

## EXTENDED FUZZY $BCK$ -SUBALGEBRAS

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**ABSTRACT.** This paper extends the notion of fuzzy  $BCK$ -subalgebras to fuzzy hyper  $BCK$ -subalgebras and defines an extended fuzzy  $BCK$ -subalgebras. This study considers a type of fuzzy hyper  $BCK$ -ideals in this hyperstructure and describes the relationship between hyper  $BCK$ -ideals and fuzzy hyper  $BCK$ -ideals. In fact, it tries to introduce a strongly regular relation on hyper  $BCK$ -algebras. Moreover, by using the fuzzy hyper  $BCK$ -ideals, it defines a congruence relation on (weak commutative) hyper  $BCK$ -algebras that under some conditions is strongly regular and the quotient of any hyper  $BCK$ -algebra via this relation is a (hyper  $BCK$ -algebra)  $BCK$ -algebra.

### 1. Introduction

The theory of hyperstructures was first introduced by Marty in 1934 during the 8<sup>th</sup> Congress of the Scandinavian Mathematicians [15]. In classical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements, therefore it can be said that the notion of hyperstructures is a generalization of classical algebraic structures, from this point of view. Marty introduced the concept of hypergroups and used it in different contexts like algebraic functions, rational fractions and non commutative groups. In recent years, many researchers have been working on hyper algebraic structures and the series of hyperalgebras branches such as superior, hypergroups, hypermoduls and hyperlattices which have appeared subsequently. Hyperstructures have many applications to several sectors of both pure and applied sciences such as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, artificial intelligence, probabilities, chemistry and physics, especially in atomic physics and harmonic analysis [4, 6, 9]. Borzooei, et al. has applied the hyperstructures to  $BCK$ -algebras which were initiated by Imai and Iseki [10] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus and introduced the concepts of hyper  $BCK$ -ideals and weak hyper  $BCK$ -ideals and discovered the relations between them [11].

Based on the superiority of fuzzy sets [21] and through simplifying the complexity of algebraic structures, many scholars have combined hyperstructures with the fuzzy set theory, and started the research about the fuzzy theory on hyperstructures [8, 14, 18]. The concept of regular relations can provide an important means to discuss hyperquasigroups theory and its analogous role to congruences in semigroup theory.

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Zhan [22] has introduced the concept of regular relations on hyperquasigroups and investigated some of its related properties. The relationships between the fuzzy sets and the algebraic hyperstructures (structures) has already been considered by Xie and Zhan and others, to which the reader can refer to [19].

Regarding these points, the aim of this paper is to generalize the notion of fuzzy *BCK*-subalgebras by considering the notion of fuzzy hyper *BCK*-subalgebras and try to define the concept of extendable fuzzy *BCK*-subalgebras. Also, we want to establish the relationship between fuzzy *BCK*-algebras and fuzzy hyper *BCK*-algebras. In the paper, we consider a type of fuzzy hyper *BCK*-ideals in this hyperstructure and describe the relationship between (*BCK*-ideals) hyper *BCK*-ideals and fuzzy hyper *BCK*-ideals. We introduce a strongly regular relation on any hyper *BCK*-algebras by using the concept of fuzzy hyper *BCK*-ideals and obtain a quotient hyper *BCK*-algebras (*BCK*-algebras). Moreover, by using fuzzy hyper *BCK*-ideals, we define a congruence relation on (weak commutative) hyper *BCK*-algebras that the quotient of any hyper *BCK*-algebra via this relation is a (hyper *BCK*-algebra) *BCK*-algebra. An isomorphism theorem of fuzzy hyper *BCK*-ideals is obtained by using the one to one fuzzy hyper *BCK*-ideals.

