ implies $\bigwedge_{k\in K} A_k \in \mathcal{E}$;

(LC3) If $\{A_j\}_{j\in J} \subseteq \mathcal{E}$ is totally ordered, then $\bigvee_{j\in J} A_j \in \mathcal{E}$.

For an L-convex structure $\mathcal{E}$ on X, the pair $(X, \mathcal{E})$ is called an L-convex space.

For partially ordered sets, in [1], the researchers present the following result.

Proposition 2.2. *A partially ordered set D is closed for directed unions if and only if each chain in D has a supremum.*

Obviously, for an L-convex structure $\mathcal{E}$ on X, since the order "$\leqslant$" on L^X is defined pointwisely, $(\mathcal{E}, \leqslant)$ is also a partially ordered set. Hence, we obtain the following result.

Proposition 2.3. *Let $\mathcal{E}$ be an L-closure system on X, that is, $\mathcal{E} \subseteq L^X$ satisfies* (LC1) *and* (LC2)*. Then the following statements are equivalent.*

(LC3) *If $\{A_j\}_{j\in J} \subseteq \mathcal{E}$ is totally ordered, then $\bigvee_{j\in J} A_j \in \mathcal{E}$.*

(LC3)$'$ *$\{A_j\}_{j\in J} \overset{dir}{\subseteq} \mathcal{E}$ implies $\bigvee_{j\in J} A_j \in \mathcal{E}$.*

In the sequel, we usually adopt (LC3)$'$ instead of (LC3).

Definition 2.4. [13] A mapping $\varphi : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ between L-convex spaces is called L-convexity-preserving (L-CEP, in short) provided that $B \in \mathcal{E}_Y$ implies $\varphi^{\leftarrow}(B) \in \mathcal{E}_X$.

It is easy to check that all L-convex spaces as objects and all L-CEP mappings as morphisms form a category, denoted by L-**CE**.

In [13], Pang and Shi provided a characterization of L-convex structures by L-hull operators, which is called convex L-closure operators in this paper.

Definition 2.5. [13] A convex L-closure operator on X is a mapping $Cl : L^X \longrightarrow L^X$ which satisfies:

(CLC1) $Cl(\underline{\perp}) = \underline{\perp}$;
(CLC2) $A \leqslant Cl(A)$;
(CLC3) $Cl(Cl(A)) = Cl(A)$;
(CLC4) $Cl(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} Cl(A_j)$ for $\{A_j \mid j \in J\} \overset{dir}{\subseteq} L^X$.

For a convex L-closure operator Cl on X, the pair (X, Cl) is called a convex L-closure space.

Definition 2.6. [13] A mapping $\varphi : (X, Cl_X) \longrightarrow (Y, Cl_Y)$ between convex L-closure spaces is called L-closure-preserving (L-CLP, in short) provided that

$$\forall A \in L^X, \quad \varphi^{\rightarrow}(Cl_X(A)) \leqslant Cl_Y(\varphi^{\rightarrow}(A)).$$

It is easy to check that all convex L-closure spaces as objects and all L-CLP mappings as morphisms form a category, denoted by L-**CLC**.

Theorem 2.7. [13] L-**CE** *is isomorphic to* L-**CLC**.

3. L-concave Spaces and Concave L-interior Spaces

In this section, we will introduce the concept of L-concave spaces, which a dual concept of L-convex spaces. Then we will propose the concept of concave L-interior spaces to characterize L-concave spaces.

Firstly, we give the definitions of L-concave structures and concave L-interior operators.

Definition 3.1. An L-concave structure $\mathcal{A}$ on X is a subset of L^X which satisfies:

(LA1) $\underline{\perp}, \underline{\top} \in \mathcal{A}$;
(LA2) $\{A_k\}_{k\in K} \subseteq \mathcal{A}$ implies $\bigvee_{k\in K} A_k \in \mathcal{A}$;
(LA3) $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} \mathcal{A}$ implies $\bigwedge_{j\in J} A_j \in \mathcal{A}$.

For an L-concave structure $\mathcal{A}$ on X, the pair $(X, \mathcal{A})$ is called an L-concave space.

Definition 3.2. A mapping $\varphi : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is called L-concavity-preserving (L-CAP, in short) provided that $B \in \mathcal{A}_Y$ implies $\varphi^{\leftarrow}(B) \in \mathcal{A}_X$.

It is easy to check that all L-concave spaces as objects and all L-CAP mappings as morphisms form a category, denoted by L-**CA**.

Definition 3.3. A concave L-interior operator on X is a mapping $Int : L^X \longrightarrow L^X$ which satisfies:

(CLI1) $Int(\underline{\top}) = \underline{\top}$;
(CLI2) $Int(A) \leqslant A$;
(CLI3) $Int(Int(A)) = Int(A)$;
(CLI4) $Int(\bigwedge_{j\in J} A_j) = \bigwedge_{j\in J} Int(A_j)$ for $\{A_j \mid j \in J\} \overset{cdir}{\subseteq} L^X$.

