CHARACTERIZATIONS OF $L$-CONVEX SPACES

B. PANG AND Y. ZHAO

Abstract. In this paper, the concepts of $L$-concave structures, concave $L$-interior operators and concave $L$-neighborhood systems are introduced. It is shown that the category of $L$-concave spaces and the category of concave $L$-interior spaces are isomorphic, and they are both isomorphic to the category of concave $L$-neighborhood systems whenever $L$ is a completely distributive lattice. Also, it is proved that these categories are all isomorphic to the category of $L$-convex spaces whenever $L$ is a completely distributive lattice with an order-reversing involution operator.

1. Introduction

Abstract convexity theory [20, 23] is an important branch of mathematics. As in the situation of topology theory, convexity theory usually deals with set-theoretic structures satisfying axioms similar to that usual convex sets fulfill. Convexities exist in many different mathematical research areas, such as convexities in lattices and in Boolean algebras [22, 24], convexities in metric spaces and graphs [8, 11, 19]. Especially, convexities appear naturally in topology and possess many topological properties, such as product spaces [4, 5], convex invariants [9, 16, 17, 18, 21] and separation [2, 3, 6, 7]. Like closure operators in topology, hull operators (also called algebraic closure operators) can be used to characterize convex structures [23].

With the development of fuzzy mathematics, convex structures have been endowed with fuzzy set theory. In [10] and [14], the authors independently introduced the concept of $L$-convex structures. Based on $L$-convex structures, Pang and Shi [12] proposed several types of $L$-convex structures and investigated their categorical relations. Since algebraic closure operators cannot be directly generalized to $L$-convex spaces, there have not been algebraic $L$-closure operators. That is to say, there have not been suitable axiomatic definition for algebraic $L$-closure operators in the framework of $L$-convex spaces. Recently, Shi and Xiu [15] provided a new approach to fuzzification of convex structures and proposed the notion of $M$-fuzzifying convex structures. In this environment, algebraic $M$-fuzzifying closure operators are introduced and they are shown to be equivalent to $M$-fuzzifying convex structures. Inspired by this, Pang and Shi [13] introduced axiomatic $L$-hull operators to characterize $L$-convex structures.

In the theory of $L$-topology, there are several characterizations of $L$-topologies, including $L$-closure operators, $L$-interior operators and $L$-neighborhood systems.
As we all know, $L$-convex structures possess so many characters of $L$-topologies. This motivates us to consider the corresponding characterizations of $L$-convex structures. In this paper, we will firstly introduce the dual concept of $L$-convex structures, which is called $L$-concave structures. And then we will consider the corresponding fuzzy interior operators and fuzzy neighborhood systems.

2. Preliminaries

Throughout this paper, let $L$ denote a complete lattice. The smallest element and the largest element in $L$ are denoted by $\bot$ and $\top$, respectively. For $a, b \in L$, we say that $a$ is wedge below $b$ in $L$, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\forall D \ni b$ implies $d \ni a$ for some $d \in D$. An element $a$ in $L$ is called coprime if $a \leq b \lor c$ implies $a \leq b$ or $a \leq c$. The set of nonzero coprime elements in $L$ is denoted by $J(L)$. Let $\beta(a) = \{b \in L \mid b \prec a\}$ and $\beta^*(a) = \beta(a) \cap J(L)$. A complete lattice $L$ is completely distributive if and only if $a = \bigvee \beta(a)$ (or $a = \bigvee \beta^*(a)$) for each $a \in L$. Also, we adopt the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$.

For a nonempty set $X$, $L^X$ denotes the set of all $L$-subsets on $X$. $L^X$ is also a complete lattice when it inherits the structure of the lattice $L$ in a natural way, by defining $\bigvee$, $\bigwedge$ and $\preceq$ pointwisely. The smallest element and the largest element in $L^X$ are denoted by $\bot$ and $\top$, respectively. We say $\{A_j\}_{j \in J}$ is a directed (resp. co-directed) subset of $L^X$, in symbols $\{A_j\}_{j \in J} \subseteq L^X$, (resp. $\{A_j\}_{j \in J} \subseteq L^X$), if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j \in J}$, there exists $A_{j_3} \in \{A_j\}_{j \in J}$ such that $A_{j_1}, A_{j_2} \preceq A_{j_3}$ (resp. $A_{j_1}, A_{j_2} \succeq A_{j_3}$).

Let $X, Y$ be two nonempty sets and $\varphi : X \rightarrow Y$ be a mapping. Define $\varphi^+: L^X \rightarrow L^Y$ and $\varphi^−: L^Y \rightarrow L^X$ by $\varphi^+(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^−(B) = B \circ \varphi$ for $B \in L^Y$, respectively.

