FUZZY TOPOLOGY GENERATED BY FUZZY NORM

M. SAHELI

Abstract. In the current paper, consider the fuzzy normed linear space (X, N) which is defined by Bag and Samanta. First, we construct a new fuzzy topology on this space and show that these spaces are Hausdorff locally convex fuzzy topological vector space. Some necessary and sufficient conditions are established to illustrate that the presented fuzzy topology is equivalent to two previously studied fuzzy topologies.

1. **Introduction**

The notion of fuzzy norm on a linear space was first introduced by Katrasas [9]. Feblin [6] gave an idea of a fuzzy norm on a linear space whose associated metric is Kalva type [7]. Cheng and Menderson [3] considered a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [8]. Felbins definition of a fuzzy norm of a linear operator between two fuzzy normed spaces was generalized by Xiao and Zhu [11]. Bag and Samanta [2] introduced a notion of boundedness of a linear operator between fuzzy normed spaces, and studied the relation between fuzzy continuity and fuzzy boundedness. They also considered fuzzy bounded linear functionals, the concept of fuzzy dual spaces, and established some fundamental theorems in the area of fuzzy functional analysis.
In [4], Das and Das defined a fuzzy topology on the fuzzy normed linear space defined by Felbin and studied some basic properties of this fuzzy topology. After, Fang [5] showd that X with this topology is not a topological vector space and modified the fuzzy topology and proved some results. Also, Xu and Fang defined another fuzzy topological space and studied these spaces [12]. Recently, we defined two fuzzy topology on the fuzzy normed linear space defined by Bag and Samanta and studied some properties of these fuzzy topologies [10].
In this paper, we define a new fuzzy topology on fuzzy normed linear space defined by Bag and Samanta and show that fuzzy normed linear space equipped with this fuzzy topology is a topological vector space. And an attempt is made to find such relation by making a comparative study of the fuzzy topology defined in this paper and [10].

2. **Preliminaries**

We give below some basic preliminaries required for this paper.

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Definition 2.1. [1] Let X be a linear space over R (real number). Let N be A fuzzy subset of $X \times R$ such that for all $x, u \in X$ and $c \in R$:
$(N1)$ $N(x,t) = 0$ for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$ for all $t \in R$,
$(N4)$ $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$ for all $s,t \in R$,
$(N5)$ $N(x,.)$ is a nondecreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$.
Then N is called a fuzzy norm on X.

We assume that
$(N6)$ $N(x,t) > 0$ for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of R and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of R.
$(N8)$ For each $x \neq 0$, there is $t_x > 0$ such that $N(x,t_x) = 0$.
$(N9)$ for each $\alpha, \beta \in (0,1)$, there is $m_{\alpha,\beta} > 0$ such that $N(x,t) \geq \alpha$ implies $N(x, tm_{\alpha,\beta}) \geq \beta$.
$(N10)$ for each $\alpha, \in (0,1)$, there is $m_\alpha > 0$ such that $N(x,t) > 0$ implies $N(x, tm_\alpha) \geq \alpha$.

Example 2.2. Let $(X, \|.\|)$ be a normed space. We define

$$N(x,t) = \begin{cases} t/(t+\|x\|) & , \quad t > 0,\ x \in X \\ 0 & , \quad t \leq 0,\ x \in X. \end{cases}$$

It is clear that (X,N) is a fuzzy normed linear space such that N satisfies $(N7)$. Now we show that N satisfies condition $(N9)$. Assume that $\alpha, \beta \in (0,1)$, $x \in X$, $t \in \mathbb{R}$ and $N(x,t) \geq \alpha$. Hence $t/(t + \|x\|) \geq \alpha$. Thus $t \geq (\alpha/(1-\alpha))\|x\|$. So $(\beta(1-\alpha)/\alpha(1-\beta))t \geq (\beta/(1-\beta))\|x\|$. Suppose that $m_{\alpha,\beta} = \beta(1-\alpha)/\alpha(1-\beta)$. Then $m_{\alpha,\beta}t \geq (\beta/(1-\beta))\|x\|$. This implies that $N(x, m_{\alpha,\beta}t) \geq \beta$. Therefore N satisfies $(N9)$.

Example 2.3. Let $(X, \|.\|)$ be a normed space. We define

$$N(x,t) = \begin{cases} 1 \quad , & t > \|x\|,\ x \in X \\ (t - \|x\|/2)/\|x\|/2 \quad , & \|x\|/2 \leq t \leq \|x\|,\ x \in X \\ 0 \quad , & t \leq \|x\|/2,\ x \in X. \end{cases}$$

It is clear that (X,N) is a fuzzy normed linear space such that N satisfies $(N7)$ and $(N8)$. Now we show that N satisfies condition $(N10)$. Assume that $\alpha, \in (0,1)$, $x \in X$, $t \in \mathbb{R}$ and $N(x,t) > 0$. let $m_\alpha = 1/(\alpha+1)$.

Case 1: If $t > \|x\|$ then $m_\alpha t = t/(1+\alpha) \geq t > \|x\|$. Hence $N(x, m_\alpha t) = 1 \geq \alpha$.

Case 2: If $\|x\|/2 \leq t \leq \|x\|$ then $m_\alpha t = t/(1+\alpha) \geq \|x\|/2$. So $(t - \|x\|/2)/\|x\|/2 \geq \alpha$. Thus $N(x, m_\alpha t) \geq \alpha$.

Therefore N satisfies $(N10)$.