For a concave L-interior operator Int on X, the pair (X, Int) is called a concave L-interior space.

Remark 3.4. By (CLI4), it is easy to check that Int is order preserving. That is to say, for $A, B \in L^X$, $A \leqslant B$ implies $Int(A) \leqslant Int(B)$.

Definition 3.5. A mapping $\varphi : (X, Int_X) \longrightarrow (Y, Int_Y)$ between concave L-interior spaces is called L-interior-preserving (L-INP, in short) provided that

$$\forall B \in L^Y, \quad \varphi^{\leftarrow}(Int_Y(B)) \leqslant Int_X(\varphi^{\leftarrow}(B)).$$

It is easy to check that all concave L-interior spaces as objects and all L-INP mappings as morphisms form a category, denoted by L-**CLI**.

Next we will establish the relations between L-concave spaces and concave L-interior spaces.

Proposition 3.6. *Let (X, Int) be a concave L-interior space and define $\mathcal{A}^{Int} \subseteq L^X$ by*

$$\mathcal{A}^{Int} = \{A \in L^X \mid A = Int(A)\}.$$

Then $\mathcal{A}^{Int}$ is an L-concave structure on X.

Proof. It is enough to verify that $\mathcal{A}^{Int}$ satisfies (LA1)–(LA3). In fact,

(LA1) Since $\underline{\top} = Int(\underline{\top})$ and $Int(\underline{\bot}) \leqslant \underline{\bot}$, we know $\underline{\bot}, \underline{\top} \in \mathcal{A}^{Int}$.

(LA2) Take any $\{A_k \mid k \in K\} \subseteq \mathcal{A}^{Int}$. Then $A_k = Int(A_k)$ for each $k \in K$. Since Int is order preserving, we have $Int(\bigvee_{k\in K} A_k) \geqslant Int(A_k)$ for each $k \in K$. Then it follows that

$$Int\left(\bigvee_{k\in K} A_k\right) \geqslant \bigvee_{k\in K} Int(A_k) = \bigvee_{k\in K} A_k.$$

By (CLI2), the inverse inequality $Int(\bigvee_{k\in K} A_k) \leqslant \bigvee_{k\in K} A_k$ holds obviously. Thus, $Int(\bigvee_{k\in K} A_k) = \bigvee_{k\in K} A_k$. This means $\bigvee_{k\in K} A_k \in \mathcal{A}^{Int}$.

(LA3) Take any $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} \mathcal{A}^{Int}$. It follows that $A_j = Int(A_j)$ for each $j \in J$. By (CLI4), we have

$$Int\left(\bigwedge_{j\in J} A_j\right) = \bigwedge_{j\in J} Int(A_j) = \bigwedge_{j\in J} A_j.$$

This means $\bigwedge_{j\in J} A_j \in \mathcal{A}^{Int}$, as desired. □

Proposition 3.7. *If $\varphi : (X, Int_X) \longrightarrow (Y, Int_Y)$ is L-INP, then $\varphi : (X, \mathcal{A}^{Int_X}) \longrightarrow (Y, \mathcal{A}^{Int_Y})$ is L-CAP.*

Proof. Since $\varphi : (X, Int_X) \longrightarrow (Y, Int_Y)$ is L-INP, it follows that

$$\forall B \in L^Y,\ \varphi^{\leftarrow}(Int_Y(B)) \leqslant Int_X(\varphi^{\leftarrow}(B)).$$

Then for each $C \in \mathcal{A}^{Int_Y}$, that is, $C = Int_Y(C)$, we have

$$Int_X(\varphi^{\leftarrow}(C)) \geqslant \varphi^{\leftarrow}(Int_Y(C)) = \varphi^{\leftarrow}(C).$$

By (LCI2), we have $Int_X(\varphi^{\leftarrow}(C)) = \varphi^{\leftarrow}(C)$. This implies $\varphi^{\leftarrow}(C) \in \mathcal{A}^{Int_X}$. Hence, $\varphi : (X, \mathcal{A}^{Int_X}) \longrightarrow (Y, \mathcal{A}^{Int_Y})$ is L-CAP. $\square$

By Propositions 3.6 and 3.7, we obtain a functor $\mathbb{A}_{\mathbb{I}} : L$-**CLI**$\longrightarrow L$-**CA** by

$$\mathbb{A}_{\mathbb{I}} : \begin{cases} L\text{-}\mathbf{CLI} & \longrightarrow & L\text{-}\mathbf{CA}, \\ (X, Int) & \longmapsto & (X, \mathcal{A}^{Int}), \\ \varphi & \longmapsto & \varphi. \end{cases}$$

Proposition 3.8. *Let $(X, \mathcal{A})$ be an L-concave space and define $Int^{\mathcal{A}} : L^X \longrightarrow L^X$ by*

$$\forall A \in L^X,\ Int^{\mathcal{A}}(A) = \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}\}.$$

Then $Int^{\mathcal{A}}$ is a concave L-interior operator on X.