**Definition 2.1.** [10, 14] An $L$-convex structure $\mathcal{E}$ on $X$ is a subset of $L^X$ which satisfies:

1. (LC1) $\bot, \top \in \mathcal{E}$;
2. (LC2) $\{A_k\}_{k \in K} \subseteq \mathcal{E}$ implies $\bigwedge_{k \in K} A_k \in \mathcal{E}$;
3. (LC3) If $\{A_j\}_{j \in J} \subseteq \mathcal{E}$ is totally ordered, then $\bigvee_{j \in J} A_j \in \mathcal{E}$.

For an $L$-convex structure $\mathcal{E}$ on $X$, the pair $(X, \mathcal{E})$ is called an $L$-convex space.

For partially ordered sets, in [1], the researchers present the following result.

**Proposition 2.2.** A partially ordered set $D$ is closed for directed unions if and only if each chain in $D$ has a supremum.

Obviously, for an $L$-convex structure $\mathcal{E}$ on $X$, since the order "$\preceq"$ on $L^X$ is defined pointwisely, $(\mathcal{E}, \preceq)$ is also a partially ordered set. Hence, we obtain the following result.

**Proposition 2.3.** Let $\mathcal{E}$ be an $L$-closure system on $X$, that is, $\mathcal{E} \subseteq L^X$ satisfies (LC1) and (LC2). Then the following statements are equivalent.

1. (LC3) If $\{A_j\}_{j \in J} \subseteq \mathcal{E}$ is totally ordered, then $\bigvee_{j \in J} A_j \in \mathcal{E}$.
2. (LC3’) $\{A_j\}_{j \in J} \subseteq \mathcal{E}$ implies $\bigvee_{j \in J} A_j \in \mathcal{E}$.
In the sequel, we usually adopt (LC3)' instead of (LC3).

**Definition 2.4.** [13] A mapping \( \varphi : (X, \mathcal{E}_X) \rightarrow (Y, \mathcal{E}_Y) \) between \( L \)-convex spaces is called \( L \)-convexity-preserving (\( L \)-CEP, in short) provided that \( B \in \mathcal{E}_Y \) implies \( \varphi^{-1}(B) \in \mathcal{E}_X \).

It is easy to check that all \( L \)-convex spaces as objects and all \( L \)-CEP mappings as morphisms form a category, denoted by \( L \text{-CE} \).

In [13], Pang and Shi provided a characterization of \( L \)-convex structures by \( L \)-hull operators, which is called convex \( L \)-closure operators in this paper.

**Definition 2.5.** [13] A convex \( L \)-closure operator on \( X \) is a mapping \( Cl : L^X \rightarrow L^X \) which satisfies:

- \((CLC1)\) \( Cl(\bot) = \bot; \)
- \((CLC2)\) \( A \subseteq Cl(A); \)
- \((CLC3)\) \( Cl(Cl(A)) = Cl(A); \)
- \((CLC4)\) \( Cl(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} Cl(A_j) \) for \( \{A_j | j \in J\} \subseteq L^X \).

For a convex \( L \)-closure operator \( Cl \) on \( X \), the pair \( (X, Cl) \) is called a convex \( L \)-closure space.

**Definition 2.6.** [13] A mapping \( \varphi : (X, Cl_X) \rightarrow (Y, Cl_Y) \) between convex \( L \)-closure spaces is called \( L \)-closure-preserving (\( L \)-CLP, in short) provided that \( \forall A \in L^X, \varphi^{-1}(Cl_Y(A)) \subseteq Cl_Y(\varphi^{-1}(A)). \)

It is easy to check that all convex \( L \)-closure spaces as objects and all \( L \)-CLP mappings as morphisms form a category, denoted by \( L \text{-CLC} \).

**Theorem 2.7.** [13] \( L \text{-CE} \) is isomorphic to \( L \text{-CLC} \).

### 3. \( L \)-concave Spaces and Concave \( L \)-interior Spaces

In this section, we will introduce the concept of \( L \)-concave spaces, which a dual concept of \( L \)-convex spaces. Then we will propose the concept of concave \( L \)-interior spaces to characterize \( L \)-concave spaces.

Firstly, we give the definitions of \( L \)-concave structures and concave \( L \)-interior operators.

**Definition 3.1.** An \( L \)-concave structure \( \mathcal{A} \) on \( X \) is a subset of \( L^X \) which satisfies:

- \((LA1)\) \( \bot, \top \in \mathcal{A}; \)
- \((LA2)\) \( \{A_k \}_{k \in K} \subseteq \mathcal{A} \) implies \( \bigvee_{k \in K} A_k \in \mathcal{A}; \)
- \((LA3)\) \( \{A_j \}_{j \in J} \subseteq \mathcal{A} \) implies \( \bigwedge_{j \in J} A_j \in \mathcal{A}. \)

For an \( L \)-concave structure \( \mathcal{A} \) on \( X \), the pair \( (X, \mathcal{A}) \) is called an \( L \)-concave space.

