MULTIPLE FUZZY REGRESSION MODEL FOR FUZZY INPUT-OUTPUT DATA

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Abstract. A novel approach to the problem of regression modeling for fuzzy input-output data is introduced. In order to estimate the parameters of the model, a distance on the space of interval-valued quantities is employed. By minimizing the sum of squared errors, a class of regression models is derived based on the interval-valued data obtained from the α-level sets of fuzzy input-output data. Then, by integrating the obtained parameters of the interval-valued regression models, the optimal values of parameters for the main fuzzy regression model are estimated. Numerical examples and comparison studies are given to clarify the proposed procedure, and to show the performance of the proposed procedure with respect to some common methods.

1. Introduction

Identification of functional relationship between one dependent variable (output variable or response variable) and a set of independent variables (input variables or explanatory variables) has been the cause of great interest in scientific researches, because through the obtained relationship, one can describe, control, and predict the values of the variable of primary interest from reported observations of the other ones. In practice, however, the observations of some variables may not measured/reported as precise quantities. Therefore, modeling and analyzing of imprecise/fuzzy data require new methods to be developed. Fuzzy set theory seems to provide appropriate tools for constructing and analyzing the relationship between imprecise/fuzzy data. Over the last decades, there have been many approaches to combine statistical methods and fuzzy set theory for regression analysis. These approaches can be classified in three general classes:

I) The Class of Possibilistic Methods. In these methods, which are based on the ideas proposed by Tanaka et al. [44, 45], using the possibilistic concepts, the fuzzy regression problem is formulated as a mathematical programming problem. In the simplest case of such problems, the objective is to minimize the total spread of the fuzzy parameters subject to the constraints that the α-level sets of the estimated responses include those of the observed responses for a certain (predetermined) α-level. This approach was investigated and improved by several authors. For example, Nasrabadi and Nasrabadi [36] defined new arithmetic operations for symmetric fuzzy numbers to provide fuzzy regression model which could avoid the

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spreads increasing problem. Modarres et al. [30, 31] proposed some fuzzy regression models called risk-neutral, risk-averse and risk-seeking fuzzy linear regression models. Their models consists of solving an optimization problem where the squared difference between observed and estimated spreads of the output is minimized and the inclusion constraints are obtained according to the concepts of possibility and necessity indices for fuzzy equality. Nasrabadi et al. [37] proposed a multi-objective fuzzy linear regression model to address the outliers problem by introducing the soft boundaries to the risk-neutral model of Modarres et al. [30, 31]. Nasrabadi et al. [35] reformulated the multi-objective fuzzy linear regression of Nasrabadi et al. [37] for computation of fuzzy linear regression models with fuzzy input-output data. Shakouri and Nadimi [42] introduced a linear fuzzy regression model for crisp input-fuzzy output observations, in which the objective function was based on a non-equality possibility index. Pourahmad et al. [38] proposed a possibilistic fuzzy logistic regression model for the case when the explanatory variables are crisp and the value of the binary response variable is reported as a number between zero and one. For a review on possibilistic regression models, see Bisserier et al. [4], in which they also introduced a modified possibilistic regression method where the optimal model can envelop all the observed data and ensure a total inclusion property.

II) The Class of Least Squares and Least Absolutes Methods. In these methods, the parameters of the model are estimated based on a distance on the space of fuzzy numbers. For instance, D'Urso [13] proposed some regression models for crisp/fuzzy input-crisp/fuzzy output data based on three sub-models. The first sub-model interpolates the centers of the fuzzy estimated values for the dependent variable, and the other two sub-models are built over the first one and yield the spread values. D'Urso et al. [15] introduced a class of fuzzy clusterwise regression models, including the fuzzy clusterwise linear regression model and the fuzzy clusterwise polynomial regression model as special cases, with crisp input-fuzzy output variables (see also [14, 17]). D'Urso et al. [16] proposed a robust fuzzy linear regression model based on the so-called least median squares-weighted least squares estimation procedure to deal with data contaminated by outliers. Coppi et al. [12] investigated a linear regression model for studying the dependence of crisp input-fuzzy output observations, along with an iterative least squares estimation procedure. Bargiela et al. [3], using the standard least squares criterion as a performance index, proposed an iterative gradient-descent optimization algorithm for calculating the coefficients of multiple regression with fuzzy variables. Based on least squares method, Ferraro et al. [18] proposed a linear regression model for imprecise responses. They also analyzed limit distribution and asymptotic properties of the estimators and applied them for determining the confidence regions and hypothesis testing procedures. By using normal equations corresponding to some least-squares models, Arabpour and Tata [1] calculated the fuzzy regression coefficients for crisp/fuzzy input-fuzzy output data. Kula and Apaydin [26] proposed a robust fuzzy least squares regression analysis based on a ranking of fuzzy sets. Taheri and Kelkinnama [24, 43] proposed some least absolutes deviation approaches to fuzzy regression model with crisp/fuzzy input and fuzzy output observations. Hassanpour et al. [19] used a least absolutes approach to calculate the crisp regression coefficients of a fuzzy regression model when the input-output data were fuzzy numbers. They also proposed a goal programming approach to determine the coefficients of fuzzy linear regression model [20]. Chachi et al. [10], by using the α-level sets of fuzzy input-fuzzy output observations, proposed a least squares method to estimate the crisp parameters of a fuzzy regression model. In addition, by applying the generalized Hausdorff-distance on the space of fuzzy numbers, Chachi and Taheri [6, 7] developed a least absolutes deviation approach to fuzzy regression analysis. Pourahmad et al. [39] and Namdari et al. [33] investigated some fuzzy logistic regression models based on the least squares method and least absolutes method, respectively, and studied their applications in the real world clinical problems. Arefi and Taheri [2] and Rabiei et al. [40] developed two least-squares regression models based on interval-valued fuzzy input-output data in which the parameters of the model are interval-valued fuzzy, too. The reader can find a review on some least squares and least absolutes methods for fuzzy regression analysis in the recent work by D'Urso et al. [16].