Proof. It is enough to show that $Int^{\mathcal{A}}$ satisfies (CLI1)–(CLI4). (CLI1) and (CLI2) are obvious. We need only verify (CLI3) and (CLI4).

(CLI3) By (LA2), we know $Int^{\mathcal{A}}(A) = \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}\} \in \mathcal{A}$. This implies $Int^{\mathcal{A}}(A) \leqslant Int^{\mathcal{A}}(Int^{\mathcal{A}}(A))$. Since the inverse inequality $Int^{\mathcal{A}}(A) \geqslant Int^{\mathcal{A}}(Int^{\mathcal{A}}(A))$ holds obviously, we obtain $Int^{\mathcal{A}}(A) = Int^{\mathcal{A}}(Int^{\mathcal{A}}(A))$.

(CLI4) By the definition of $Int^{\mathcal{A}}$, we know $Int^{\mathcal{A}}$ is order preserving. That is to say, $A \leqslant B$ implies $Int^{\mathcal{A}}(A) \leqslant Int^{\mathcal{A}}(B)$. This means $Int^{\mathcal{A}}\left(\bigwedge_{j\in J} A_j\right) \leqslant \bigwedge_{j\in J} Int^{\mathcal{A}}(A_j)$. Since $\{A_j\}_{j\in J}$ is co-directed, it follows that $\{Int^{\mathcal{A}}(A_j)\}_{j\in J}$ is co-directed. Further, by (LA2), it follows that $Int^{\mathcal{A}}(A) = \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}\} \in \mathcal{A}$. This implies $\bigwedge_{j\in J} Int^{\mathcal{A}}(A_j) \in \mathcal{A}$. As $\bigwedge_{j\in J} Int^{\mathcal{A}}(A_j) \leqslant \bigwedge_{j\in J} A_j$, we have $\bigwedge_{j\in J} Int^{\mathcal{A}}(A_j) \leqslant Int^{\mathcal{A}}\left(\bigwedge_{j\in J} A_j\right)$. This proves $\bigwedge_{j\in J} Int^{\mathcal{A}}(A_j) = Int^{\mathcal{A}}\left(\bigwedge_{j\in J} A_j\right)$, as desired. $\square$

Proposition 3.9. *If $\varphi : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is L-CAP, then $\varphi : (X, Int^{\mathcal{A}_X}) \longrightarrow (Y, Int^{\mathcal{A}_Y})$ is L-INP.*

Proof. Since $\varphi : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is L-CAP, it follows that

$$\forall B \in L^Y,\ B \in \mathcal{A}_Y \text{ implies } \varphi^{\leftarrow}(B) \in \mathcal{A}_X.$$

Then we have

$$\begin{aligned}
& \varphi^{\leftarrow}(Int^{\mathcal{A}_Y}(B)) \\
= \; & \varphi^{\leftarrow}(\bigvee\{C \in L^Y \mid C \leqslant B, C \in \mathcal{A}_Y\}) \\
\leqslant \; & \bigvee\{\varphi^{\leftarrow}(C) \in L^X \mid \varphi^{\leftarrow}(C) \leqslant \varphi^{\leftarrow}(B), \varphi^{\leftarrow}(C) \in \mathcal{A}_X\} \\
\leqslant \; & \bigvee\{A \in L^X \mid A \leqslant \varphi^{\leftarrow}(B), A \in \mathcal{A}_X\} \\
= \; & Int^{\mathcal{A}_X}(\varphi^{\leftarrow}(B)).
\end{aligned}$$

This proves that $\varphi : (X, Int^{\mathcal{A}_X}) \longrightarrow (Y, Int^{\mathcal{A}_Y})$ is L-INP. $\square$

By Propositions 3.8 and 3.9, we obtain a functor $\mathbb{I}_{\mathbb{A}} : L$-**CA**$\longrightarrow L$-**CLI** by

$$\mathbb{I}_{\mathbb{A}} : \begin{cases} L\text{-}\mathbf{CA} & \longrightarrow & L\text{-}\mathbf{CLI}, \\ (X, \mathcal{A}) & \longmapsto & (X, Int^{\mathcal{A}}), \\ \varphi & \longmapsto & \varphi. \end{cases}$$

Theorem 3.10. *L-**CLI** is isomorphic to L-**CA**.*

Proof. It suffices to show that $\mathbb{I}_{\mathbb{A}} \circ \mathbb{A}_{\mathbb{I}} = \mathbb{I}_{L-\mathbf{CLI}}$ and $\mathbb{A}_{\mathbb{I}} \circ \mathbb{I}_{\mathbb{A}} = \mathbb{I}_{L-\mathbf{CA}}$. That is to say, we need only verify (1) $Int^{\mathcal{A}^{Int}} = Int$ and (2) $\mathcal{A}^{Int^{\mathcal{A}}} = \mathcal{A}$.