**Definition 3.2.** A mapping \( \varphi : (X, \mathcal{A}_X) \rightarrow (Y, \mathcal{A}_Y) \) is called \( L \)-concavity-preserving (\( L \)-CAP, in short) provided that \( B \in \mathcal{A}_Y \) implies \( \varphi^{-1}(B) \in \mathcal{A}_X \).

It is easy to check that all \( L \)-concave spaces as objects and all \( L \)-CAP mappings as morphisms form a category, denoted by \( L \text{-CA} \).
Proposition 3.7. If \( \bigwedge \) \( \bigvee \) \( \bigcap \) \( \bigcup \)

This means

By (CLI4), we have

By (CLI2), the inverse inequality

Then it follows that

Since

by Proposition 3.6.

Let

A mapping

Definition 3.5.

Proof. It is enough to verify that Int is order preserving. That is to say, for \( A, B \in L^X \), \( A \leq B \) implies \( Int(A) \leq Int(B) \).

Definition 3.3.

A concave \( L \)-interior operator on \( X \) is a mapping \( Int : L^X \rightarrow L^X \) which satisfies:

- (CLI1) \( Int(\top) = \top \);
- (CLI2) \( Int(A) \leq A \);
- (CLI3) \( Int(Int(A)) = Int(A) \);
- (CLI4) \( Int(\bigwedge_{j \in J} A_j) = \bigwedge_{j \in J} Int(A_j) \) for \( \{ A_j \mid j \in J \} \subset L^X \).

For a concave \( L \)-interior operator \( Int \) on \( X \), the pair \( (X, Int) \) is called a concave \( L \)-interior space.

Remark 3.4. By (CLI4), it is easy to check that \( Int \) is order preserving. That is to say, for \( A, B \in L^X \), \( A \leq B \) implies \( Int(A) \leq Int(B) \).

Definition 3.5. A mapping \( \varphi : (X, Int_X) \rightarrow (Y, Int_Y) \) between concave \( L \)-interior spaces is called \( L \)-interior-preserving \( (L \text{INP}, \text{in short}) \) provided that

\[ \forall B \in L^Y, \varphi^c(dir)(Int_Y(B)) \leq Int_X(\varphi^c(dir)(B)). \]

It is easy to check that all concave \( L \)-interior spaces as objects and all \( L \text{INP} \) mappings as morphisms form a category, denoted by \( L \text{CLI} \).

Next we will establish the relations between \( L \)-concave spaces and concave \( L \)-interior spaces.

Proposition 3.6. Let \( (X, Int) \) be a concave \( L \)-interior space and define \( A^{Int} \subset L^X \) by

\[ A^{Int} = \{ A \in L^X \mid A = Int(A) \}. \]

Then \( A^{Int} \) is an \( L \)-concave structure on \( X \).

Proof. It is enough to verify that \( A^{Int} \) satisfies (LA1)–(LA3). In fact,

- (LA1) Since \( \top = Int(\top) \) and \( Int(\bot) \leq \bot \), we know \( \bot, \top \subset A^{Int} \).
- (LA2) Take any \( \{ A_k \mid k \in K \} \subset A^{Int} \). Then \( A_k = Int(A_k) \) for each \( k \in K \). Since \( Int \) is order preserving, we have \( Int(\bigvee_{k \in K} A_k) \geq Int(A_k) \) for each \( k \in K \). Then it follows that

\[ Int(\bigvee_{k \in K} A_k) \geq \bigvee_{k \in K} Int(A_k) = \bigvee_{k \in K} A_k. \]

By (CLI2), the inverse inequality \( Int(\bigvee_{k \in K} A_k) \leq \bigvee_{k \in K} A_k \) holds obviously. Thus, \( Int(\bigvee_{k \in K} A_k) = \bigvee_{k \in K} A_k \). This means \( \bigvee_{k \in K} A_k \in A^{Int} \).

- (LA3) Take any \( \{ A_j \}_{j \in J} \subset A^{Int} \). It follows that \( A_j = Int(A_j) \) for each \( j \in J \).

By (CLI4), we have

\[ Int(\bigwedge_{j \in J} A_j) = \bigwedge_{j \in J} Int(A_j) = \bigwedge_{j \in J} A_j. \]

This means \( \bigwedge_{j \in J} A_j \in A^{Int} \), as desired.