## 2. Preliminaries

**Definition 2.1.** [10] Let  $X$  be a set with a binary operation “ $*$ ” and a constant “ $0$ ”. Then  $(X, *, 0)$  is called a *BCK-algebra* if it satisfies the following conditions: for all  $x, y, z \in X$ ,

- (*BCI-1*)  $((x * y) * (x * z)) * (z * y) = 0$ ,
- (*BCI-2*)  $(x * (x * y)) * y = 0$ ,
- (*BCI-3*)  $x * x = 0$ ,
- (*BCI-4*)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$ ,
- (*BCK-5*)  $0 * x = 0$ .

We define a binary relation “ $\leq$ ” on  $X$  by  $x \leq y$  if and only if  $x * y = 0$ . A *BCK*-algebra  $(X, *, 0)$  is called *commutative* if for any  $x, y \in X$ ,  $x * (x * y) = y * (y * x)$ .

**Theorem 2.2.** [10] Let  $(X, *, 0)$  be a *BCK*-algebra. Then we have the following properties: for all  $x, y, z \in X$ ,

- (a)  $x \leq y$  implies  $z * y \leq z * x$ ,
- (b)  $x \leq y$  implies  $x * z \leq y * z$ ,
- (c)  $x \leq y$  and  $y \leq z$  imply  $x \leq z$ ,
- (d)  $(x * y) * z = (x * z) * y$ ,
- (e)  $x * y \leq z$  implies  $x * z \leq y$ ,
- (f)  $(x * z) * (y * z) \leq x * y$ ,
- (g)  $x * y \leq x$ ,
- (h)  $x * 0 = x$ .

**Definition 2.3.** [5] Let  $X$  be a nonempty set and  $P^*(X)$  be the family of all nonempty subsets of  $X$ . Functions  $\circ_{i_x} : X \times X \rightarrow P^*(X)$ , where  $i \in \{1, 2, \dots, n\}$  and  $n \in \mathbb{N}$ , are called binary hyperoperations. For all  $x, y \in X$ ,  $\circ_{i_x}(x, y)$  is called the hyperproduct of  $x$  and  $y$ . An algebraic system  $(X, \circ_{1_x}, \circ_{2_x}, \dots, \circ_{n_x})$  is called

an  $n$ -algebraic hyperstructure, and the structure  $(X, \circ_X)$  endowed with only one hyperoperation is called a hypergroupoid. For any two nonempty subsets  $A$  and  $B$  of hypergroupoid  $X$ , we define  $A \circ_X B = \bigcup_{a \in A, b \in B} a \circ_X b$

**Definition 2.4.** [2, 11] Let  $X$  be a non-empty set, endowed with a binary hyperoperation “ $\circ$ ” and a constant “ $0$ ”. Then,  $(X, \circ, 0)$  is called a *hyper BCK-algebra* if satisfies the following axioms: for all  $x, y, z \in X$

(H1)  $(x \circ z) \circ (y \circ z) \ll x \circ y$ ,

(H2)  $(x \circ y) \circ z = (x \circ z) \circ y$ ,

(H3)  $x \circ X \ll x$ ,

(H4)  $x \ll y$  and  $y \ll x$  imply  $x = y$ .

where  $x \ll y$  is defined by  $0 \in x \circ y$ , and for every  $A, B \subseteq H$ ,  $A \ll B$  is defined by: for any  $a \in A$ , there exists  $b \in B$  such that  $a \ll b$ .

A hyper BCK-algebra  $X$  is called *weak commutative* if for any  $x, y \in X$ ,

$$(x \circ (x \circ y)) \cap (y \circ (y \circ x)) \neq \emptyset.$$

**Theorem 2.5.** [11] In any hyper BCK-algebra  $X$ , the following statements hold: for all  $x, y, z \in X$  and  $A, B \subseteq X$ ,

(a1)  $0 \circ 0 = \{0\}$ ,

(a2)  $0 \ll x$ ,

(a3)  $x \ll x$ ,

(a4)  $A \ll A$ ,

(a5)  $A \ll 0$  implies  $A = \{0\}$ ,

(a6)  $A \subseteq B$  implies  $A \ll B$ ,

(a7)  $0 \circ x = \{0\}$ ,

(a8)  $x \circ y \ll x$ ,

(a9)  $x \in x \circ 0$ ,

(a10)  $y \ll z$  implies that  $x \circ z \ll x \circ y$ ,

(a11)  $A \circ B \ll A$ .