For (1), take each $A \in L^X$. Then

$$\begin{aligned} Int^{\mathcal{A}^{Int}}(A) &= \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}^{Int}\} \\ &= \bigvee\{B \in L^X \mid B \leqslant A, B = Int(B)\}. \end{aligned}$$

On one hand, take each $B \in L^X$ such that $B \leqslant A$ and $B = Int(B)$, we have $B = Int(B) \leqslant Int(A)$. This means $Int^{\mathcal{A}^{Int}}(A) \leqslant Int(A)$. On the other hand, let $B = Int(A)$. Then $B \leqslant A$ and $Int(B) = Int(Int(A)) = Int(A) = B$. This means that $Int(A) \leqslant Int^{\mathcal{A}^{Int}}(A)$, as desired.

(2) Take each $A \in L^X$. Then

$$\begin{aligned} & A \in \mathcal{A}^{Int^{\mathcal{A}}} \\ \Longleftrightarrow\ & A = Int^{\mathcal{A}}(A) = \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}\} \\ \Longleftrightarrow\ & A = \bigvee\{B \in L^X \mid B \leqslant A, B \in \mathcal{A}\} \\ \Longleftrightarrow\ & A \in \mathcal{A}. \end{aligned}$$

This means $\mathcal{A}^{Int^{\mathcal{A}}} = \mathcal{A}$. $\square$

Theorem 3.11. *Suppose that L is equipped with an order-reversing involution operator $'$. Then L-**CE** is isomorphic to L-**CA**.*

Proof. Given an L-convex space $(X, \mathcal{E})$, define $\mathcal{A}^{\mathcal{E}} \subseteq L^X$ by

$$\mathcal{A}^{\mathcal{E}} = \{A \in L^X \mid A' \in \mathcal{E}\}$$

Then $(X, \mathcal{A}^{\mathcal{E}})$ is an L-concave space. Similarly, given an L-concave space $(X, \mathcal{A})$, define $\mathcal{E}^{\mathcal{A}} \subseteq L^X$ by

$$\mathcal{E}^{\mathcal{A}} = \{A \in L^X \mid A' \in \mathcal{A}\}$$

Then $(X, \mathcal{E}^{\mathcal{A}})$ is an L-convex space. Since $'$ is an order-reversing involution operator on L, it can be easily checked that L-**CE** is isomorphic to L-**CA**. $\square$

By Theorems 2.7, 3.10 and 3.11, we obtain

Corollary 3.12. *Suppose that L is equipped with an order-reversing involution operator $'$. Then L-**CE**, L-**CA**, L-**CLC** and L-**CLI** are isomorphic.*

4. **Concave L-neighborhood Systems**

In this section, we introduce the concept of concave L-neighborhood systems and study its relations with L-concave structures. In this section, L is assumed to be a completely distributive lattice.

Definition 4.1. A concave L-neighborhood system on X is a set $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $\mathcal{N}_{x_\lambda} \subseteq L^X$ satisfies:

(CLN1) $\top_L^X \in \mathcal{N}_{x_\lambda}$, $\perp_L^X \notin \mathcal{N}_{x_\lambda}$;
(CLN2) $\forall A \in \mathcal{N}_{x_\lambda}$, $x_\lambda \leqslant A$;
(CLN3) $\forall A \in \mathcal{N}_{x_\lambda}$, $A \leqslant B$ implies $B \in \mathcal{N}_{x_\lambda}$;
(CLN4) $\forall \{A_j\}_{j\in J} \overset{cdir}{\subseteq} \mathcal{N}_{x_\lambda}$, $\bigwedge_{j\in J} A_j \in \mathcal{N}_{x_\lambda}$;
(CLN5) $A \in \mathcal{N}_{x_\lambda}$ if and only if $\exists B \in L^X$ s.t. $x_\lambda \leqslant B \leqslant A$ and $\forall y_\mu \prec B$, $B \in \mathcal{N}_{y_\mu}$.

For a concave L-neighborhood system $\mathcal{N}$ on X, the pair $(X, \mathcal{N})$ is called a concave L-neighborhood space.