Proposition 3.7. If \( \varphi : (X, Int_X) \rightarrow (Y, Int_Y) \) is \( L \text{INP} \), then \( \varphi : (X, A^{Int_X}) \rightarrow (Y, A^{Int_Y}) \) is \( L \text{CAP} \).
Proof. Since $\varphi : (X, \text{Int}_X) \to (Y, \text{Int}_Y)$ is L-INP, it follows that
$$\forall B \in L^Y, \varphi^\rightarrow(\text{Int}_Y(B)) \subseteq \text{Int}_X(\varphi^\rightarrow(B)).$$

Then for each $C \in A^{\text{Int}_Y}$, that is, $C = \text{Int}_Y(C)$, we have
$$\text{Int}_X(\varphi^\rightarrow(C)) \supseteq \varphi^\rightarrow(\text{Int}_Y(C)) = \varphi^\rightarrow(C).$$

By (LCI2), we have $\text{Int}_X(\varphi^\rightarrow(C)) = \varphi^\rightarrow(C)$. This implies $\varphi^\rightarrow(C) \in A^{\text{Int}_X}$. Hence, $\varphi : (X, A^{\text{Int}_X}) \to (Y, A^{\text{Int}_Y})$ is L-CAP. \hfill $\square$

By Propositions 3.6 and 3.7, we obtain a functor $A_3 : L-\text{CLI} \to L-\text{CA}$ by
$$A_3 : \begin{cases}
L-\text{CLI} &\to L-\text{CA},
(X, \text{Int}) &\to (X, A^{\text{Int}}),
\varphi &\mapsto \varphi.
\end{cases}$$

**Proposition 3.8.** Let $(X, A)$ be an L-concave space and define $\text{Int}^A : L^X \to L^X$ by
$$\forall A \in L^X, \text{Int}^A(A) = \bigvee \{B \in L^X \mid B \leq A, B \in A\}.$$ 

Then $\text{Int}^A$ is a concave interior operator on $X$.

**Proof.** It is enough to show that $\text{Int}^A$ satisfies (CLI1)–(CLI4). (CLI1) and (CLI2) are obvious. We need only verify (CLI3) and (CLI4).

(CLII3) By (LA2), we know $\text{Int}^A(A) = \bigvee \{B \in L^X \mid B \leq A, B \in A\} \in A$. This implies $\text{Int}^A(A) \subseteq \text{Int}^A(\text{Int}^A(A))$. Since the inverse inequality $\text{Int}^A(A) \supseteq \text{Int}^A(\text{Int}^A(A))$ holds obviously, we obtain $\text{Int}^A(A) = \text{Int}^A(\text{Int}^A(A))$.

(CLII4) By the definition of $\text{Int}^A$, we know $\text{Int}^A$ is order preserving. That is to say, $A \leq B$ implies $\text{Int}^A(A) \leq \text{Int}^A(B)$. This means $\text{Int}^A \left(\bigsqcup_{j\in I} A_j\right) \leq \bigsqcup_{j\in I} \text{Int}^A(A_j)$. Since $\{A_j\}_{j\in I}$ is co-directed, it follows that $\{\text{Int}^A(A_j)\}_{j\in I}$ is co-directed. Further, by (LA2), it follows that $\text{Int}^A(A) = \bigvee \{B \in L^X \mid B \leq A, B \in A\} \in A$. This implies $\bigsqcup_{j\in I} \text{Int}^A(A_j) \in A$. As $\bigsqcup_{j\in I} \text{Int}^A(A_j) \leq \bigsqcup_{j\in I} A_j$, we have $\bigsqcup_{j\in I} \text{Int}^A(A_j) \leq \text{Int}^A \left(\bigsqcup_{j\in I} A_j\right)$. This proves $\bigsqcup_{j\in I} \text{Int}^A(A_j) = \text{Int}^A \left(\bigsqcup_{j\in I} A_j\right)$, as desired. \hfill $\square$

**Proposition 3.9.** If $\varphi : (X, A_X) \to (Y, A_Y)$ is L-CAP, then $\varphi : (X, \text{Int}^{A_X}) \to (Y, \text{Int}^{A_Y})$ is L-INP.

**Proof.** Since $\varphi : (X, A_X) \to (Y, A_Y)$ is L-CAP, it follows that
$$\forall B \in L^Y, B \in A_Y \text{ implies } \varphi^\rightarrow(B) \in A_X.$$ 

Then we have
$$\varphi^\rightarrow(\text{Int}^{A_Y}(B)) = \varphi^\rightarrow(\bigvee \{C \in L^Y \mid C \leq B, C \in A_Y\}) \leq \bigvee \{\varphi^\rightarrow(C) \in L^X \mid \varphi^\rightarrow(C) \leq \varphi^\rightarrow(B), \varphi^\rightarrow(C) \in A_X\} \leq \bigvee \{A \in L^X \mid A \leq \varphi^\rightarrow(B), A \in A_X\} = \text{Int}^{A_X}(\varphi^\rightarrow(B)).$$
This proves that \( \varphi : (X, \text{Int}^{A_{X}}) \rightarrow (Y, \text{Int}^{A_{Y}}) \) is L-INP. \( \square \)

By Propositions 3.8 and 3.9, we obtain a functor \( I_{A} : \text{L-CA} \rightarrow \text{L-CLI} \) by

\[
I_{A} : \begin{align*}
&\text{L-CA} \quad \rightarrow \quad \text{L-CLI}, \\
&(X, A) \quad \mapsto \quad (X, \text{Int}^{A}), \\
&\varphi \quad \mapsto \quad \varphi.
\end{align*}
\]

Theorem 3.10. \( \text{L-CLI} \) is isomorphic to \( \text{L-CA} \).