**Definition 2.6.** [11] Let  $I$  be a nonempty subset of a hyper BCK-algebra  $X$ . Then  $I$  is called:

- (i) a hyper BCK-subalgebra of  $X$ , if for all  $x, y \in I$ ,  $x \circ y \subseteq I$ ;
- (ii) a weak hyper BCK-ideal of  $X$  if (1)  $0 \in I$ , (2) for all  $x, y \in X$ ,  $x \circ y \subseteq I$  and  $y \in I$  imply  $x \in I$ ;
- (iii) a hyper BCK-ideal of  $X$  if (1)  $0 \in I$ , (2) for all  $x, y \in X$ ,  $x \circ y \ll I$  and  $y \in I$  imply  $x \in I$ .

**Definition 2.7.** [2] Let  $(X; \circ, 0)$  be a hyper BCK-algebra,  $R$  be an equivalence relation on  $X$  and  $A, B \subseteq X$ .

- (i)  $ARB$  means that there exist  $a \in A$  and  $b \in B$  such that  $aRb$  and  $\overline{\overline{ARB}}$  means that for all  $a \in A$  and  $b \in B$ , we have  $aRb$ ;
- (ii)  $R$  is called *strongly regular* if for all  $x, a, b$  of  $X$ , from  $aRb$  it follows that  $(a \circ x) \overline{\overline{R}} (b \circ x)$  and  $(x \circ a) \overline{\overline{R}} (x \circ b)$ ;

- (iii)  $\mathcal{L}(a_1, a_2, \dots, a_n)$  will denote the set of all finite combinations of elements  $\{a_1, a_2, \dots, a_n\}$  where  $a_1, a_2, \dots, a_n \in X$ ;  
 (iv)  $\beta_1 = \{(x, x) \mid x \in X\}$  and for every integer  $n \geq 1$ , let  $\beta_n$  be defined as follows:

$$x\beta_n y \iff \exists (a_1, \dots, a_n) \in X^n \text{ and } \exists u \in \mathcal{L}(a_1, \dots, a_n) \text{ such that } \{x, y\} \subseteq u.$$

For any  $n \geq 1$ , the relations  $\beta_n$  are symmetric, the relation  $\beta = \bigcup_{n \geq 1} \beta_n$  is a reflexive and symmetric relation. Let  $\beta^*$  be the transitive closure of  $\beta$  (the smallest transitive relation such that contains  $\beta$ ).

**Theorem 2.8.** [2] Let  $(X; \circ, 0)$  be a weak commutative hyper BCK-algebra. Then  $\beta^*$  is a strongly regular relation on  $X$  and  $(X/\beta^*; *, \bar{0})$  is a BCK-algebra.

**Definition 2.9.** [7] Let  $(X, \circ, 0)$  be a hyper BCK-algebra. A fuzzy set  $\mu : X \rightarrow [0, 1]$ , is called a fuzzy hyper BCK-subalgebra if for all  $x, y \in X$ ,

$$\inf(\mu(x \circ y)) \geq \min(\mu(x), \mu(y)).$$

**Lemma 2.10.** [7] Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $\mu$  be a fuzzy hyper BCK-subalgebra on  $X$ . Then for any  $x \in X$ , we have  $\mu(x) \leq \mu(0)$ .

**Definition 2.11.** [12] Let  $(X, \circ, 0)$  be a hyper BCK-algebra. A fuzzy set  $\nu$  on  $X$  is called a fuzzy hyper BCK-ideal, if for all  $x, y \in X$  it satisfies the following properties:

- (FH1)  $x \ll y \implies \nu(x) \geq \nu(y)$ ,  
 (FH2)  $\nu(y) \geq \min\{\nu(x), \inf(\nu(y \circ x))\}$ .