Definition 4.2. A mapping $\varphi : (X, \mathcal{N}^X) \longrightarrow (Y, \mathcal{N}^Y)$ between concave L-neighborhood spaces is called L-neighborhood-preserving (L-NEP, in short) provided that

$$\forall x_\lambda \in J(L^X), \forall B \in L^Y, \quad B \in \mathcal{N}^Y_{\varphi(x)_\lambda} \text{ implies } \varphi^{\leftarrow}(B) \in \mathcal{N}^X_{x_\lambda}.$$

It is easy to check that all concave L-neighborhood spaces as objects and all L-NEP mappings as morphisms form a category, denoted by L-**CN**.

Next we will investigate the relations between L-**CN** and L-**CA**.

Proposition 4.3. *Let $(X, \mathcal{N})$ be a concave L-neighborhood space and define $\mathcal{A}^{\mathcal{N}} \subseteq L^X$ as follows:*

$$\mathcal{A}^{\mathcal{N}} = \{A \in L^X \mid \forall x_\lambda \prec A, A \in \mathcal{N}_{x_\lambda}\}.$$

Then $\mathcal{A}^{\mathcal{N}}$ is an L-concave structure on X.

Proof. It suffices to verify that $\mathcal{A}^{\mathcal{N}}$ satisfies (LA1), (LA2) and (LA3). In fact,

(LA1) Obviously.

(LA2) Take each $\{A_k\}_{k\in K} \subseteq \mathcal{A}^{\mathcal{N}}$. Then for each $k \in K$, it follows that $A_k \in \mathcal{N}_{x_\lambda}$ for each $x_\lambda \prec A_k$. For each $x_\lambda \in J(L^X)$ satisfying $x_\lambda \prec \bigvee_{k\in K} A_k$, there exists $k_0 \in K$ such that $x_\lambda \prec A_{k_0}$. This implies that $A_{k_0} \in \mathcal{N}_{x_\lambda}$. By (CLN3), we have $\bigvee_{k\in K} A_k \in \mathcal{N}_{x_\lambda}$. This shows for each $x_\lambda \prec \bigvee_{k\in K} A_k$, $\bigvee_{k\in K} A_k \in \mathcal{N}_{x_\lambda}$. Thus, we get $\bigvee_{k\in K} A_k \in \mathcal{A}^{\mathcal{N}}$.

(LA3) Take each $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} \mathcal{A}^{\mathcal{N}}$. Then for each $j \in J$, it follows that $A_j \in \mathcal{N}_{x_\lambda}$ for each $x_\lambda \prec A_j$. For each $x_\lambda \in J(L^X)$ satisfying $x_\lambda \prec \bigwedge_{j\in J} A_j$, we have $x_\lambda \prec A_j$ for each $j \in J$. This means $A_j \in \mathcal{N}_{x_\lambda}$ for each $j \in J$. By (CLN4), we obtain $\bigwedge_{j\in J} A_j \in \mathcal{N}_{x_\lambda}$. This proves that for each $x_\lambda \prec \bigwedge_{j\in J} A_j$, $\bigwedge_{j\in J} A_j \in \mathcal{N}_{x_\lambda}$. Thus, we obtain $\bigwedge_{j\in J} A_j \in \mathcal{A}^{\mathcal{N}}$. □

Proposition 4.4. *If $\varphi : (X, \mathcal{N}^X) \longrightarrow (Y, \mathcal{N}^Y)$ is L-NEP, then $\varphi : (X, \mathcal{A}^{\mathcal{N}^X}) \longrightarrow (Y, \mathcal{A}^{\mathcal{N}^Y})$ is L-CAP.*

Proof. Since $\varphi : (X, \mathcal{N}^X) \longrightarrow (Y, \mathcal{N}^X)$ is L-NEP, it follows that

$$\forall x_\lambda \in J(L^X), \forall B \in L^Y,\ B \in \mathcal{N}^Y_{\varphi(x)_\lambda} \text{ implies } \varphi^{\leftarrow}(B) \in \mathcal{N}^X_{x_\lambda}.$$

Then for each $C \in L^Y$, we obtain

$$\begin{aligned} C \in \mathcal{A}^{\mathcal{N}^Y} &\iff \forall y_\mu \prec C,\ C \in \mathcal{N}^Y_{y_\mu} \\ &\Longrightarrow \forall \varphi(x)_\lambda \prec C,\ C \in \mathcal{N}^Y_{\varphi(x)_\lambda} \\ &\Longrightarrow \forall x_\lambda \prec \varphi^{\leftarrow}(C), \varphi^{\leftarrow}(C) \in \mathcal{N}^X_{x_\lambda} \\ &\iff \varphi^{\leftarrow}(C) \in \mathcal{A}^{\mathcal{N}^X}. \end{aligned}$$