Proof. It suffices to show that \( I_{A} \circ A_{1} = I_{L-\text{CLI}} \) and \( A_{1} \circ I_{A} = I_{L-\text{CA}} \). That is to say, we need only verify (1) \( \text{Int}^{A_{\text{Int}}} = \text{Int} \) and (2) \( A^{\text{Int}} = A \).

For (1), take each \( A \in L^{X} \). Then

\[
\text{Int}^{A_{\text{Int}}} = \bigvee \{ B \in L^{X} \mid B \leq A, B \in A^{\text{Int}} \} = \bigvee \{ B \in L^{X} \mid B \leq A, B = \text{Int}(B) \}.
\]

On one hand, take each \( B \in L^{X} \) such that \( B \leq A \) and \( B = \text{Int}(B) \), we have \( B = \text{Int}(B) \leq \text{Int}(A) \). This means \( \text{Int}^{A_{\text{Int}}} \leq \text{Int}(A) \). On the other hand, let \( B = \text{Int}(A) \). Then \( B \leq A \) and \( \text{Int}(B) = \text{Int}(\text{Int}(A)) = \text{Int}(A) = B \). This means that \( \text{Int}(A) \leq \text{Int}^{A_{\text{Int}}} \), as desired.

(2) Take each \( A \in L^{X} \). Then

\[
A \in A^{\text{Int}} \iff A = \text{Int}^{A}(A) = \bigvee \{ B \in L^{X} \mid B \leq A, B \in A \}
\]

\[
\iff A = \bigvee \{ B \in L^{X} \mid B \leq A, B \in A \}
\]

\[
\iff A \in A.
\]

This means \( A^{\text{Int}} = A \). \( \square \)

Theorem 3.11. Suppose that \( L \) is equipped with an order-reversing involution operator \( ' \). Then \( \text{L-CE} \) is isomorphic to \( \text{L-CA} \).

Proof. Given an \( L \)-convex space \((X, \mathcal{E})\), define \( \mathcal{A}^{\mathcal{E}} \subseteq L^{X} \) by

\[
\mathcal{A}^{\mathcal{E}} = \{ A \in L^{X} \mid A' \in \mathcal{E} \}
\]

Then \((X, \mathcal{A}^{\mathcal{E}})\) is an \( L \)-concave space. Similarly, given an \( L \)-concave space \((X, \mathcal{A})\), define \( \mathcal{E}^{A} \subseteq L^{X} \) by

\[
\mathcal{E}^{A} = \{ A \in L^{X} \mid A' \in \mathcal{A} \}
\]

Then \((X, \mathcal{E}^{A})\) is an \( L \)-convex space. Since \( ' \) is an order-reversing involution operator on \( L \), it can be easily checked that \( \text{L-CE} \) is isomorphic to \( \text{L-CA} \). \( \square \)

By Theorems 2.7, 3.10 and 3.11, we obtain

Corollary 3.12. Suppose that \( L \) is equipped with an order-reversing involution operator \( ' \). Then \( \text{L-CE}, \text{L-CA}, \text{L-CLC} \) and \( \text{L-CLI} \) are isomorphic.
4. Concave \(L\)-neighborhood Systems

In this section, we introduce the concept of concave \(L\)-neighborhood systems and study its relations with \(L\)-concave structures. In this section, \(L\) is assumed to be a completely distributive lattice.

**Definition 4.1.** A concave \(L\)-neighborhood system on \(X\) is a set \(\mathcal{N} = \{N_{x_\lambda} \mid x_\lambda \in J(L^X)\}\), where \(N_{x_\lambda} \subseteq L^X\) satisfies:

- (CLN1) \(\top^X \in N_{x_\lambda}, \bot^X \notin N_{x_\lambda}\);
- (CLN2) \(\forall A \in N_{x_\lambda}, x_\lambda \leq A\);
- (CLN3) \(\forall A \in N_{x_\lambda}, A \leq B\) implies \(B \in N_{x_\lambda}\);
- (CLN4) \(\forall\{A_j\}_{j \in J} \subseteq N_{x_\lambda}, \bigwedge_{j \in J} A_j \in N_{x_\lambda}\);
- (CLN5) \(A \in N_{x_\lambda}\) if and only if \(\exists B \in L^X\) s.t. \(x_\lambda \leq B \leq A\) and \(\forall y_\mu \prec B, B \in N_{y_\mu}\).

For a concave \(L\)-neighborhood system \(\mathcal{N}\) on \(X\), the pair \((X, \mathcal{N})\) is called a concave \(L\)-neighborhood space.