Definition 2.4. [4] A fuzzy subset μ of a vector space X is said to be convex if

$$\mu(kx + (1-k)y) \geq \min(\mu(x), \mu(y)), \text{ for all } x, y \in X \text{ and } k \in [0,1].$$

Definition 2.5. [5] Let X be a vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $A, B \in I^X$ and $t \in \mathbb{K}$. Then $A + B$ and tA are defined by

$$(A + B)(x) = \sup_{u+v=x} (A(u) \wedge B(v))$$

and

$$(tA)(x) = A(x/t),\ t \neq 0,$$

$$(0A)(x) = \begin{cases} \sup_{y \in X} A(y) & , \quad x = 0 \\ 0 & , \quad x \neq 0. \end{cases}$$

Definition 2.6. [4] A fuzzy topology on a set X is a family τ of fuzzy subsets of X satisfying the following:
(i) The fuzzy subsets 1 and 0 are in τ,
(ii) τ is closed under finite intersection of fuzzy subsets,
(iii) τ is closed under arbitrary union of fuzzy subsets.
The pair (X, τ) is called a fuzzy topological space.

Definition 2.7. [4] A fuzzy set μ in a fuzzy topological space (X, τ) is called a neighborhood of a point $x \in X$ if and only if there is ρ in τ such that $\rho \subseteq \mu$ and $\mu(x) = \rho(x) > 0$.

Definition 2.8. [4] A fuzzy topological space (X, τ) is said to be fuzzy Hausdorff if for $x, y \in X$ and $x \neq y$ there exist $\eta, \mu \in \tau$ with $\mu(x) = \eta(y) = 1$ and $\eta \cap \mu = \emptyset$.

Definition 2.9. [5] A stratified fuzzy topology τ on a vector space X is said to be an fuzzy vector topology, if the following two mappings

$$f : X \times X \longrightarrow X,\ (x, y) \longrightarrow x + y \text{ and } g : \mathbb{K} \times X \longrightarrow X,\ (t, x) \longrightarrow tx,$$

are continuous, where $\mathbb{K}$ is equipped with the fuzzy topology induced by the usual topology and $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product fuzzy topologies. A vector space X with an fuzzy vector topology τ, denoted by (X, τ) is called an fuzzy topological vector space (FTVS).

Definition 2.10. [5] Let (X, τ) be fuzzy topological space and $x_\alpha \in Pt(I^X)$.
(i) A fuzzy set U on X is called Q-neighborhood of x_α iff there exists $G \in \tau$ such that $x_\alpha \widetilde{\in} G \subseteq U$.
(ii) A family $\mathfrak{U}_{x_\alpha}$ of Q-neighborhoods of x_α is called a Q-neighborhood base of x_α iff for every Q-neighborhood A of x_α, there exists $U \in \mathfrak{U}_{x_\alpha}$ such that $U \subseteq A$.

Definition 2.11. [5] A fuzzy topological vector space (X, τ) is said to be of QL-type, if there exists a family $\mathfrak{U}$ of fuzzy sets on X such that for each $\alpha \in (0, 1]$,

$$\mathfrak{U}_\alpha = \{U \cap \underline{r} : U \in \mathfrak{U},\ r \in (1 - \alpha, 1]\}$$

is a Q-neighborhood base of 0_α in (X, τ). The family $\mathfrak{U}$ is called a Q-prebase for τ.

Theorem 2.12. [5] *Let (X, τ) be a fuzzy topological space, $U \in I^X$ and $x \in X$. Then U is a neighborhood of x if and only if U is a Q-neighborhood of x_α for each $\alpha \in (1 - U(x), 1]$.*

Theorem 2.13. [5] *Let (X,τ) be a fuzzy topological vector spaces. Then*
(i) U is an (open) Q-neighborhood of O_α iff $x+U$ is an (open) Q-neighborhood of x_α, where $x \in X$.
(ii) U is an (open) Q-neighborhood of x_α iff tU is an (open) Q-neighborhood of $(tx)_\alpha$, where $t \in \mathbb{K}$, $t \neq 0$.

Lemma 2.14. [5] *Let (X,τ) be an FTVS and $\mathfrak{U}_\alpha$ a Q-neighborhood base of 0_α in X, $\alpha \in (0,1]$. Then the following conclusions hold*
(i) If $U \in \mathfrak{U}_\alpha$ or $U = \underline{r}$, where $r \in (1-\alpha, 1]$, then there exists $\alpha_0 \in (0,\alpha)$ such that for each $\mu \in [\alpha_0, 1]$ there exists a $V \in \mathfrak{U}_\mu$ such that $V \subseteq U$,
(ii) If $U, V \in \mathfrak{U}_\alpha$, then there exists $W \in \mathfrak{U}_\alpha$ such that $W \subseteq U \cap V$,
(iii) If $U \in \mathfrak{U}_\alpha$, then there exists $V \in \mathfrak{U}_\alpha$ such that $V + V \subseteq U$,
(iv) If $U \in \mathfrak{U}_\alpha$, then there exists $V \in \mathfrak{U}_\alpha$ such that $tV \subseteq U$ for all $t \in \mathbb{K}$ with $|t| \leq 1$,
(v) If $U \in \mathfrak{U}_\alpha$ and $x \in X$, there exists $\lambda > 0$ such that $x_\alpha \widetilde{\in} \lambda U$.
Conversely, let X be a vector space over $\mathbb{K}$ such that every $\alpha \in (0,1]$ has a family $\mathfrak{U}_\alpha$ of fuzzy sets on X satisfying the conditions (i)-(v), then there exists a unique fuzzy topology τ on X such that (X,τ) is an FTVS and $\mathfrak{U}_\alpha$ is a Q-neighborhood base of 0_α.

Definition 2.15. [5] A fuzzy topological vector space (X,τ) is said to be locally convex, if for each $\alpha \in (0,1]$, there is a base of Q-neighborhoods of 0_α consisting of convex fuzzy sets.