**Proposition 2.12.** [12] Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $\nu$  be a fuzzy hyper BCK-ideal of  $X$ . Then for any  $x \in X$ ,  $\nu(0) \geq \nu(x)$ .

**Definition 2.13.** [17] Let  $X$  be a nonempty set. A fuzzy set  $\mu : X \rightarrow [0, 1]$  is said to be normal if there exists  $x \in X$  such that  $\mu(x) = 1$ .

### 3. Fuzzy Hyper BCK-subalgebras

In this section, the concept of fuzzy hyper BCK-subalgebras will be considered as a generalization of fuzzy BCK-subalgebras and some of its properties will be investigated. We will also prove that a fuzzy hyper BCK-subalgebra and fuzzy BCK-subalgebras of fuzzy hyper BCK-subalgebras can be constructed via the fundamental relation. We will define the concept of extendable fuzzy BCK-subalgebras and will show that any infinite set is an extended fuzzy BCK-subalgebra.

**Theorem 3.1.** Let  $X$  be a nonempty set and  $0 \notin X$ . Then there exist a hyperoperation “ $\circ$ ” and a fuzzy set  $\mu$  on  $X' = X \cup \{0\}$  such that  $(X', \circ, 0)$  is a hyper BCK-subalgebra and  $\mu$  is a fuzzy hyper BCK-subalgebra on  $X'$ .

*Proof.* Let  $X$  be a nonempty set and  $0 \notin X$ . For any  $x, y \in X'$ , define the binary hyperoperation “ $\circ$ ” on  $X'$  as follows:

$$x \circ y = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ x, & \text{otherwise.} \end{cases} \quad (1)$$

Clearly  $(X', \circ, 0)$  is a hyper BCK-algebra. Now, it is easy to see that any fuzzy set  $\mu : X' \rightarrow [0, 1]$  such that  $\mu(0) = 1$ , is a fuzzy hyper BCK-subalgebra on  $X'$ .  $\square$

**Corollary 3.2.** *For any nonempty set, we can construct at least a hyper BCK-algebra.*

**Theorem 3.3.** *Let  $(X, \circ)$  be a hyper BCK-subalgebra which is defined in Theorem 3.1 and  $\mathcal{Fh} = \{\mu \mid \mu \text{ is a fuzzy hyper BCK-subalgebra on } (X, \circ)\}$ . If  $|X| \geq 1$ , then  $|\mathcal{Fh}| = |\mathbb{R}|$ .*

*Proof.* (i) Let  $X = \{x\}$ . Then  $(X, \circ, x)$  is a hyper BCK-algebra such that  $x \circ x = \{x\}$ . Then  $\mu : X \rightarrow [0, 1]$  such that it is defined by  $\mu(x) = \alpha$  is a fuzzy hyper BCK-subalgebra on  $X$  where  $\alpha \in [0, 1]$ .

(ii) Let  $|X| \geq 2$ . Then by Corollary 3.2, we can construct at least a hyper BCK-subalgebra on  $X$ . If  $(X, \circ, 0)$  is one of the hyper BCK-algebras which are defined in Theorem 3.1, then for all  $\alpha \in [0, 1]$  we define  $\mu_\alpha : X \rightarrow [0, 1]$  by

$$\mu_\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0. \end{cases}$$

Now for any  $y \in X$  and by (1), we get that

$$\inf(\mu(x \circ y)) = \mu(y) \geq \min(\mu(y), \mu(x)).$$

Hence  $\mu \in \mathcal{Fh}$  and so  $|\mathcal{Fh}| = |[0, 1]|$ .  $\square$

**Theorem 3.4.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $\mu$  be a fuzzy hyper BCK-subalgebra on  $X$ . Then for any  $x \in X$ , we have  $\mu(x) = \inf(\mu(x \circ 0))$ .*