**Definition 4.2.** A mapping \(\varphi : (X, \mathcal{N}^X) \to (Y, \mathcal{N}^Y)\) between concave \(L\)-neighborhood spaces is called \(L\)-neighborhood-preserving (\(L\)-NEP, in short) provided that

\[
\forall x_\lambda \in J(L^X), \forall B \in L^Y, B \in N_{\varphi(x_\lambda)} \implies \varphi^{-1}(B) \in N_{x_\lambda}.
\]

It is easy to check that all concave \(L\)-neighborhood spaces as objects and all \(L\)-NEP mappings as morphisms form a category, denoted by \(L-CN\).

Next we will investigate the relations between \(L-CN\) and \(L-CA\).

**Proposition 4.3.** Let \((X, \mathcal{N})\) be a concave \(L\)-neighborhood space and define \(\mathcal{A}^\mathcal{N} \subseteq L^X\) as follows:

\[
\mathcal{A}^\mathcal{N} = \{A \in L^X \mid \forall x_\lambda \prec A, A \in N_{x_\lambda}\}.
\]

Then \(\mathcal{A}^\mathcal{N}\) is an \(L\)-concave structure on \(X\).

**Proof.** It suffices to verify that \(\mathcal{A}^\mathcal{N}\) satisfies (LA1), (LA2) and (LA3). In fact,

- (LA1) Obviously.
- (LA2) Take each \(\{A_k\}_{k \in K} \subseteq \mathcal{A}^\mathcal{N}\). Then for each \(k \in K\), it follows that \(A_k \in N_{x_\lambda}\) for each \(x_\lambda \prec A_k\). For each \(x_\lambda \in J(L^X)\) satisfying \(x_\lambda \prec \bigvee_{k \in K} A_k\), there exists \(k_0 \in K\) such that \(x_\lambda \prec A_{k_0}\). This implies that \(A_{k_0} \in N_{x_\lambda}\). By (CLN3), we have \(\bigvee_{k \in K} A_k \in N_{x_\lambda}\). This shows that \(A \in \mathcal{A}^\mathcal{N}\).
- (LA3) Take each \(\{A_j\}_{j \in J} \subseteq \mathcal{A}^\mathcal{N}\). Then for each \(j \in J\), it follows that \(A_j \in N_{x_\lambda}\) for each \(x_\lambda \prec A_j\). For each \(x_\lambda \in J(L^X)\) satisfying \(x_\lambda \prec \bigwedge_{j \in J} A_j\), we have \(x_\lambda \prec A_j\) for each \(j \in J\). This means \(A_j \in N_{x_\lambda}\) for each \(j \in J\). By (CLN4), we obtain \(\bigwedge_{j \in J} A_j \in N_{x_\lambda}\). This proves that \(A \in \mathcal{A}^\mathcal{N}\).

**Proposition 4.4.** If \(\varphi : (X, \mathcal{N}^X) \to (Y, \mathcal{N}^Y)\) is \(L\)-NEP, then \(\varphi : (X, \mathcal{A}^\mathcal{N}^X) \to (Y, \mathcal{A}^\mathcal{N}^Y)\) is \(L\)-CAP.
Proof. Since \( \varphi : (X, \mathcal{N}^X) \rightarrow (Y, \mathcal{N}^Y) \) is L-NEP, it follows that 
\[ \forall x_\lambda \in J(L^X), \forall B \in L^Y, B \in \mathcal{N}^Y_{\varphi(x)_\lambda} \text{ implies } \varphi^{-1}(B) \in \mathcal{N}^X_{x_\lambda}. \]

Then for each \( C \in L^Y \), we obtain 
\[
C \in \mathcal{A}^{N^Y} \iff \forall y_\mu \prec C, \ C \in \mathcal{N}^Y_{y_\mu} \\
\iff \forall \varphi(x)_\lambda \prec C, \ C \in \mathcal{N}^Y_{\varphi(x)_\lambda} \\
\iff \forall x_\lambda \prec \varphi^{-1}(C), \varphi^{-1}(C) \in \mathcal{N}^X_{x_\lambda} \\
\iff \varphi^{-1}(C) \in \mathcal{A}^{N^X}. \]

This proves that \( \varphi : (X, \mathcal{A}^{N^X}) \rightarrow (Y, \mathcal{A}^{N^Y}) \) is L-CAP. \( \square \)

By Propositions 4.3 and 4.4, we obtain a functor \( \mathcal{A}_N : L\text{-}CN \rightarrow L\text{-CA} \) by 
\[
\mathcal{A}_N : \begin{cases} 
L\text{-}CN \rightarrow L\text{-CA,} \\
(X, \mathcal{N}) \quad \mapsto \quad (X, \mathcal{A}^N), \\
\varphi \quad \mapsto \quad \varphi.
\end{cases}
\]

**Proposition 4.5.** Let \( (X, \mathcal{A}) \) be an L-concave space and define \( \mathcal{N}^A_{x_\lambda} \subseteq L^X \) by 
\[ \mathcal{N}^A_{x_\lambda} = \{ A \in L^X \mid \exists B \in \mathcal{A} \text{ s.t. } x_\lambda \preceq B \preceq A \}. \]

Then \( \mathcal{N}^A = \{ \mathcal{N}^A_{x_\lambda} \mid x_\lambda \in J(L^X) \} \) is a concave L-neighborhood system on \( X \).