Definition 2.16. [10] Let (X,N) be a fuzzy normed linear space and let $x \in X$, $\alpha \in (0,1)$ and $\epsilon > 0$ the fuzzy set $\mu_\alpha(x,\epsilon)$ defined on X by

$$\mu_\alpha(x,\epsilon)(y) = \begin{cases} 1-\alpha & , \quad N(x-y,\epsilon) > \alpha \\ 0 & , \quad o.w. \end{cases}$$

is said to be an α-open sphere in X.

Definition 2.17. [10] Let (X,N) be a fuzzy normed linear space. A fuzzy set μ on X is called N-open if for $\mu(x) > 0$, there exists $\epsilon > 0$ such that $\mu_\alpha(x,\epsilon) \subseteq \mu$, for some $\alpha \in (0,1)$.

Theorem 2.18. [10] *Let (X,N) be a fuzzy normed linear space. Then a family $\tau'_N = \{\mu \in I^X : \mu \ is \ N - open\}$ is a fuzzy topology on X.*

Theorem 2.19. [10] *Let (X,N) be a fuzzy normed linear space such that N satisfies $(N7)$. Then*
(i) the mapping $f : (X,\tau'_N) \times (X,\tau'_N) \longrightarrow (X,\tau'_N)$, $(x,y) \longrightarrow x+y$, is continuous,
(ii) the mapping $g : \mathbb{R} \times (X,\tau'_N) \longrightarrow (X,\tau'_N)$, $(t,x) \longrightarrow tx$, is not continuous.

Definition 2.20. [10] Let (X,N) be a fuzzy normed linear space. A fuzzy set μ on X is said to be N-linearly open if for every $x \in supp\mu$ and $\alpha \in (1-\mu(x),1)$, there exists $\epsilon > 0$ such that $\mu_\alpha(x,\epsilon) \subseteq \mu$.

Theorem 2.21. [10] *Let (X,N) be a fuzzy normed linear space. Then a family*

$$\tau^*_N = \{\mu \in I^X : \mu \ is \ N - linearly \ open\}$$

is a fuzzy topology on X.

Theorem 2.22. [10] *Let (X, N) be a fuzzy normed linear space such that N satisfies (N7). Then α-open sphere is an N-linearly open, for all $\alpha \in (0, 1)$.*

Theorem 2.23. [10] *Let (X, N) be a fuzzy normed linear space such that N satisfies (N7). Then (X, τ_N^*) is a locally convex FTVS and for every $\alpha \in (0, 1)$,*

$$\mathfrak{U}_\alpha = \{U_{\beta,\epsilon} \cap \underline{(1-\beta)} : \epsilon > 0, \beta \in (0, \alpha)\} = \{\mu_\beta(0, \epsilon) : \epsilon > 0, \beta \in (0, \alpha)\}$$

is a Q-neighborhood base of 0_α, where $U_{\alpha,\epsilon} = \{z \in X : N(z, \varepsilon) > \alpha\}$.

Theorem 2.24. [10] *Let (X, N) be fuzzy normed linear space such that N satisfies (N7). Then the fuzzy topological space (X, τ_N^*) is fuzzy Hausdorff.*

Definition 2.25. [2] Let (X, N) be a fuzzy normed linear space. We define a set $B(x, \alpha, t)$ as $B(x, \alpha, t) = \{y : N(x - y, t) > 1 - \alpha\}$.

Theorem 2.26. [2] *Let (X, N) be a fuzzy normed linear space. If we define*

$$\tau_N^\dagger = \{G \subseteq X : x \in G \textit{ iff } \exists t > 0 \textit{ and } 0 < \alpha < 1 \textit{ such that } B(x, \alpha, t) \subseteq G\}.$$

Then $\tau_N^\dagger$ is a topology on (X, N).

3. **Fuzzy Topology on Fuzzy Normed Linear Space**

First, we construct a new fuzzy topology on fuzzy normed linear space defined by Bag and Samanta.

Definition 3.1. Let (X, N) be a fuzzy normed linear space and $\epsilon > 0$. The fuzzy set $B_\epsilon : X \longrightarrow [0, 1]$ defined on X by

$$B_\epsilon(x) = \sup\{\alpha \in (0, 1] : N(x, \epsilon) \geq \alpha\}, \text{ for all } x \in X,$$

is said to be a fuzzy sphere with center 0 and radius ϵ in X.

Theorem 3.2. *Let (X, N) be a fuzzy normed linear space. Then a family*

$$\tau_N = \{\mu \in I^X : \forall x \in supp\mu \textit{ and } 0 < r < \mu(x) \textit{ there is } \epsilon > 0 \textit{ s.t. } x + B_\epsilon \cap \underline{r} \subseteq \mu\}$$

is a fuzzy topology on X.

Proof. i) it is clear that $1, 0 \in \tau_N$.
ii) Let $\mu_1, ..., \mu_n \in \tau_N$ and $(\bigcap_{i=1}^n \mu_i)(x) > r > 0$. Hence $\mu_i(x) > r > 0$, for all $1 \leq i \leq n$. So there are $\epsilon_i > 0$ such that $x + B_{\epsilon_i} \cap \underline{r} \subseteq \mu_i$, for all $1 \leq i \leq n$. Assume that $\epsilon = \min\{\epsilon_i : 1 \leq i \leq n\}$. We have $\epsilon \leq \epsilon_i$, for all $1 \leq i \leq n$. This implies that $N(x, \epsilon) \leq N(x, \epsilon_i)$, for all $1 \leq i \leq n$. Thus $x + B_\epsilon \cap \underline{r} \subseteq x + B_{\epsilon_i} \cap \underline{r} \subseteq \mu_i$, for all $1 \leq i \leq n$. Therefore $x + B_\epsilon \cap \underline{r} \subseteq \bigcap_{i=1}^n \mu_i$. Hence $\bigcap_{i=1}^n \mu_i \in \tau_N$.
iii) Let $\mu_i \in \tau_N$, for all $i \in I$ and $(\bigcup_{i \in I} \mu_i)(x) > r > 0$. Hence there exists $i_0 \in I$ such that $\mu_{i_0}(x) > r > 0$. Thus there exists an $\epsilon > 0$ such that $x + B_\epsilon \cap \underline{r} \subseteq \mu_{i_0}$. Therefore $x + B_\epsilon \cap \underline{r} \subseteq \bigcup_{i \in I} \mu_i$. So $\bigcup_{i \in I} \mu_i \in \tau_N$. □

Now, we show that fuzzy topological space defined in Theorem 3.2 is a fuzzy topological vector space.