III) The Class of Heuristic Methods. This class includes some novel methods or some methods which combine the possibilistic, least squares, and least absolutes methods. For instance, Kao and Chyu [22] proposed a two-stage methodology to construct the fuzzy regression model. In the first stage, crisp coefficients of the model are estimated by applying the classical least squares method to the defuzzified input-output data. In the second stage, fuzzy error term is determined by a mathematical programming procedure. They also investigated a least squares method in fuzzy regression analysis for crisp/fuzzy input-fuzzy output data based on a ranking of fuzzy numbers [23]. Chen and Dang [11] proposed a three-phase method to investigate a variable spread fuzzy linear regression model. In the first phase, the membership functions of the least squares estimates of regression coefficients were constructed. In the second phase, the fuzzy regression coefficients were defuzzified to obtain the crisp regression coefficients, and in the third phase, fuzzy error terms were determined by a mathematical programming procedure. Lu and Wang [27] proposed an enhanced fuzzy linear regression model which could avoid the spreads increasing problem. Using tabu search and harmony search methods, Mashinchi et al. [28] proposed a metaheuristic unconstrained global continuous optimization approach to the fuzzy regression problem. Hu [21] suggested a genetic-algorithm-based method for determining two functional-link nets for a robust nonlinear interval regression model. Nasrabadi and Hashemi [34] suggested a robust fuzzy regression model using multilayered feed-forward neural networks where weights, biases, input and output variables were assumed to be fuzzy numbers. Recently, Chachi et al. [8] introduced a hybrid fuzzy regression model and investigated its application to hydrology engineering, (see also Chachi et al. [9] and Chachi and Roozbeh [5]).

In the present paper, we develop a new method to identify the functional relationship between some variables by means of a fuzzy regression model where both input and output observations are given as fuzzy data. Our proposed method consists of minimizing the sum of squared errors which is based on a distance on the space of interval-valued quantities. To do this, we will first propose linear regression models in terms of the interval-valued data of the corresponding α-level sets of fuzzy input-output data. Then, by aggregating the parameters obtained from the interval regression models, we estimate the parameters of the fuzzy regression model. The proposed method is followed by a couple of numerical examples to illustrate performance of the proposed method compared to some existing methods.

The rest of this paper is organized as follows. In Section 2, some concepts and preliminary results that will be used in this paper are recalled. Section 3 presents our new method to construct a regression model for fuzzy input-output data. In Section 4, two criteria are introduced to evaluate the goodness-of-fit of the proposed model. Section 5, will be devoted to numerical examples and comparison studies to clarify the theoretical results, to illustrate the advantages of the method, and to show some possible applications. Finally, in Section 6, some concluding remarks are given.

2. **Preliminary Concepts**

A fuzzy set $\widetilde{A}$ on the universal set $\mathbb{X}$ is described by its membership function $\widetilde{A}(x) : \mathbb{X} \to [0,1]$. Through this paper we consider $\mathbb{X} = \mathbb{R}$ (the real line). The α-cut of a fuzzy set $\widetilde{A}$ is defined as the crisp set $A_\alpha = \{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\}$, $\alpha \in (0,1]$. For $\alpha = 0$, define A_0 as the closure of the set $\{x \in \mathbb{R} : \widetilde{A}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if A_α is a non-empty compact interval, for any $\alpha \in [0,1]$. Such interval is represented by $A_\alpha = [A^l_\alpha, A^r_\alpha]$. An specific type of fuzzy number, which is rich and flexible enough to cover most of the applications, is the so-called LR-fuzzy number denoted by $\widetilde{N} = (n,l,r)_{LR}$ with central value $n \in \mathbb{R}$, left and right spread values $l, r \in \mathbb{R}^+$, decreasing left and right shape functions $L, R : \mathbb{R}^+ \to [0,1]$, with $L(0) = R(0) = 1$. Such LR-fuzzy numbers have the following membership function [46]

$$\widetilde{N}(x) = \begin{cases} L(\frac{n-x}{l}) & \text{if} \quad x \leq n, \\ R(\frac{x-n}{r}) & \text{if} \quad x > n. \end{cases}$$

An special type of LR-fuzzy number is the so-called triangular fuzzy number, denoted by $\widetilde{N} = (n,l,r)_T$. For $l = r$, the triangular fuzzy number $\widetilde{N}$ is called symmetric and is abbreviated by $\widetilde{N} = (n,l)_T$. The membership function and the α-cut of the triangular fuzzy number $\widetilde{N}$ are as follows

$$\begin{aligned} \widetilde{N}(x) &= \frac{x-(n-l)}{l}\mathcal{I}_{[n-l,n]}(x) + \frac{(n+r)-x}{r}\mathcal{I}_{(n,n+r]}(x), \quad x \in \mathbb{R}, \\ N_\alpha &= [N^l_\alpha, N^r_\alpha] = [n-(1-\alpha)l, n+(1-\alpha)r], \quad \alpha \in [0,1], \end{aligned}$$

where $\mathcal{I}_A$ stands the characteristic function of a crisp set A.

Theorem 2.1. [46] *Let $\widetilde{M} = (m, l_m, r_m)_{LR}$ and $\widetilde{N} = (n, l_n, r_n)_{LR}$ be two LR fuzzy numbers, and λ be a real number. Then*

$$\lambda \otimes \widetilde{M} = \begin{cases} (\lambda m, \lambda l_m, \lambda r_m)_{LR} & if \quad \lambda > 0, \\ \mathcal{I}_{\{0\}} & if \quad \lambda = 0, \\ (\lambda m, |\lambda| r_m, |\lambda| l_m)_{RL} & if \quad \lambda < 0, \end{cases}$$

$$\widetilde{M} \oplus \widetilde{N} = (m+n, l_m + l_n, r_m + r_n)_{LR}.$$

Note that in the special case, when $I = [i_1, i_2]$ and $J = [j_1, j_2]$ be two closed intervals, then [32]

$$\lambda \otimes I = \begin{cases} [\lambda i_1, \lambda i_2] & \lambda > 0, \\ 0 & \lambda = 0, \\ [\lambda i_2, \lambda i_1] & \lambda < 0. \end{cases}$$

$$I \oplus J = [i_1 + j_1, i_2 + j_2].$$

Definition 2.2. [32] The distance between two intervals $I = [i_1, i_2]$ and $J = [j_1, j_2]$ is defined as

$$\mathcal{D}(I, J) = (i_1 - j_1)^2 + (i_2 - j_2)^2.$$

The reader is refereed to Moore et al. [32] for more details on interval arithmetic.