This proves that $\varphi : (X, \mathcal{A}^{\mathcal{N}_X}) \longrightarrow (Y, \mathcal{A}^{\mathcal{N}_Y})$ is L-CAP. □

By Propositions 4.3 and 4.4, we obtain a functor $\mathbb{A}_{\mathbb{N}} : L$-**CN**$\longrightarrow L$-**CA** by

$$\mathbb{A}_{\mathbb{N}} : \begin{cases} L\text{-}\mathbf{CN} & \longrightarrow & L\text{-}\mathbf{CA}, \\ (X, \mathcal{N}) & \longmapsto & (X, \mathcal{A}^{\mathcal{N}}), \\ \varphi & \longmapsto & \varphi. \end{cases}$$

Proposition 4.5. *Let $(X, \mathcal{A})$ be an L-concave space and define $\mathcal{N}^{\mathcal{A}}_{x_\lambda} \subseteq L^X$ by*

$$\mathcal{N}^{\mathcal{A}}_{x_\lambda} = \{A \in L^X \mid \exists B \in \mathcal{A} \text{ s.t. } x_\lambda \leqslant B \leqslant A\}.$$

Then $\mathcal{N}^{\mathcal{A}} = \{\mathcal{N}^{\mathcal{A}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is a concave L-neighborhood system on X.

Proof. It is enough to show that $\mathcal{N}^{\mathcal{A}}$ satisfies (CLN1)–(CLN5). (CLN1)–(CLN3) are obvious.

(CLN4) Take $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} \mathcal{N}^{\mathcal{A}}_{x_\lambda}$. Then for each $j \in J$, there exists $B_j \in \mathcal{A}$ such that $x_\lambda \leqslant B_j \leqslant A_j$. By (LA2), we know $Int^{\mathcal{A}}(A_j) \in \mathcal{A}$ and $x_\lambda \leqslant B_j \leqslant Int^{\mathcal{A}}(A_j) \leqslant A_j$. Since $Int^{\mathcal{A}}$ is order preserving, we obtain $\{Int^{\mathcal{A}}(A_j) \mid j \in J\} \overset{cdir}{\subseteq} \mathcal{A}$. Put $B = \bigwedge_{j\in J} Int^{\mathcal{A}}(A_j)$. By (LA3), it follows that $B \in \mathcal{A}$. Further, $x_\lambda \leqslant B \leqslant \bigwedge_{j\in J} A_j$. Thus, we get $\bigwedge_{j\in J} A_j \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$.

(CLN5) *Necessity.* Take each $A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$. Then there exists $B \in \mathcal{A}$ such that $x_\lambda \leqslant B \leqslant A$. Since $B \in \mathcal{A}$, for each $y_\mu \prec B$, there exists $C(= B) \in \mathcal{A}$ such that $y_\mu \leqslant C \leqslant B$. This means $B \in \mathcal{N}^{\mathcal{A}}_{y_\mu}$ for each $y_\mu \prec B$, as desired.

Sufficiency. We first prove the following result.

$$\text{(CLN0)} \quad A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda} \iff \forall \mu \in \beta^*(\lambda), A \in \mathcal{N}^{\mathcal{A}}_{x_\mu}.$$

On one hand, if $A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$, then there exists $B \in \mathcal{A}$ such that $x_\lambda \leqslant B \leqslant A$. Take each $\mu \in \beta^*(\lambda)$. Then $x_\mu \prec x_\lambda \leqslant B \leqslant A$. This implies $A \in \mathcal{N}^{\mathcal{A}}_{x_\mu}$ for $\mu \in \beta^*(\lambda)$.

On the other hand, since $A \in \mathcal{N}^{\mathcal{A}}_{x_\mu}$ for $\mu \in \beta^*(\lambda)$, it follows that for each $\mu \in \beta^*(\lambda)$, there exists $B_\mu \in \mathcal{A}$ such that $x_\mu \leqslant B_\mu \leqslant A$. Let $B = \bigvee_{\mu\in\beta^*(\lambda)} B_\mu$. Then by (LA2), we have $B \in \mathcal{A}$. Further,

$$x_\lambda = \bigvee_{\mu\in\beta^*(\lambda)} x_\mu \leqslant \bigvee_{\mu\in\beta^*(\lambda)} B_\mu = B \leqslant A.$$

This implies that $A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$.