Theorem 3.3. *Let (X, N) be a fuzzy normed linear space. Then (X, τ_N) is a FTVS and for every $\lambda \in (0, 1)$,*

$$\mathfrak{U}_\lambda = \{B_\epsilon \cap \underline{r} : \epsilon > 0,\ r \in (1-\lambda, 1]\}$$

is a Q-neighborhood base of 0_λ.

Proof. First, we show that $\mathfrak{U}_\lambda$ satisfies conditions (i)-(v) of Lemma 2.14, for all $\lambda \in (0, 1)$.
i) Let $U = B_\epsilon \cap \underline{r} \in \mathfrak{U}_\lambda$. We have $1 - \lambda < r$. So there exists $\lambda_0 \in (0, \lambda)$ such that $1-\lambda_0 < r$. Suppose that $\nu \in [\lambda_0, 1]$. Since $r \in (1-\lambda_0, 1]$. Hence $1-\nu \le 1-\lambda_0 < r$. Thus $V = B_\epsilon \cap \underline{r} \in \mathfrak{U}_\nu$ and $V \subseteq U$.
Let $U = \underline{r}$ with $r \in (1 - \lambda, 1]$. Then there exists $\lambda_0 \in (0, \lambda)$ such that $r > 1 - \lambda_0$. So $V = B_\epsilon \cap \underline{r} \in \mathfrak{U}_\nu$ and $V \subseteq U$, for all $\nu \in [\lambda_0, 1]$.
ii) Let $B_{\epsilon_1} \cap \underline{r_1}, B_{\epsilon_2} \cap \underline{r_2} \in \mathfrak{U}_\lambda$. Suppose that $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $r = \min\{r_1, r_2\}$. Since $1 - \lambda < r_1, r_2$ it follows that $1 - \lambda < r$. Hence $B_\epsilon \cap \underline{r} \in \mathfrak{U}_\lambda$ and $B_\epsilon \cap \underline{r} \subseteq (B_{\epsilon_1} \cap \underline{r_1}) \cap (B_{\epsilon_2} \cap \underline{r_2})$.
(iii) Let $B_\epsilon \cap \underline{r} \in \mathfrak{U}_\lambda$. Since $1 - \lambda < r$ it follows that $B_{\epsilon/2} \cap \underline{r} \in \mathfrak{U}_\lambda$. Assume that $x = z + y$ and $\alpha < \min\{B_{\epsilon/2}(z), B_{\epsilon/2}(y)\}$. So $\alpha < B_{\epsilon/2}(z)$ and $\alpha < B_{\epsilon/2}(y)$. Hence $\alpha \le N(z, \epsilon/2)$ and $\alpha \le N(y, \epsilon/2)$. Thus $N(x, \epsilon) \ge \min\{N(y, \epsilon/2), N(z, \epsilon/2)\} \ge \alpha$. Therefore $B_\epsilon(x) \ge \alpha$. This implies that $\min\{B_{\epsilon/2}(z), B_{\epsilon/2}(y)\} \le B_\epsilon(x)$. So

$$(B_{\epsilon/2} + B_{\epsilon/2})(x) = \sup\{B_{\epsilon/2}(y) \wedge (B_{\epsilon/2}(z) : x = z + y\} \le B_\epsilon(x).$$

Hence $B_{\epsilon/2} \cap \underline{r} + B_{\epsilon/2} \cap \underline{r} \subseteq B_\epsilon \cap \underline{r}$.
(iv) Let $B_\epsilon \cap \underline{r} \in \mathfrak{U}_\lambda$. We have $N(x/t, \epsilon) = N(x, |t|\epsilon) \le N(x, \epsilon)$, for all $x \in X$ and all $t \in \mathbb{R}$ with $0 < |t| \le 1$. Hence

$$\begin{aligned} (t(B_\epsilon \cap \underline{r}))(x) &= (B_\epsilon \cap \underline{r})(x/t) \\ &= \sup\{\alpha \wedge r : N(x/t, \epsilon) \ge \alpha\} \\ &= \sup\{\alpha \wedge r : N(x, |t|\epsilon) \ge \alpha\} \\ &\le \sup\{\alpha \wedge r : N(x, \epsilon) \ge \alpha\} \\ &= (B_\epsilon \cap \underline{r})(x), \text{ for all } 0 < |t| \le 1. \end{aligned}$$

Therefore $t(B_\epsilon \cap \underline{r}) \subseteq B_\epsilon \cap \underline{r}$, for all $t \in \mathbb{R}$ with $|t| \le 1$.
(v) Let $B_\epsilon \cap \underline{r} \in \mathfrak{U}_\lambda$ and $x \in X$. By $(N5)$, we have $\lim_{t \longrightarrow \infty} N(x, t) = 1$. Thus there exists $t > 0$ such that $N(x, t\epsilon) > 1 - \lambda$. So

$$\begin{aligned} (t(B_\epsilon \cap \underline{r}))(x) &= (B_\epsilon \cap \underline{r})(x/t) \\ &= \sup\{\alpha \wedge r : N(x/t, \epsilon) \ge \alpha\} \\ &= \sup\{\alpha \wedge r : N(x, t\epsilon) \ge \alpha\} \\ &\ge (1 - \lambda) \wedge r \\ &> 1 - \lambda. \end{aligned}$$

Hence $x_\lambda \widetilde{\in} t(B_\epsilon \cap \underline{r})$.
By Lemma 2.14, there exists a unique fuzzy topology τ on X such that (X, τ) is a fuzzy topological vector space and $\mathfrak{U}_\lambda$ is a Q-neighborhood base of 0_λ.
Now we prove $\tau = \tau_N$. Let $\mu \in \tau_N$, $\mu(x) > 0$, $\lambda > 1 - \mu(x)$ and $1 - \lambda < r < \mu(x)$.