Proof. It is enough to show that \( \mathcal{N}^A \) satisfies (CLN1)–(CLN5). (CLN1)–(CLN3) are obvious.

( CLN4) Take \( \{ A_j \}_{j \in J} \subseteq \mathcal{N}^A_{x_\lambda} \). Then for each \( j \in J \), there exists \( B_j \in \mathcal{A} \) such that \( x_\lambda \preceq B_j \preceq A_j \). By (LA2), we know \( \text{Int}^A(A_j) \in \mathcal{A} \) and \( x_\lambda \preceq B_j \preceq \text{Int}^A(A_j) \preceq A_j \). Since \( \text{Int}^A \) is order preserving, we obtain \( \{ \text{Int}^A(A_j) \mid j \in J \} \subseteq \mathcal{A} \). Put \( B = \bigwedge_{j \in J} \text{Int}^A(A_j) \). By (LA3), it follows that \( B \in \mathcal{A} \). Further, \( x_\lambda \preceq B \preceq \bigwedge_{j \in J} A_j \).

Thus, we get \( \bigwedge_{j \in J} A_j \in \mathcal{N}^A_{x_\lambda} \).

( CLN5) Necessity. Take each \( A \in \mathcal{N}^A_{x_\lambda} \). Then there exists \( B \in \mathcal{A} \) such that \( x_\lambda \preceq B \preceq A \). Since \( B \in \mathcal{A} \), for each \( y_\mu \prec B \), there exists \( C(= B) \in \mathcal{A} \) such that \( y_\mu \preceq C \preceq B \). This means \( B \in \mathcal{N}^A_{y_\mu} \) for each \( y_\mu \prec B \), as desired.

Sufficiency. We first prove the following result.

( CLN0) \( A \in \mathcal{N}^A_{x_\lambda} \iff \forall \mu \in \beta^*(\lambda), A \in \mathcal{N}^A_{y_\mu} \).

On one hand, if \( A \in \mathcal{N}^A_{x_\lambda} \), then there exists \( B \in \mathcal{A} \) such that \( x_\lambda \preceq B \preceq A \). Take each \( \mu \in \beta^*(\lambda) \). Then \( x_\mu \prec x_\lambda \preceq B \preceq A \). This implies \( A \in \mathcal{N}^A_{y_\mu} \) for \( \mu \in \beta^*(\lambda) \).

On the other hand, since \( A \in \mathcal{N}^A_{x_\lambda} \) for \( \mu \in \beta^*(\lambda) \), it follows that for each \( \mu \in \beta^*(\lambda) \), there exists \( B_\mu \in \mathcal{A} \) such that \( x_\mu \preceq B_\mu \preceq A \). Let \( B = \bigvee_{\mu \in \beta^*(\lambda)} B_\mu \). Then by (LA2), we have \( B \in \mathcal{A} \). Further, 
\[
x_\lambda = \bigvee_{\mu \in \beta^*(\lambda)} x_\mu \preceq \bigvee_{\mu \in \beta^*(\lambda)} B_\mu = B \preceq A.
\]

This implies that \( A \in \mathcal{N}^A_{x_\lambda} \).
Now we know there exists \( B \in L^X \) such that \( x_\lambda \leq B \leq A \) and for each \( y_\mu \prec B \), \( B \in N^A_{y_\mu} \). Thus, for each \( \mu \in \beta^*(\lambda) \), it follows that \( x_\mu \prec x_\lambda \leq B \). This means \( B \in N^A_{x_\mu} \). By (CLN3), we know \( A \in N^A_{x_\mu} \). By (CLN0), we get \( A \in N^A_{x_\lambda} \). This proves the sufficiency of (CLN5).

\[ \square \]

**Proposition 4.6.** If \( \varphi : (X, A_X) \rightarrow (Y, A_Y) \) is \( L \)-CAP, then \( \varphi : (X, N^{A_X}) \rightarrow (Y, N^{A_Y}) \) is \( L \)-NEP.

**Proof.** Since \( \varphi : (X, A_X) \rightarrow (Y, A_Y) \) is \( L \)-CAP, it follows that \( \forall B \in L^Y, B \in A_Y \) implies \( \varphi^+(B) \in A_X \).