3. The Proposed Model

3.1. **Fuzzy simple linear regression.** Assume that the set of observed data $(\widetilde{x}_i, \widetilde{y}_i)$, $i = 1, \ldots, n$, are given in which $\widetilde{y}_i$ and $\widetilde{x}_i$ are LR-fuzzy numbers as observations of the dependent and independent variables, respectively. We wish to construct a regression model for the aforementioned data as follows

$$\widetilde{y} = \beta_0 \oplus \beta_1 \otimes \widetilde{x}.$$

In order to estimate the crisp coefficients β_0 and β_1, we employ the following procedure.

Step 1: First, based on the α-level sets of the fuzzy input-output data, we transform the regression model into a set of regression models. In fact, using the α-level sets $x_{i\alpha} = [x^l_{i\alpha}, x^r_{i\alpha}]$ and $y_{i\alpha} = [y^l_{i\alpha}, y^r_{i\alpha}]$, $i = 1, \ldots, n$, we provide the following interval-valued regression model for each $\alpha \in [0, 1]$

$$\left[y^l_\alpha, y^r_\alpha\right] = b_0(\alpha) \oplus b_1(\alpha) \otimes \left[x^l_\alpha, x^r_\alpha\right].$$

Note that, the regression coefficients $b_0(\alpha)$ and $b_1(\alpha)$ are real-valued functions of α.

Step 2: Now, we try to estimate $b_0(\alpha)$ and $b_1(\alpha)$ in the above interval-valued regression model. To do this, the following optimality criterion being used which consists of minimizing the sum of the squared distances between the α-level sets of observations and estimations of the dependent variable. (For simplicity, b_0 and b_1 are used instead of $b_0(\alpha)$ and $b_1(\alpha)$, respectively).

$$\mathcal{D}(b_0, b_1) = \sum_{i=1}^{n} \mathcal{D}([y^l_{i\alpha}, y^r_{i\alpha}], b_0 \oplus b_1 \otimes [x^l_{i\alpha}, x^r_{i\alpha}])$$

$$= \begin{cases} \mathcal{D}^+(b_0, b_1) = \sum_{i=1}^n \left[(y^l_{i\alpha} - b_0 - b_1 x^l_{i\alpha})^2 + (y^r_{i\alpha} - b_0 - b_1 x^r_{i\alpha})^2\right], & b_1 \geq 0, \\ \mathcal{D}^-(b_0, b_1) = \sum_{i=1}^n \left[(y^l_{i\alpha} - b_0 - b_1 x^r_{i\alpha})^2 + (y^r_{i\alpha} - b_0 - b_1 x^l_{i\alpha})^2\right], & b_1 < 0, \end{cases}$$

Setting the partial derivatives of $\mathcal{D}(b_0, b_1)$ with respect to b_0 and b_1 to zero, we have

$$\frac{\partial}{\partial b_0}\mathcal{D}^+(b_0, b_1) = 0, \qquad \frac{\partial}{\partial b_1}\mathcal{D}^+(b_0, b_1) = 0, \qquad \text{if } b_1 \geq 0,$$

or

$$\frac{\partial}{\partial b_0}\mathcal{D}^-(b_0, b_1) = 0, \qquad \frac{\partial}{\partial b_1}\mathcal{D}^-(b_0, b_1) = 0, \qquad \text{if } b_1 < 0.$$

In the positive case, we obtain

$$\begin{aligned} b_0^+(\alpha) &= \bar{Y}(\alpha) - b_1^+(\alpha)\bar{X}(\alpha), \\ b_1^+(\alpha) &= \frac{\sum_{i=1}^n [x_{i\alpha}^l y_{i\alpha}^l + x_{i\alpha}^r y_{i\alpha}^r] - 2n\bar{X}(\alpha)\bar{Y}(\alpha)}{\sum_{i=1}^n [(x_{i\alpha}^l)^2 + (x_{i\alpha}^r)^2] - 2n\bar{X}^2(\alpha)}, \end{aligned}$$

where, $\bar{Y}(\alpha) = \frac{1}{n}\sum_{i=1}^n \frac{y_{i\alpha}^l + y_{i\alpha}^r}{2}$ and $\bar{X}(\alpha) = \frac{1}{n}\sum_{i=1}^n \frac{x_{i\alpha}^l + x_{i\alpha}^r}{2}$.
In the negative case, the regression parameters are estimated as

$$\begin{aligned} b_0^-(\alpha) &= \bar{Y}(\alpha) - b_1^-(\alpha)\bar{X}(\alpha), \\ b_1^-(\alpha) &= \frac{\sum_{i=1}^n [x_{i\alpha}^l y_{i\alpha}^r + x_{i\alpha}^r y_{i\alpha}^l] - 2n\bar{X}(\alpha)\bar{Y}(\alpha)}{\sum_{i=1}^n [(x_{i\alpha}^l)^2 + (x_{i\alpha}^r)^2] - 2n\bar{X}^2(\alpha)}. \end{aligned}$$

Step 3: Finally, we aggregate the set of functions $b_0^+(\alpha)$ and $b_1^+(\alpha)$ (or $b_0^-(\alpha)$ and $b_1^-(\alpha)$) to estimate the regression coefficients β_0 and β_1, as follows

$$\widehat{\beta}_0 = \int_0^1 b_0^+(\alpha)d\alpha, \qquad \widehat{\beta}_1 = \int_0^1 b_1^+(\alpha)d\alpha,$$

or

$$\widehat{\beta}_0 = \int_0^1 b_0^-(\alpha)d\alpha, \qquad \widehat{\beta}_1 = \int_0^1 b_1^-(\alpha)d\alpha.$$

Therefore, the optimal model is obtained as $\widehat{\widetilde{y}} = \widehat{\beta}_0 \oplus \widehat{\beta}_1 \otimes \widetilde{x}$.