Then there exists $\epsilon > 0$ such that $x + B_\epsilon \cap \underline{r} \subseteq \mu$. Thus by Theorem 2.13, $x + B_\epsilon \cap \underline{r}$ is a Q-neighborhood of x_λ for τ. Hence μ is a Q-neighborhood of x_λ for τ. By Theorem 2.12, μ is a neighborhood of x for τ. Thus $\mu \in \tau$. So $\tau_N \subseteq \tau$.
On the other hand, let $\mu \in \tau$, $x \in supp\mu$ and $0 < r < \mu(x)$. Assume that $\lambda = 1 - r$. Then we have $x_\lambda \widetilde{\in} \mu$. Since $\mathfrak{U}_\lambda$ is a Q-neighborhood base of 0_λ, there exists $\epsilon > 0$ and $1 - \lambda < r_0 < \mu(x)$ such that $x + B_\epsilon \cap \underline{r_0} \subseteq \mu$. Thus $x + B_\epsilon \cap \underline{r} \subseteq x + B_\epsilon \cap \underline{r_0} \subseteq \mu$. This shows that $\mu \in \tau_N$. So $\tau \subseteq \tau_N$. Thus $\tau = \tau_N$. □

Theorem 3.4. *Let (X, N) be a fuzzy normed linear space. Then $B_\epsilon \cap \underline{r}$ is a fuzzy convex set, for all $\epsilon > 0$ and $r \in [0, 1]$.*

Proof. Let $\epsilon > 0$, $r \in [0, 1]$, $y, z \in X$ and $k \in [0, 1]$. Suppose that $\alpha < \min\{B_\epsilon(y), B_\epsilon(z)\}$. Therefore

$$\begin{aligned} N((ky + (1-k)z), \epsilon) &\geq \min\{N(ky, k\epsilon), N((1-k)z, (1-k)\epsilon)\} \\ &= \min\{N(y, \epsilon), N(z, \epsilon)\} \\ &\geq \alpha. \end{aligned}$$

Hence $B_\epsilon(ky + (1-k)z) \geq \alpha$. Thus $B_\epsilon(ky + (1-k)z) \geq \min\{B_\epsilon(y), B_\epsilon(z)\}$. So B_ϵ is a fuzzy convex set. This implies that $B_\epsilon \cap \underline{r}$ is a fuzzy convex set. □

Corollary 3.5. *Let (X, N) be a fuzzy normed linear space. Then the fuzzy topological space (X, τ_N) is a locally convex fuzzy topological vector space.*

Theorem 3.6. *Let (X, N) be a fuzzy normed linear space. Then the fuzzy sphere with center 0 and radius ϵ is an open set, for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. Suppose that $B_\epsilon(x) > r > 0$. Therefore $B_\epsilon \cap \underline{r} \subseteq B_\epsilon$. Hence B_ϵ is a fuzzy open set in (X, τ_N). □

Theorem 3.7. *Let (X, N) be a fuzzy normed linear space such that N satisfies (N8). Then the fuzzy topological space (X, τ_N) is fuzzy Hausdorff.*

Proof. Let $x, y \in X$ and $x \neq y$. By $(N8)$, there exists $t_0 > 0$ such that $N(x - y, t_0) = 0$. Suppose that $\epsilon < t_0$. If $(x + B_{\epsilon/2}) \cap (y + B_{\epsilon/2}) \neq \emptyset$, then there exists $z \in X$ such that $((x + B_{\epsilon/2}) \cap (y + B_{\epsilon/2}))(z) > 0$. Hence $N(x - z, \epsilon/2) > 0$ and $N(x - z, \epsilon/2) > 0$. Therefore

$$\begin{aligned} N(x - y, t_0) &\geq \min\{N(x - z, t_0/2), N(y - z, t_0/2)\} \\ &\geq \min\{N(x - z, \epsilon/2), N(y - z, \epsilon/2)\} \\ &> 0. \end{aligned}$$

This is a contradiction. Hence $(x + B_{\epsilon/2}) \cap (y + B_{\epsilon/2}) = \emptyset$. So (X, τ_N) is fuzzy Hausdorff. □

4. **Relations Among Fuzzy Topologies On Fuzzy Normed Linear Spaces**

In this section, Some necessary and sufficient conditions are established to illustrate that the presented fuzzy topology is equivalent to previously studied fuzzy topologies defined in [10].

Theorem 4.1. *Let (X, N) be a fuzzy normed linear space. Then $\tau_N^* \subseteq \omega(\tau_N^\dagger) \subseteq \tau_N'$.*

Proof. Let $\mu \in \tau_N^*$, $r \in [0,1)$ and $x \in \sigma_r(\mu)$. Hence $\mu(x) > r$. So $1-\mu(x) < 1-r$. Suppose that $\alpha \in (1-\mu(x), 1-r)$. Since $\mu \in \tau_N^*$, there exists $\epsilon > 0$ such that $\mu_\alpha(x,\epsilon) \subseteq \mu$. Thus $(x + U_{\alpha,\epsilon}) \cap \underline{(1-\alpha)} \subseteq \mu$. Therefore

$$B(x, 1-\alpha, \epsilon) = x + U_{\alpha,\epsilon} = \sigma_r((x + U_{\alpha,\epsilon}) \cap \underline{(1-\alpha)}) \subseteq \sigma_r(\mu).$$