Now we know there exists $B \in L^X$ such that $x_\lambda \leqslant B \leqslant A$ and for each $y_\mu \prec B$, $B \in \mathcal{N}^{\mathcal{A}}_{y_\mu}$. Thus, for each $\mu \in \beta^*(\lambda)$, it follows that $x_\mu \prec x_\lambda \leqslant B$. This means $B \in \mathcal{N}^{\mathcal{A}}_{x_\mu}$. By (CLN3), we know $A \in \mathcal{N}^{\mathcal{A}}_{x_\mu}$. By (CLN0), we get $A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$. This proves the sufficiency of (CLN5). □

Proposition 4.6. *If $\varphi : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is L-CAP, then $\varphi : (X, \mathcal{N}^{\mathcal{A}_X}) \longrightarrow (Y, \mathcal{N}^{\mathcal{A}_Y})$ is L-NEP.*

Proof. Since $\varphi : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is L-CAP, it follows that

$$\forall B \in L^Y,\ B \in \mathcal{A}_Y \text{ implies } \varphi^{\leftarrow}(B) \in \mathcal{A}_X.$$

Then for each $x_\lambda \in J(L^X)$ and $B \in L^Y$, we have

$$\begin{aligned} & B \in \mathcal{N}^{\mathcal{A}_Y}_{\varphi(x)_\lambda} \\ \Longleftrightarrow\ & \exists C \in \mathcal{A}_Y \ s.t.\ \varphi(x)_\lambda \leqslant C \leqslant B \\ \Longrightarrow\ & \exists \varphi^{\leftarrow}(C) \in \mathcal{A}_X \ s.t.\ x_\lambda \leqslant \varphi^{\leftarrow}(C) \leqslant \varphi^{\leftarrow}(B) \\ \Longrightarrow\ & \exists A \in \mathcal{A}_X \ s.t.\ x_\lambda \leqslant A \leqslant \varphi^{\leftarrow}(B) \\ \Longleftrightarrow\ & \varphi^{\leftarrow}(B) \in \mathcal{N}^{\mathcal{A}_X}_{x_\lambda}. \end{aligned}$$

This shows that $\varphi : (X, \mathcal{N}^{\mathcal{A}_X}) \longrightarrow (Y, \mathcal{N}^{\mathcal{A}_Y})$ is L-NEP. □

By Propositions 4.5 and 4.6, we obtain a functor $\mathbb{N}_{\mathbb{A}} : L$-**CA**$\longrightarrow L$-**CN** by

$$\mathbb{N}_{\mathbb{A}} : \begin{cases} L\text{-}\mathbf{CA} & \longrightarrow & L\text{-}\mathbf{CN}, \\ (X, \mathcal{A}) & \longmapsto & (X, \mathcal{N}^{\mathcal{A}}), \\ \varphi & \longmapsto & \varphi. \end{cases}$$

Next we present the main result in this section.

Theorem 4.7. L-**CN** *is isomorphic to* L-**CA**.

Proof. It suffices to show that $\mathbb{N}_{\mathbb{A}} \circ \mathbb{A}_{\mathbb{N}} = \mathbb{I}_{L-\mathbf{CLI}}$ and $\mathbb{A}_{\mathbb{N}} \circ \mathbb{N}_{\mathbb{A}} = \mathbb{I}_{L-\mathbf{CA}}$. That is to say, we need only verify (1) $\mathcal{N}^{\mathcal{A}^{\mathcal{N}}} = \mathcal{N}$ and (2) $\mathcal{A}^{\mathcal{N}^{\mathcal{A}}} = \mathcal{A}$.

For (1), take each $A \in L^X$. Then

$$\begin{aligned} A \in \mathcal{N}^{\mathcal{A}^{\mathcal{N}}}_{x_\lambda} \ &\Longleftrightarrow\ \exists B \in \mathcal{A}^{\mathcal{N}} \ s.t.\ x_\lambda \leqslant B \leqslant A \\ &\Longleftrightarrow\ \exists B \in L^X \ s.t.\ x_\lambda \leqslant B \leqslant A \ and\ \forall y_\mu \prec B,\ B \in \mathcal{N}_{y_\mu} \\ &\Longleftrightarrow\ A \in \mathcal{N}_{x_\lambda}.\ \text{ (by (CLN5))} \end{aligned}$$

This shows $\mathcal{N}^{\mathcal{A}^{\mathcal{N}}} = \mathcal{N}$.

(2) Take each $A \in L^X$. Then

$$\begin{aligned} & A \in \mathcal{A}^{\mathcal{N}^{\mathcal{A}}} \\ \Longleftrightarrow\ & \forall x_\lambda \prec A, A \in \mathcal{N}^{\mathcal{A}}_{x_\lambda} \\ \Longleftrightarrow\ & \forall x_\lambda \prec A, \exists B \in \mathcal{A} \ s.t.\ x_\lambda \leqslant B \leqslant A \\ \Longleftrightarrow\ & A = \bigvee_{x_\lambda \prec A} \bigvee_{x_\lambda \leqslant B \leqslant A} B \\ \Longleftrightarrow\ & A \in \mathcal{A}. \end{aligned}$$

This means $\mathcal{A}^{\mathcal{N}^{\mathcal{A}}} = \mathcal{A}$. $\square$

Corollary 4.8. *Suppose that L is a completely distributive lattice with an order-reversing involution operator $'$. Then L-**CE**, L-**CA**, L-**CLC**, L-**CLI** and L-**CN** are isomorphic.*

5. Conclusions

In this paper, we provided several characterizations of L-convex structures from a categorical viewpoint, including L-concave structures, concave L-interior operators and concave L-neighborhood systems. As in the situation of L-topology, all these resulting categories are isomorphic whenever L is a completely distributive lattice with an order-reversing involution operator. Based on these new concepts, we will consider "topological" properties of L-convex spaces, such as compactness and separation property.