3.2. **Extension to the Multiple Case.** The proposed procedure for fuzzy simple linear regression model could be extended to general cases with several independent variables as follows.

Step 1: By considering the α-level sets of the observed data, the following form of the multiple interval-valued regression model is considered

$$[y_\alpha^l, y_\alpha^r] = b_0(\alpha) \oplus b_1(\alpha) \otimes [x_{1\alpha}^l, x_{1\alpha}^r] \oplus \ldots \oplus b_p(\alpha) \otimes [x_{p\alpha}^l, x_{p\alpha}^r].$$

Step 2: The unknown vector parameter $\mathbf{b} = [b_0(\alpha), b_1(\alpha), \cdots, b_p(\alpha)]^t$ is evaluated by minimizing the following objective function

$$\mathcal{D}(\mathbf{b}) = \sum_{i=1}^n \mathcal{D}(y_{i\alpha}, \mathbf{x}_{i\alpha}\mathbf{b}).$$

This objective function is the sum of squared distances between the α-level sets of observations of the dependent variable $\widetilde{y}$ and their estimated values, i.e. $\mathbf{x}_\alpha \mathbf{b}$, where $\mathbf{x}_{i\alpha} = [1, x_{1i\alpha}, \ldots, x_{pi\alpha}]$ is the vector of the α-level sets of independent variables.

Since we are using the interval multiplication, it is necessary to ensure that the minimum and maximum values of the α-level sets are properly considered for both positive and negative cases of $b_0(\alpha)$, $b_1(\alpha)$, ..., and $b_p(\alpha)$. The discussion on the optimal solution would require to consider 2^p different objective functions. In each case, we can formalize such a requirement by introducing the following substitution variables

$$L_{ij,\alpha} = \begin{cases} x^l_{ji,\alpha} & b_j(\alpha) \geq 0, \\ x^r_{ji,\alpha} & b_j(\alpha) < 0, \end{cases} \qquad R_{ij,\alpha} = \begin{cases} x^r_{ji,\alpha} & b_j(\alpha) \geq 0, \\ x^l_{ji,\alpha} & b_j(\alpha) < 0, \end{cases}$$

where, $j = 0, 1, \ldots, p$, $i = 1, \ldots, n$, and $x^l_{0i,\alpha} = x^r_{0i,\alpha} = 1$. Now, we can rewrite the objective function $\mathcal{D}(\mathbf{b})$ as

$$\mathcal{D}(\mathbf{b}) = \parallel \mathbf{y}^l_\alpha - \mathbf{L}_\alpha \mathbf{b} \parallel^2 + \parallel \mathbf{y}^r_\alpha - \mathbf{R}_\alpha \mathbf{b} \parallel^2,$$

where, $\parallel \cdot \parallel$ is the Euclidean norm, and $\mathbf{L}_\alpha$ and $\mathbf{R}_\alpha$ are matrices with general elements $L_{ij,\alpha}$ and $R_{ij,\alpha}$, respectively. By equating to zero the partial derivative of $\mathcal{D}(\mathbf{b})$ with respect to the unknown parameter $\mathbf{b}$, the best value of $\mathbf{b}$ which minimizes the objective function $\mathcal{D}(\mathbf{b})$, is found. In fact the following system of linear equations must be solved:

$$\mathbf{L}^t_\alpha \mathbf{L}_\alpha \mathbf{b} - \mathbf{L}^t_\alpha \mathbf{y}^l_\alpha + \mathbf{R}^t_\alpha \mathbf{R}_\alpha \mathbf{b} - \mathbf{R}^t_\alpha \mathbf{y}^r_\alpha = 0,$$

where, $\mathbf{L}^t_\alpha$ and $\mathbf{R}^t_\alpha$ are the transposes of matrices $\mathbf{L}_\alpha$ and $\mathbf{R}_\alpha$, respectively. Subject to the existence of $(\mathbf{L}^t_\alpha \mathbf{L}_\alpha + \mathbf{R}^t_\alpha \mathbf{R}_\alpha)^{-1}$, the least squares estimate of $\mathbf{b}$ is obtained as

$$\widehat{\mathbf{b}} = (\mathbf{L}^t_\alpha \mathbf{L}_\alpha + \mathbf{R}^t_\alpha \mathbf{R}_\alpha)^{-1} (\mathbf{L}^t_\alpha \mathbf{y}^l_\alpha + \mathbf{R}^t_\alpha \mathbf{y}^r_\alpha).$$

Step 3: Finally, we aggregate the optimal solutions $b_j(\alpha)$, $j = 0, 1, \ldots, p$, to estimate the regression coefficients β_j, as follows

$$\widehat{\beta}_j = \int_0^1 b_j(\alpha) d\alpha, \qquad j = 0, 1, \ldots, p.$$

The optimal model, therefore, is obtained as

$$\widehat{\widetilde{y}} = \widehat{\beta}_0 \oplus \widehat{\beta}_1 \otimes \widetilde{x}_1 \oplus \cdots \oplus \widehat{\beta}_p \otimes \widetilde{x}_p.$$

Remark 3.1. Generally, we may have no idea about the signs of b_js. Therefore, we need to consider all 2^p cases for the objective function in order to find the optimal model. In such a case, we consider the amounts of b_j's, $j = 0, 1, \ldots, p$, for which the related objective function is smaller than the other ones. However, in many practical studies we may pre-determine the signs of b_js, for example, by using some expert opinions or by fitting a classical regression model to the centers of the fuzzy observations.