This implies that $\sigma_r(\mu) \in \tau_N^\dagger$. Hence $\mu \in \omega(\tau_N^\dagger)$.
Let $\mu \in \omega(\tau_N^\dagger)$ and $\mu(x) > 0$. Suppose that $r \in [0, \mu(x))$. Hence $x \in \sigma_r(\mu)$. Since $\sigma_r(\mu) \in \tau_N^\dagger$, There exist $t > 0$ and $\alpha \in (0,1)$ such that

$$x + U_{1-\alpha,t} = B(x, \alpha, t) \subseteq \sigma_r(\mu).$$

Therefore $(x + U_{1-\alpha,t}) \cap \underline{r} \subseteq \mu$. Assume that $\beta = \min\{\alpha, r\}$. Now, we obtain that $\mu_{1-\beta}(x,t) = (x + U_{1-\beta,t}) \cap \underline{\beta} \subseteq (x + U_{1-\alpha,t}) \cap \underline{r} \subseteq \mu$. Hence $\mu \in \tau_N'$. □

Lemma 4.2. *Let (X, N) be a fuzzy normed linear space such that N satisfies $(N7)$. Then $B(x, \alpha, t) \in \tau_N^\dagger$, for all $x \in X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$.*

Proof. Let $x \in X$, $t \in \mathbb{R}$, $\alpha \in (0,1)$ and $y \in B(x, \alpha, t)$. Hence $N(x-y, t) > 1-\alpha$. By $(N7)$, there exists $t_0 \in (0,t)$ Such that $N(x-y, t_0) > 1-\alpha$. Suppose that $s = t - t_0$ and $z \in B(y, \alpha, s)$. Thus $N(z-y, s) > 1-\alpha$. Now we have

$$N(z-x, t) \geq \min\{N(x-y, t_0), N(z-y, s)\} > 1-\alpha.$$

So $z \in B(x, \alpha, t)$. This implies that $B(y, \alpha, s) \subseteq B(x, \alpha, t)$. Hence $B(x, \alpha, t) \in \tau_N^\dagger$. □

Lemma 4.3. *Let (X, N) be a fuzzy normed linear space such that N satisfies $(N7)$. Then $U_{\alpha,\epsilon} \cap \underline{\beta} \in \omega(\tau_N^\dagger)$, for all $\beta \in [0,1]$, $\epsilon > 0$ and $\alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $r \in [0,1)$. If $r < \beta$ then

$$\sigma_r(U_{\alpha,\epsilon} \cap \underline{\beta}) = U_{\alpha,\epsilon} = B(0, 1-\alpha, \epsilon).$$

By Lemma 4.2, $\sigma_r(U_{\alpha,\epsilon} \cap \underline{\beta}) \in \tau_N^\dagger$.
If $r \geq \beta$ then $\sigma_r(U_{\alpha,\epsilon} \cap \underline{\beta}) = \emptyset$. So $\sigma_r(U_{\alpha,\epsilon} \cap \underline{\beta}) \in \tau_N^\dagger$. Therefore $U_{\alpha,\epsilon} \cap \underline{\beta} \in \omega(\tau_N^\dagger)$. □

Theorem 4.4. *Let (X, N) be a fuzzy normed linear space such that N satisfies $(N7)$. Then $\omega(\tau_N^\dagger) \subseteq \tau_N^*$ if and only if N satisfies condition $(N9)$.*

Proof. Let $\tau_N^* = \omega(\tau_N^\dagger)$ and $\alpha, \beta \in (0,1)$.

- **Case 1:** Suppose that $\alpha \geq \beta$. If $N(x,t) \geq \alpha$ then $N(x,t) \geq \beta$.
- **Case 2:** Suppose that $\alpha < \beta$. Assume that $1-\alpha < r$. By Lemma 4.3, we have $U_{\beta,1} \cap \underline{r} \in \omega(\tau_N^\dagger)$. So $U_{\beta,1} \cap \underline{r} \in \tau_N^*$. Suppose that $1-r < \alpha_0 < \alpha$. Since $1-r < \alpha_0$, there exists $\epsilon > 0$ such that $\mu_{\alpha_0}(0,\epsilon) \subseteq U_{\beta,1} \cap \underline{r}$. Thus $U_{\alpha_0,\epsilon} \cap \underline{(1-\alpha_0)} \subseteq U_{\beta,1} \cap \underline{r}$. This implies that

$$U_{\alpha_0,\epsilon} = \sigma_{1-\alpha}(U_{\alpha_0,\epsilon} \cap \underline{(1-\alpha_0)}) \subseteq \sigma_{1-\alpha}(U_{\beta,1} \cap \underline{r}) = U_{\beta,1}.$$

If $N(x,t) \geq \alpha$. By $(N7)$, we obtain that $N(\epsilon x/t, \epsilon) > \alpha_0$. Hence $\epsilon x/t \in U_{\alpha_0,\epsilon}$. Thus $\epsilon x/t \in U_{\beta,1}$. So $N(\epsilon x/t, 1) > \beta$. Therefore $N(x, t/\epsilon) > \beta$. Then $N(x, t/\epsilon) \geq \beta$.