In the classical case, convex structures can be characterized by algebraic closure operators (also called domain finite closure operators) in [23]. Convex L-closure operators in Definition 2.5 are not direct generalization of algebraic closure operators. Since algebraic closure operators cannot be generalized to the fuzzy case directly, we firstly found an equivalent form of L-convex spaces in Proposition 2.3 and then used this equivalent form to propose Definition 2.5. Consequently, the concept of concave L-interior operators can also be defined dually and the concept of concave L-neighborhood systems can be proposed. In the future, we will consider how to define algebraic L-closure operators in the framework of L-convex spaces and study its relations with convex L-closure operators.

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References

[1] S. Abramsky and A. Jung, *Domain theory*, S. Abramsky, D. Gabbay, T.S.E. Mailbaum (Eds.), Handbook of Logic in Computer Science, Oxford University Press, Oxford (1994), 1–168.
[2] V. Chepoi, *Separation of two convex sets in convexity structures*, J. Geom., **50** (1994), 30–51.
[3] E. Ellis, *A general set-separation theorem*, Duke Math. J., **19** (1952), 417–421.
[4] J. Eckhoff, *Radon's theorem in convex product structures I*, Monatsh. Math., **72** (1968), 303–314.
[5] J. Eckhoff, *Radon's theorem in convex product structures II*, Monatsh. Math., **73** (1969), 17–30.
[6] R. E. Jamison, *A general theory of convexity*, Dissertation, University of Washington, Seattle, Washington, 1974.
[7] D. C. Kay and E. W. Womble, *Axiomatic convexity theory and the relationship between the Carathéodory, Helly and Radon numbers*, Pacific J. Math., **38** (1971), 471–485.
[8] M. Lassak, *On metric B-convexity for which diameters of any set and its hull are equal*, Bull. Acad. Polon. Sci., **25** (1977), 969–975.

[9] F. W. Levi, *On Helly's theorem and the axioms of convexity*, J. Indian Math. Soc., **15** Part A (1951), 65–76.
[10] Y. Maruyama, *Lattice-valued fuzzy convex geometry*, RIMS Kokyuroku, **164** (2009), 22–37.
[11] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann., **100**(1928), 75–163.
[12] B. Pang and F. G. Shi, *Subcategories of the category of L-convex spaces*, Fuzzy Sets Syst., (2016), http://dx.doi.org/10.1016/j.fss.2016.02.014.
[13] B. Pang and F. G. Shi, *L-hull operators and L-interval operators in L-convex spaces*, Submitted.
[14] M. V. Rosa, *On fuzzy topology fuzzy convexity spaces and fuzzy local convexity*, Fuzzy Sets Syst., **62** (1994), 97–100.
[15] F. G. Shi and Z. Y. Xiu, *A new approach to the fuzzification of convex structures*, J. Appl. Math., **2014** (2014), 12 pages.
[16] G. Sierkama, *Carathéodory and Helly-numbers of convex-product-structures*, Pacific J. Math., **61** (1975), 272–282.
[17] G. Sierkama, *Relationships between Carathéodory, Helly, Radon and Exchange numbers of convex spaces*, Nieuw Archief Wisk, **25** (1977), 115–132.
[18] V. P. Soltan, *Some questions in the abstract theory of convexity*, Soviet Math. Dokl., **17** (1976), 730–733.
[19] V. P. Soltan, *D-convexity in graphs*, Soviet Math. Dokl., **28** (1983), 419–421.
[20] V. P. Soltan, *Introduction to the axiomatic theory of convexity*, (Russian) Shtiinca, Kishinev 1984.
[21] M. Van De Vel, *Finite dimensional convex structures II: the invariants*, Topology Appl., **16** (1983), 81–105.
[22] M. Van De Vel, *Binary convexities and distributive lattices*, Proc. London Math. Soc., **48** (1984), 1–33.
[23] M. Van De Vel, *Theory of convex structures*, North-Holland, Amsterdam 1993.
[24] J. C. Varlet, *Remarks on distributive lattices*, Bull. Acad. Polon. Sci., **23** (1975), 1143–1147.

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