Remark 3.2. It should be noted that, Bargiela et al. [3] considered a more or less similar approach to regression modeling for fuzzy data, based on the following distance

$$D^* = \sum_{i=1}^{n} \int_0^1 \left([y^l_i(\alpha) - \widehat{y}^l_i(\alpha)]^2 + [y^r_i(\alpha) - \widehat{y}^r_i(\alpha)]^2 \right) d\alpha,$$

to estimate the parameters of the regression model. However, the proposed method in this paper, is based on the distance $\mathcal{D}$ between the α-level sets of fuzzy observations and fuzzy estimations of $\widetilde{y}$ to provide an interval-valued regression model. Then, we integrate the parameters of the interval-valued regression model to estimate the parameters of the main fuzzy regression model.

Remark 3.3. The proposed method uses of α-level sets in the first stage, so that there is no limitation for the shape of fuzzy observations. Therefore, the method is applicable when the observations have different shapes of membership functions. Also, when the fuzzy observations are reduced to crisp values, then the objective function becomes $2 \parallel \mathbf{y} - \mathbf{X}\beta \parallel^2$, which results in the conventional least squares estimates for the coefficients of the model.

4. Methods of Evaluations of the Model

To evaluate the fuzzy regression models, several criteria have been proposed by authors. Here, we use two well known criteria to evaluate our proposed approach and to compare it with some common approaches. The first one is the common index to evaluate the goodness-of-fit of the fuzzy regression models. The second one is proposed to evaluate the predictability of the fuzzy regression models.

Definition 4.1. (Kim and Bishu [25]) The mean of error in estimation for a fuzzy regression model is defined by $\mathcal{E} = \frac{1}{n}\sum_{i=1}^{n}\mathcal{E}(\widetilde{y}_i, \widehat{\widetilde{y}}_i)$, where

$$\mathcal{E}(\widetilde{y}_i, \widehat{\widetilde{y}}_i) = \int \frac{|\widetilde{y}_i(x) - \widehat{\widetilde{y}}_i(x)|}{\int \widetilde{y}_i(x)\, dx} dx.$$

Definition 4.2. The mean of predictive ability of a fuzzy regression model is defined by $MPA = \frac{1}{n}\sum_{i=1}^{n} PA(i)$, where

$$PA(i) = \mathcal{E}(\widetilde{y}_i, \widehat{\widetilde{y}}^{\,i}),$$

in which $\widehat{\widetilde{y}}^{\,i}$ is the predicted value of the dependent variable with the model for which the ith observation is left out from the data set while the remaining observations are used to develop the fuzzy regression model.

5. Comparison Studies

Here, we provide numerical examples to explain how the proposed method is applicable to obtain a suitable regression model for fuzzy observations. We also compare the goodness-of-fit of the proposed model with some common fuzzy regression models. The syntax of the codes used in the following examples is written in MATLAB [29].

5.1. Simple Case.

Example 5.1. Consider the fuzzy input-output data set in Table 1, given by Sakawa and Yano [41]. The data set consists of eight pairs of symmetric triangular fuzzy numbers. Based on such fuzzy data, we compare the proposed method in this paper with some other methods. Using the computational procedure proposed

i	$\widetilde{x}_i = (x_i, l_i)_T$	$\widetilde{y}_i = (y_i, s_i)_T$
1	$(2.0, 0.5)_T$	$(4.0, 0.5)_T$
2	$(3.5, 0.5)_T$	$(5.5, 0.5)_T$
3	$(5.5, 1.0)_T$	$(7.5, 1.0)_T$
4	$(7.0, 0.5)_T$	$(6.5, 0.5)_T$
5	$(8.5, 0.5)_T$	$(8.5, 0.5)_T$
6	$(10.5, 1.0)_T$	$(8.0, 1.0)_T$
7	$(11.0, 0.5)_T$	$(10.5, 0.5)_T$
8	$(12.5, 0.5)_T$	$(9.5, 0.5)_T$

TABLE 1. Fuzzy Data in Example 5.1

in Section 3.1, we consider first the following interval-valued regression model for each $\alpha \in [0, 1]$

$$[y^l_\alpha, y^r_\alpha] = b_0(\alpha) \oplus b_1(\alpha) \otimes [x^l_\alpha, x^r_\alpha].$$

In order to determine the sign of the coefficient $b_1(\alpha)$, we provide the classical regression model based on the centers of the observations

$$\widehat{y}_i = 3.5724 + 0.5193\, x_i, \qquad i = 1, \ldots, 8.$$

The above classical regression model confirms the positive relationship between the dependent and independent variables. Therefore, we consider the sign of the coefficient $b_1(\alpha)$ as positive for each $\alpha \in [0, 1]$. Now using the procedure described in Step 2, the coefficients $b_0(\alpha)$ and $b_1(\alpha)$ are estimated as

$$\widehat{\mathbf{b}}(\alpha) = \begin{bmatrix} \widehat{b}_0(\alpha) \\ \widehat{b}_1(\alpha) \end{bmatrix} = \begin{bmatrix} \dfrac{-7\alpha^2 + 14\alpha + 11164}{112\alpha^2 - 224\alpha + 3239} & \dfrac{112\alpha^2 - 224\alpha + 1736}{112\alpha^2 - 224\alpha + 3239} \end{bmatrix}^t.$$

Finally, the estimated coefficients are derived as

$$\begin{aligned} \widehat{\beta}_0 &= \int_0^1 \frac{-7\alpha^2 + 14\alpha + 11164}{112\alpha^2 - 224\alpha + 3239}\, d\alpha = 3.530, \\ \widehat{\beta}_1 &= \int_0^1 \frac{112\alpha^2 - 224\alpha + 1736}{112\alpha^2 - 224\alpha + 3239}\, d\alpha = 0.525. \end{aligned}$$

So, the fuzzy linear regression model is obtained as follows

$$\widehat{\widetilde{y}} = 3.530 \oplus 0.525 \otimes \widetilde{x}.$$

We compare the proposed method with five well known methods proposed for modeling the fuzzy input-output data. The index $\mathcal{E}$ (mean of errors) is employed to evaluate the goodness-of-fit of the models. Different fuzzy regression models and the amounts of index $\mathcal{E}$ for such models are summarized in Table 2. It is clear that the proposed method offers an improved performance over all the other models.