Let $m_{\alpha,\beta} = \max\{1, 1/\epsilon\}$. If $N(x,t) \geq \alpha$ then $N(x, tm_{\alpha,\beta}) \geq \beta$. Thus N satisfies in condition $(N9)$.
Conversely, let $\mu \in \omega(\tau_N^\dagger)$ and $x \in supp\mu$ and $\alpha \in (1-\mu(x), 1)$. Hence $x \in \sigma_{1-\alpha}(\mu)$. Since $\sigma_{1-\alpha}(\mu) \in \tau_N^\dagger$, there exists $\epsilon > 0$ and $\beta \in (0,1)$ such that

$$x + U_{\beta,\epsilon} = B(x, 1-\beta, \epsilon) \subseteq \sigma_{1-\alpha}(\mu).$$

If $\beta \leq \alpha$ then $U_{\alpha,\epsilon} \subseteq U_{\beta,\epsilon}$. Therefore $x + U_{\alpha,\epsilon} \subseteq x + U_{\beta,\epsilon} \subseteq \sigma_{1-\alpha}(\mu)$. Thus

$$\mu_\alpha(x,\epsilon) = (x + U_{\alpha,\epsilon}) \cap \underline{(1-\alpha)} \subseteq \mu.$$

This implies that $\mu \in \tau_N^*$.
If $\alpha < \beta$ and $N(x, \epsilon/(m_{\alpha,\beta}+1)) > \alpha$. By $(N9)$, we have $N(x, \epsilon m_{\alpha,\beta}/(m_{\alpha,\beta}+1)) \geq \beta$. By $(N7)$, we get $N(x,\epsilon) > \beta$. So $U_{\alpha,\epsilon/m_{\alpha,\beta}} \subseteq U_{\beta,\epsilon}$. Hence

$$x + U_{\alpha,\epsilon/(m_{\alpha,\beta}+1)} \subseteq x + U_{\beta,\epsilon} \subseteq \sigma_{1-\alpha}(\mu).$$

Thus

$$\mu_\alpha(x, \epsilon/(m_{\alpha,\beta}+1)) = (x + U_{\alpha,\epsilon/(m_{\alpha,\beta}+1)}) \cap \underline{(1-\alpha)} \subseteq \mu.$$

This implies that $\mu \in \tau_N^*$. So $\omega(\tau_N^\dagger) \subseteq \tau_N^*$. □

Corollary 4.5. *Let (X,N) be a fuzzy normed linear space such that N satisfies (N7). Then $\omega(\tau_N^\dagger) = \tau_N^*$ if and only if N satisfies condition (N9).*

Theorem 4.6. *Let (X,N) be a fuzzy normed linear space such that N satisfies (N7). Then $\tau_N^* \subseteq \tau_N$ if and only if N satisfies condition (N10).*

Proof. Let $\tau_N^* \subseteq \tau_N$ and $\alpha \in (0,1)$. We have $U_{\alpha,1} \cap \underline{(1-\alpha)} = \mu_\alpha(0,1) \in \tau_N^*$. Hence $U_{\alpha,1} \cap \underline{(1-\alpha)} \in \tau_N$. Thus there exist $0 < r < 1-\alpha$ and $\epsilon > 0$ such that $B_\epsilon \cap \underline{r} \subseteq U_{\alpha,1} \cap \underline{(1-\alpha)}$. So

$$\sigma_0(B_\epsilon) = \sigma_0(B_\epsilon \cap \underline{r}) \subseteq \sigma_0(U_{\alpha,1} \cap \underline{(1-\alpha)}) = U_{\alpha,1}.$$

Suppose that $m_\alpha = 1/\epsilon$. If $N(x,t) > 0$ then $N(\epsilon x/t, \epsilon) > 0$. Hence $B_\epsilon(\epsilon x/t) > 0$. Therefore $\epsilon x/t \in U_{\alpha,1}$. This implies that $N(\epsilon x/t, 1) \geq \alpha$. Then $N(x, t/\epsilon) \geq \alpha$. So $N(x, m_\alpha t) \geq \alpha$. Thus N satisfies condition $(N10)$.
Conversely, let $\mu \in \tau_N^*$, $0 < r < \mu(x)$ and $\alpha \in (1-\mu(x), 1-r)$. Hence there exists $\epsilon > 0$ such that $\mu_\alpha(x,\epsilon) \subseteq \mu$. If $(x + B_{\epsilon/(m_\alpha+1)} \cap \underline{r})(y) > 0$ then $N(x-y, \epsilon/(m_\alpha + 1)) > 0$. By $(N10)$, we obtain that $N(x-y, \epsilon m_\alpha/(m_\alpha+1)) \geq \alpha$. By $(N7)$, we have $N(x-y,\epsilon) > \alpha$. Hence $y \in x + U_{\alpha,\epsilon}$. Thus

$$x + B_{\epsilon/(m_\alpha+1)} \cap \underline{r} \subseteq (x + U_{\alpha,\epsilon}) \cap \underline{(1-\alpha)} = \mu_\alpha(x,\epsilon) \subseteq \mu.$$

Therefore $\mu \in \tau_N$. □

Theorem 4.7. *Let (X,N) be a fuzzy normed linear space such that N satisfies (N7). Then $\tau_N \subseteq \tau_N^*$ if and only if N satisfies in condition (N9).*

Proof. Let $\mu \in \tau_N$, $x \in suup\mu$, $\alpha \in (1-\mu(x), 1)$ and $r \in (1-\alpha, \mu(x))$. Then there exists $\epsilon > 0$ such that $x + B_\epsilon \cap \underline{r} \subseteq \mu$. If $\mu_\alpha(x, \epsilon/m_{\alpha,1-\alpha})(y) = 1-\alpha$ then $N(x-y, \epsilon/m_{\alpha,1-\alpha}) > \alpha$. By $(N9)$, we have $N(x-y,\epsilon) \geq 1-\alpha$. Hence $x + B_\epsilon(y) \geq 1-\alpha$. Therefore $\mu_\alpha(x, \epsilon/m_{\alpha,1-\alpha}) \subseteq x + B_\epsilon$. Thus $\mu_\alpha(x, \epsilon/m_{\alpha,1-\alpha}) \subseteq x + B_\epsilon \cap \underline{r} \subseteq \mu$. So $\mu \in \tau_N^*$.
Conversely, let $\alpha, \beta \in (0,1)$ and $N(x,t) \geq \alpha$. By $(N7)$, we have $N(x, 2t) > \alpha$.