Then for each \( x_\lambda \in J(L^X) \) and \( B \in L^Y \), we have

\[
\begin{align*}
B \in N^{A_X}_{\varphi(x_\lambda)} & \iff \exists C \in A_Y \text{ s.t. } \varphi(x_\lambda) \leq C \leq B \\
& \iff \exists \varphi^+(C) \in A_X \text{ s.t. } x_\lambda \leq \varphi^+(C) \leq \varphi^+(B) \\
& \iff \exists A \in A_X \text{ s.t. } x_\lambda \leq A \leq \varphi^+(B) \\
& \iff \varphi^+(B) \in N^A_{x_\lambda}.
\end{align*}
\]

This shows that \( \varphi : (X, N^{A_X}) \rightarrow (Y, N^{A_Y}) \) is \( L \)-NEP.

By Propositions 4.5 and 4.6, we obtain a functor \( N_A : L-\text{CA} \rightarrow L-\text{CN} \) by

\[
N_A : \begin{cases}
L-\text{CA} \rightarrow L-\text{CN}, \\
(X, A) \mapsto (X, N^A), \\
\varphi \mapsto \varphi.
\end{cases}
\]

Next we present the main result in this section.

**Theorem 4.7.** \( L-\text{CN} \) is isomorphic to \( L-\text{CA} \).

**Proof.** It suffices to show that \( N_A \circ A_N = \mathbb{I}_{L-\text{CLN}} \) and \( A_N \circ N_A = \mathbb{I}_{L-\text{CA}} \). That is to say, we need only verify (1) \( N^{A_X} = N \) and (2) \( A^{N_A} = A \).

For (1), take each \( A \in L^X \). Then \( A \in N^{A_X} \iff \exists B \in A \text{ s.t. } x_\lambda \leq B \leq A \)

\[
\begin{align*}
\iff \exists B \in L^X \text{ s.t. } x_\lambda \leq B \leq A \text{ and } \forall y_\mu \prec B, \ B \in N^A_{y_\mu} \\
\iff A \in N^A_{x_\lambda}. \quad \text{(by (CLN5))}
\end{align*}
\]

This shows \( N^{A_X} = N \).

(2) Take each \( A \in L^X \). Then

\[
\begin{align*}
A \in A^{N_A} & \iff \forall x_\lambda \prec A, A \in N^A_{x_\lambda} \\
& \iff \forall x_\lambda \prec A, \exists B \in A \text{ s.t. } x_\lambda \leq B \leq A \\
& \iff A = \bigvee_{x_\lambda \prec A, x_\lambda \leq B \leq A} B \\
& \iff A \in A.
\end{align*}
\]
This means $A^{\mathcal{A}^{\mathcal{A}}}=A$. □

**Corollary 4.8.** Suppose that $L$ is a completely distributive lattice with an order-reversing involution operator $'$. Then $L$-CE, $L$-CA, $L$-CLC, $L$-CLI and $L$-CN are isomorphic.

5. Conclusions

In this paper, we provided several characterizations of $L$-convex structures from a categorical viewpoint, including $L$-concave structures, concave $L$-interior operators and concave $L$-neighborhood systems. As in the situation of $L$-topology, all these resulting categories are isomorphic whenever $L$ is a completely distributive lattice with an order-reversing involution operator. Based on these new concepts, we will consider “topological” properties of $L$-convex spaces, such as compactness and separation property.

In the classical case, convex structures can be characterized by algebraic closure operators (also called domain finite closure operators) in [23]. Convex $L$-closure operators in Definition 2.5 are not direct generalization of algebraic closure operators. Since algebraic closure operators cannot be generalized to the fuzzy case directly, we firstly found an equivalent form of $L$-convex spaces in Proposition 2.3 and then used this equivalent form to propose Definition 2.5. Consequently, the concept of concave $L$-interior operators can also be defined dually and the concept of concave $L$-neighborhood systems can be proposed. In the future, we will consider how to define algebraic $L$-closure operators in the framework of $L$-convex spaces and study its relations with convex $L$-closure operators.

**Acknowledgements.** The authors would like to express their sincere thanks to the anonymous reviewers for their careful reading and constructive comments. This work is supported by National Nature Science Foundation Committee (NSFC) of China (No. 61573119), China Postdoctoral Science Foundation (No. 2015M581434), Fundamental Research Project of Shenzhen (No. JCYJ20120613144110654 and No. JCYJ2014041712417109).

**References**

Characterizations of $L$-convex Spaces


BIN PANG, SHENZHEN GRADUATE SCHOOL, HARBIN INSTITUTE OF TECHNOLOGY, 518055 SHENZHEN, P.R. CHINA
E-mail address: pangbin1205@163.com

Yi Zhao*, SHENZHEN GRADUATE SCHOOL, HARBIN INSTITUTE OF TECHNOLOGY, 518055 SHENZHEN, P.R. CHINA
E-mail address: zhaoyisz420@souhu.com

*Corresponding author