Example 5.2. In this example, we consider the performances of the methods studied in the previous example by discussing the accuracy of forecasting. To illustrate the forecasting accuracy of the proposed model, suppose that the first pair of observations, i.e. $(\widetilde{x}_1, \widetilde{y}_1) = ((2.0, 0.5)_T, (4.0, 0.5)_T)$, is unknown and we

Proposed by	Model	$\mathcal{E}$
Nasrabadi and Nasrabadi [36]	$\widehat{\widetilde{y}}_i = (3.5767 + 0.5467\, x_i, l_i)_T$	1.48
Bargiela et al. [3]	$\widehat{\widetilde{y}}_i = (3.446 + 0.536\, x_i, 0.536\, l_i)_T$	1.34
Kao and Chyu [22]	$\widehat{\widetilde{y}}_i = (3.5724 + 0.5193\, x_i, 0.24 + 0.5193\, l_i)_T$	1.50
Kao and Chyu [23]	$\widehat{\widetilde{y}}_i = (3.3936 + 0.543\, x_i, 0.9644 + 0.543 l_i)_T$	2.01
Chen and Dang [11]	$\widehat{\widetilde{y}}_i = 3.5284 + 0.5298\, \widetilde{x}_i + \widetilde{E}_i^{\dagger}$	1.44
The proposed model	$\widehat{\widetilde{y}}_i = (3.530 + 0.525\, x_i, 0.525 l_i)_T$	1.32

† For the values of $\widetilde{E}_i$ see [11]

Table 2. Fuzzy Regression Models and Their Mean of Errors in Example 5.1

i	Removed pair	Our Model	$PA(i)$	$PA(i)$ [3]	$PA(i)$ [22]	$PA(i)$ [23]	$PA(i)$ [36]
1	$((2.0, 0.5), (4.0, 0.5))$	$4.10 \oplus 0.46\, \widetilde{x}$	1.4662	1.4663	2.0000	2.6875	1.7823
2	$((3.5, 0.5), (5.5, 0.5))$	$3.45 \oplus 0.53\, \widetilde{x}$	0.6713	0.6724	0.5739	1.6365	0.0392
3	$((5.5, 1.0), (7.5, 1.0))$	$3.17 \oplus 0.55\, \widetilde{x}$	1.5093	1.5094	1.6406	1.9187	1.4130
4	$((7.0, 0.5), (6.5, 0.5))$	$3.66 \oplus 0.52\, \widetilde{x}$	1.5200	1.5200	1.9293	2.4652	1.9981
5	$((8.5, 0.5), (8.5, 0.5))$	$3.50 \oplus 0.51\, \widetilde{x}$	1.4404	1.4404	1.6679	2.0003	0.9527
6	$((10.5, 1.0), (8.0, 1.0))$	$3.40 \oplus 0.56\, \widetilde{x}$	1.5242	1.5242	1.6521	2.0578	1.7668
7	$((11.0, 0.5), (10.5, 0.5))$	$3.75 \oplus 0.46\, \widetilde{x}$	1.4694	1.4695	2.0000	3.3550	2.0000
8	$((12.5, 0.5), (9.5, 0.5))$	$3.29 \oplus 0.57\, \widetilde{x}$	1.5715	1.5716	1.9808	3.1250	2.0000
		MPA$=\frac{1}{8}\sum_{i=1}^{8} PA(i)$	1.3965	1.3967	1.6806	2.4058	1.4940

Table 3. Removed Pair of Observation and Forecasting Performance in Example 5.2

would like to use the other seven pairs of observations to predict it. By applying the proposed method stated in Section 3.1, the constructed model is obtained as

$$\widehat{\widetilde{y}} = 4.1040 + 0.4662\, \widetilde{x},$$

which is shown in the first row of the third column of Table 3. For the removed independent observation $\widetilde{x}_1 = (2.0, 0.5)_T$, its corresponding estimate is $\widehat{\widetilde{y}}^{1} = (5.0364, 0.2331)_T$. Referring to Table 1, the real response for $\widetilde{x}_1$ is $\widetilde{y}_1 = (4.0, 0.5)_T$. Using the index $PA(1) = \mathcal{E}(\widetilde{y}_1, \widehat{\widetilde{y}}^{1})$ as the measure of the forecasting performance, the error in estimation for the proposed model is 1.4662 which is shown in the first row of the forth column of Table 3. The third and fourth columns of Table 3 show models and the errors in predicting the removed pair of the observations. The mean of the forth column shows the mean of predictive ability (MPA) as the index for forecasting performance of the proposed model, which is 1.3965.

By applying the index $PA(i) = \mathcal{E}(\widetilde{y}_i, \widehat{\widetilde{y}}^{i})$, the errors in forecasting are calculated for the other methods. The results are shown in Table 3. The mean of predictive ability (MPA) for various methods are given in the last row of this table. As expected, the forecasting performance of the proposed method is better than that of the other methods. It is remarkable that, we could not employ the Chen and Dang's method [11] in this comparison. In fact, for the second removed observation, the estimated center for the fuzzy dependent variable has no degree of membership in any observed dependent variable. So, we could not use the fuzzy inference system introduced by Chen and Dang to predict the fuzzy error term (for more details see Chen and Dang [11]).