Case 1: Assume that $\beta \leq \alpha$ then $N(x,t) \geq \beta$.

Case 2: Assume that $\alpha < \beta$. By Theorem 3.6, we get $B_1 \in \tau_N$. Hence $B_1 \in \tau_N^*$ and $B_1(0) = 1$. Thus there exist $\epsilon_1, \epsilon_2 > 0$ such that $\mu_\alpha(0, \epsilon_1) \subseteq B_1$ and $\mu_{1-\beta}(0, \epsilon_2) \subseteq B_1$. Since $N(x, 2t) > \alpha$ it follows that $N(\epsilon_1 x/2t, \epsilon_1) > \alpha$. So

$$1-\alpha = \mu_\alpha(0, \epsilon_1)(\epsilon_1 x/2t) \leq B_1(\epsilon_1 x/2t).$$

Hence $N(\epsilon_1 x/2t, 1) \geq 1-\alpha$. Thus $N(\epsilon_1\epsilon_2 x/2t, \epsilon_2) \geq 1-\alpha > 1-\beta$. Therefore

$$\beta = \mu_{1-\beta}(0, \epsilon_2)(\epsilon_1\epsilon_2 x/2t) \leq B_1(\epsilon_1\epsilon_2 x/2t).$$

So $N(\epsilon_1\epsilon_2 x/2t, 1) \geq \beta$. This implies that $N(x, 2t/\epsilon_1\epsilon_2) \geq \beta$.
Suppose that $m_{\alpha,\beta} = \max\{1, 2/\epsilon_1\epsilon_2\}$. Hence $N(x, m_{\alpha,\beta} t) \geq \beta$. Thus N satisfies condition $(N9)$. □

Lemma 4.8. *Let (X, N) be a fuzzy normed linear space such that N satisfyies $(N10)$. Then N satisfies condition $(N9)$.*

Proof. Let $\alpha, \beta \in (0,1)$, $x \in X$, $t \in \mathbb{R}$ and $N(x,t) \geq \alpha$. Hence $N(x,t) > 0$. Thus there exists $m_\beta > 0$ such that $N(x, m_\beta t) \geq \beta$. Suppose that $m_{\alpha,\beta} = m_\beta$. This implies that $N(x, m_{\alpha,\beta} t) \geq \beta$. Therefore N satisfies condition $(N9)$. □

Corollary 4.9. *Let (X, N) be a fuzzy normed linear space such that N satisfies $(N7)$. Then $\tau_N = \tau_N^*$ if and only if N satisfies condition $(N10)$.*

Theorem 4.10. *Let (X, N) be a fuzzy normed linear space. Then $\tau_N \subseteq \omega(\tau_N^\dagger)$.*

Proof. Let $\mu \in \tau_N$, $\alpha \in [0,1)$ and $x \in \sigma_\alpha(\mu)$. Hence $\mu(x) > \alpha$. Suppose that $\alpha < r < \mu(x)$. So there exists $\epsilon > 0$ such that $x + B_\epsilon \cap \underline{r} \subseteq \mu$. Therefore

$$B(x, 1-\alpha, \epsilon) = x + U_{\alpha,\epsilon} \subseteq x + \sigma_\alpha(B_\epsilon) = x + \sigma_\alpha(B_\epsilon \cap \underline{r}) = \sigma_\alpha(x + B_\epsilon \cap \underline{r}) \subseteq \sigma_\alpha(\mu).$$

Hence $\sigma_\alpha(\mu) \in \tau^\dagger$. Thus $\mu \in \omega(\tau_N^\dagger)$. □

Theorem 4.11. *Let (X, N) be a fuzzy normed linear space such that N satisfying $(N7)$. Then $\omega(\tau_N^\dagger) \subseteq \tau_N$ if and only if N satisfies condition $(N10)$.*

Proof. Let $\omega(\tau_N^\dagger) \subseteq \tau_N$ and $\alpha \in (0,1)$. By Lemma 4.3, we have $U_{\alpha,1} \cap \underline{1} \in \omega(\tau_N^\dagger)$. This implies that $U_{\alpha,1} \cap \underline{1} \in \tau_N$. Therefore there exist $r \in (0,1)$ and $\epsilon > 0$ such that $B_\epsilon \cap \underline{r} \subseteq U_{\alpha,1} \cap \underline{1}$. Thus $\sigma_0(B_\epsilon) \subseteq \sigma_0(B_\epsilon \cap \underline{r}) \subseteq \sigma_0(U_{\alpha,1} \cap \underline{1}) = U_{\alpha,1}$.
Suppose that $m_\alpha = 1/\epsilon$. If $N(x,t) > 0$ then $N(\epsilon x/t, \epsilon) > 0$. So $B_\epsilon(\epsilon x/t) > 0$. Hence $\epsilon x/t \in \sigma_0(B_\epsilon)$. This implies that $\epsilon x/t \in U_{\alpha,1}$. Thus $N(\epsilon x/t, 1) \geq \alpha$. Therefore $N(x, m_\alpha t) = N(x, t/\epsilon) \geq \alpha$. So N satisfies condition $(N10)$.
Conversely, let N satisfies condition $(N10)$. By Lemma 4.8, N satisfies condition

$(N9)$. By Theorem 4.4, We obtain that $\tau_N^* = \omega(\tau_N^\dagger)$. By Theorem 4.6, We have $\tau_N^* \subseteq \tau_N$. Therefore $\omega(\tau_N^\dagger) \subseteq \tau_N$. □

Corollary 4.12. *Let (X, N) be a fuzzy normed linear space such that N satisfies (N7). Then $\tau_N = \omega(\tau_N^\dagger)$ if and only if N satisfies condition $(N10)$.*

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M. Saheli, Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

E-mail address: saheli@mail.vru.ac.ir