i	$\widetilde{x}_{i1} = (x_{i1}, l_{i1}, r_{i1})_T$	$\widetilde{x}_{i2} = (x_{i2}, l_{i2}, r_{i2})_T$	$\widetilde{y}_i = (y_i, l_i, r_i)_T$
1	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(8, 0.75, 1.00)_T$
2	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
3	$(6, 0.25, 0.50)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
4	$(8, 0.75, 1.00)_T$	$(9, 0.00, 1.00)_T$	$(9, 0.00, 1.00)_T$
5	$(8, 0.75, 1.00)_T$	$(8, 0.75, 1.00)_T$	$(8, 0.75, 1.00)_T$
6	$(6, 0.25, 0.50)_T$	$(7, 0.50, 1.25)_T$	$(5, 0.00, 1.00)_T$
7	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(7, 0.50, 1.25)_T$
8	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(5, 0.00, 1.00)_T$
9	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(7, 0.50, 1.25)_T$
10	$(6, 0.25, 0.50)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
11	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(8, 0.75, 1.00)_T$
12	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$	$(6, 0.25, 0.50)_T$
13	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(9, 0.00, 1.00)_T$
14	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(8, 0.75, 1.00)_T$
15	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$
16	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$
17	$(6, 0.25, 0.50)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
18	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(7, 0.50, 1.25)_T$
19	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$
20	$(7, 0.50, 1.25)_T$	$(9, 0.00, 1.00)_T$	$(7, 0.50, 1.25)_T$
21	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(7, 0.50, 1.25)_T$
22	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(6, 0.25, 0.50)_T$
23	$(7, 0.50, 1.25)_T$	$(9, 0.00, 1.00)_T$	$(7, 0.50, 1.25)_T$
24	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$
25	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
26	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$
27	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$
28	$(7, 0.50, 1.25)_T$	$(8, 0.75, 1.00)_T$	$(7, 0.50, 1.25)_T$
29	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$	$(7, 0.50, 1.25)_T$
30	$(6, 0.25, 0.50)_T$	$(7, 0.50, 1.25)_T$	$(6, 0.25, 0.50)_T$

TABLE 4. The Triangular Fuzzy Input-output Data in Example 5.3

5.2. **Multiple Case.**

Example 5.3. In this example, we consider a fuzzy multiple linear regression model. Consider the data set given in Table 4 (see also D'Urso [13, pp. 62, Table 3]), which consists of two fuzzy input variables (decision on cooking $\widetilde{x}_1$ and decision on environment $\widetilde{x}_2$) and a fuzzy output variable (decision on cellar $\widetilde{y}$). This data are the performances of the 30 good-quality Roman restaurants. The proposed model for such data is obtained as follows. First, we consider the following interval-valued regression model

$$[y^l_\alpha, y^r_\alpha] = b_0(\alpha) \oplus b_1(\alpha) \otimes [x^l_{1\alpha}, x^r_{1\alpha}] \oplus b_2(\alpha) \otimes [x^l_{2\alpha}, x^r_{2\alpha}], \forall \alpha \in [0, 1].$$

The classical regression model based on the centers of the fuzzy data is as follows

$$\widehat{y} = -3.3379 + 0.9846\, x_1 + 0.4572\, x_2,$$

we suggest that the signs of the coefficients $\widehat{b}_1(\alpha)$ and $\widehat{b}_2(\alpha)$ are positive. Now, we consider the following matrices

$$\mathbf{L}_\alpha = \begin{bmatrix} 1 & x^l_{11,\alpha} & x^l_{21,\alpha} \\ 1 & x^l_{12,\alpha} & x^l_{22,\alpha} \\ \vdots & \vdots & \vdots \\ 1 & x^l_{130,\alpha} & x^l_{230,\alpha} \end{bmatrix}, \qquad \mathbf{R}_\alpha = \begin{bmatrix} 1 & x^r_{11,\alpha} & x^r_{21,\alpha} \\ 1 & x^r_{12,\alpha} & x^r_{22,\alpha} \\ \vdots & \vdots & \vdots \\ 1 & x^r_{130,\alpha} & x^r_{230,\alpha} \end{bmatrix}.$$

Proposed by	$\widehat{\widetilde{y}}_i = (\widehat{y}_i, \widehat{l}_i, \widehat{r}_i)_T$	Performances
Lu and Wang [27]	$\widehat{y}_i = -1.66 + 1.33\,x_{i1}$ $\widehat{l}_i = 0.8 - 0.91\,l_{i1} - 0.65\,l_{i2} + 0.63\,x_{i1}$ $-0.28\,x_{i2} - 1.48\,r_{i2}$ $\widehat{r}_i = 0.08 + 0.37\,l_{i1} + 0.09\,l_{i2} - 0.12\,x_{i1}$ $+0.03\,x_{i2} + r_{i1} + 0.2\,r_{i2}$	$\delta^2 = 64.56_{(2)}$ $S = 0.56_{(1)}$ $D^* = 60.66_{(3)}$
D'Urso [13]	$\widehat{y}_i = [1\ x_{i1}\ x_{i2}]\mathbf{x} + [1\ l_{i1}\ l_{i2}]\mathbf{l} + [1\ r_{i1}\ r_{i2}]\mathbf{r}$ $\mathbf{x} = [0.6498399\ 0.4542534\ 0.4924441]^t$ $\mathbf{l} = [-1.868527\ 2.3604004\ 0.7392849]^t$ $\mathbf{r} = [-0.233325\ \ -0.13392\ 0.1271022]^t$ $\widehat{l}_i = -0.401173 + 0.1173197\,\widehat{y}_i$ $\widehat{r}_i = -0.650102 + 0.2306911\,\widehat{y}_i$	$\delta^2 = 73.69_{(3)}$ $S = 0.22_{(3)}$ $D^* = 32.54_{(1)}$
Proposed model	$\widehat{y}_i = -1.7638 + 0.7864\,x_{i1} + 0.4245\,x_{i2}$ $\widehat{l}_i = 0.7864\,l_{i1} + 0.4245\,l_{i2}$ $\widehat{r}_i = 0.7864\,r_{i1} + 0.4245\,r_{i2}$	$\delta^2 = 60.07_{(1)}$ $S = 0.38_{(2)}$ $D^* = 37.03_{(2)}$

Table 5. The Performance of Various Models for Restaurants Data in Example 5.3

Therefore, according to the computational procedure given in Section 3.2, the parameters of the interval-valued regression model are estimated as

$$\widehat{\mathbf{b}}(\alpha) = \begin{bmatrix} \widehat{b}_0(\alpha) \\ \widehat{b}_1(\alpha) \\ \widehat{b}_2(\alpha) \end{bmatrix} = (\mathbf{L}_\alpha^t \mathbf{L}_\alpha + \mathbf{R}_\alpha^t \mathbf{R}_\alpha)^{-1}(\mathbf{L}_\alpha^t \mathbf{y}_\alpha^l + \mathbf{R}_\alpha^t \mathbf{y}_\alpha^r)$$

$$= \begin{bmatrix} \frac{-\frac{52505}{4}\alpha^5 + \frac{7138661}{4}\alpha^4 - \frac{14456797}{2}\alpha^3 + \frac{31135525}{2}\alpha^2 - \frac{57087357}{4}\alpha - \frac{17181023}{4}}{1874767\alpha^4 - 8061200\alpha^3 + 21457094\alpha^2 - 26689688\alpha + 13952403} \\ \\ \frac{1105999\alpha^4 - 4767980\alpha^3 + 12105210\alpha^2 - 14677740\alpha + 8728975}{1874767\alpha^4 - 8061200\alpha^3 + 21457094\alpha^2 - 26689688\alpha + 13952403} \\ \\ \frac{475056\alpha^4 - 2066080\alpha^3 + 6485984\alpha^2 - 9016736\alpha + 5279920}{1874767\alpha^4 - 8061200\alpha^3 + 21457094\alpha^2 - 26689688\alpha + 13952403} \end{bmatrix}.$$

Finally, by integrating $\widehat{b}_i(\alpha)$, $\alpha \in (0,1]$, $i = 0,1,2$, the parameters of the fuzzy regression model are estimated and then, the optimal model is obtained as

$$\widehat{\widetilde{y}} = -1.7638 \oplus 0.7864 \otimes \widetilde{x}_1 \oplus 0.4245 \otimes \widetilde{x}_2.$$

Comparison with D'Urso [13] **and Lu and Wang** [27] **Methods:** To clarify the performance of the proposed model, here, following the comparison provided in Example 4 by Lu and Wang [27], we compare our method with the methods introduced by D'Urso [13] and Lu and Wang [27]. Lu and Wang [27] have noted that: "... the solution of a model depends on the objective function to be optimized. Thus, it is not easy to compare the solutions of different models that optimize different objective functions. However, if the evaluation results of model M_A are better than those of model M_B when both objective functions of M_A and M_B are used as the evaluation measurements, we can then say M_A outperforms M_B in terms of these two evaluation measurements." So, in order to do a perfect comparison between the performance of various models, we employ three different criteria: the sum of squared distances between fuzzy numbers, δ^2, the sum of similarity measure S, and the distance D^*, which are defined as follows

$$\delta^2 = \sum_{i=1}^{n}[y_i - \widehat{y}_i]^2 + [(y_i - l_i) - (\widehat{y}_i - \widehat{l}_i)]^2 + [(y_i + r_i) - (\widehat{y}_i + \widehat{r}_i)]^2,$$

$$S = \sum_{i=1}^{n} \frac{\int \min\{\widetilde{y}_i(x), \widehat{\widetilde{y}}_i(x)\}\, dx}{\int \max\{\widetilde{y}_i(x), \widehat{\widetilde{y}}_i(x)\}\, dx},$$

$$D^* = \sum_{i=1}^{n} \int_0^1 \left([y_i^l(\alpha) - \widehat{y}_i^l(\alpha)]^2 + [y_i^r(\alpha) - \widehat{y}_i^r(\alpha)]^2\right) d\alpha.$$

Note that, i) the objective function of D'Urso's model [13] is to minimize the sum of squared distances between $\widetilde{y}$ and $\widehat{\widetilde{y}}$, i.e. δ^2, ii) Lu and Wang [27] maximize the sum of similarity measures between $\widetilde{y}$ and $\widehat{\widetilde{y}}$. In their study, Lu and Wang [27] showed that their model dominated D'Urso's model [13] based on the criteria δ^2 and S.

Now, we use these three criteria to show the performance of the proposed model and to compare it with the models provided by D'Urso [13] and Lu and Wang [27]. The results are given in Table 5. From this table, we can see that the sum of squared distance is smaller than those of the other methods, so that, on the basis of criterion δ^2, the proposed method provides a better fuzzy model than those of D'Urso [13] and Lu and Wang [27]. With a general point of view, at least based on the results of this example the proposed model (with ranks 1, 2, and 2) is superior to the model obtained by Lu and Wang's method (with ranks 2, 1, and 3). The proposed method has also better ranks with respect to the model proposed by D'Urso (with ranks 3, 3, and 1). It should be mentioned, it is shown that the models proposed by D'Urso [13] and Lu and Wang [27] dominated some other fuzzy regression models with different criteria as their objective functions. Therefore, at least by the results provided in this example, it is concluded that the proposed model, not only is better than these two models but also is better than a lot of models dominated by the above three models.

6. **Concluding Remarks**

A least squares approach to fuzzy regression model was investigated in this paper for modeling fuzzy input-output data. A two-stage procedure was described to construct the model. First, by utilizing the concept of least squares as a fitting criterion and applying it to the interval-valued data obtained from the α-level sets of fuzzy data, a regression model was provided at each level of α. The goal of estimating the coefficients of this interval-valued regression analysis was to find a regression model having a minimal total difference between the α-level sets of observations and estimations of fuzzy dependent variable. Finally, by integrating the obtained parameters of the interval-valued regression models, the optimal values of parameters for the main fuzzy regression model were estimated.

Since the proposed method is using the α-level sets in the first stage, there is no limitation for the fuzzy observations to have the same type of membership functions and this method is applicable when observations have different types of membership functions. Also, when the fuzzy observations are reduced to crisp values, the proposed method is reduced to the conventional least squares method. By comparing the proposed method with some well-known methods, applied to simple and multiple models, the performance of the method was investigated, based on three goodness of fit criteria. The claim about the better performance of the proposed method is true for the results of the solved numerical examples in this paper. Note that generally talking about the performance of any fuzzy regression model needs variety different examples. The investigation of the proposed approach to non linear regression (see, e.g. D'Urso and Gastaldi [14]), based on the least absolutes deviation would be a potential topic for the future work.

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