*Proof.* Since  $x \in x \circ 0$ , by Lemma 2.10, we get that

$$\mu(x) = \min(\mu(0), \mu(x)) \leq \inf(\mu(x \circ 0)) \leq \mu(x).$$

It follows that  $\mu(x) = \inf(\mu(x \circ 0))$ .  $\square$

**Corollary 3.5.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra,  $\mu$  be a fuzzy hyper BCK-subalgebra on  $X$ ,  $\alpha \in [0, 1]$  and  $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ . Then*

- (i)  $0 \in \mu_\alpha$ ,
- (ii)  $\mu_\alpha$  is a hyper BCK-subalgebra of  $(X, \circ, 0)$ ,
- (iii) if  $0 \leq \alpha \leq \alpha' \leq 1$ , then  $\mu_{\alpha'} \subseteq \mu_\alpha$ .

*Proof.* (i) Let  $x \in \mu_\alpha$ . Since by Lemma 2.10,  $\mu(0) \geq \mu(x) \geq \alpha$ , we get that  $0 \in \mu_\alpha$ .

(ii) Let  $x, y \in \mu_\alpha$ . Then  $\min(\mu(x), \mu(y)) \geq \alpha$ . Now, for any  $a \in x \circ y$ ,  $\mu(a) \geq \inf(\mu(x \circ y)) \geq \min(\mu(x), \mu(y)) \geq \alpha$ . Hence  $a \in \mu_\alpha$  and so  $x \circ y \subseteq \mu_\alpha$ . Then  $\mu_\alpha$  is a hyper BCK-subalgebra of  $(X, \circ, 0)$ .

(iii) It is straightforward by definition.  $\square$

**Theorem 3.6.** Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $S$  be a hyper subalgebra of  $X$ . Define  $\mu : X \rightarrow [0, 1]$  by,

$$\mu(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases} \text{ where } 0 < \alpha < \alpha' < 1.$$

Then  $\mu$  is a fuzzy hyper BCK-subalgebra on  $X$ .

*Proof.* Let  $x, y \in X$ , then  $\inf(\mu(x \circ y)) = \alpha$  or  $\alpha'$ . If  $x, y \in S$ , then  $x \circ y \subseteq S$  and so  $\inf(\mu(x \circ y)) = \alpha' = \min(\mu(x), \mu(y))$ . If  $x \in S$  and  $y \notin S$ , then  $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha) = \min(\mu(x), \mu(y))$ . If  $x \notin S$  and  $y \in S$ , then  $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha') = \min(\mu(x), \mu(y))$ . If  $x \notin S$  and  $y \notin S$ , then  $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha) = \min(\mu(x), \mu(y))$ . It follows that  $\mu$  is a fuzzy hyper BCK-subalgebra on  $X$ .  $\square$

**Lemma 3.7.** If  $(X, \circ, 0)$  is a weak commutative hyper BCK-subalgebra and  $\mu$  is a fuzzy hyper BCK-algebra on  $X$ , then there exists a fuzzy set  $\bar{\mu}$  on BCK-algebra  $(X/\beta^*, *, \bar{0})$  such that: for any  $x, y \in X$ , (i)  $\bar{\mu}(\beta^*(0)) \geq \bar{\mu}(\beta^*(x))$ ;

- (ii) if  $x\beta^*y$ , then  $\bar{\mu}(\beta^*(x)) = \bar{\mu}(\beta^*(y))$ ;
- (iii)  $\min(\mu(x), \mu(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$ ;
- (iv) there exists  $t \in x \circ y$  such that  $\bar{\mu}(\beta^*(x \circ y)) = \mu(t)$ .

*Proof.* By Theorem 2.8,  $(X/\beta^*, *, \bar{0})$  is a BCK-algebra. We define a map  $\bar{\mu} : X/\beta^* \rightarrow [0, 1]$  by

$$\bar{\mu}(\beta^*(t)) = \begin{cases} \mu(0), & \text{if } 0 \in \beta^*(x), \\ \bigwedge_{t\beta^*x} \mu(x), & \text{otherwise,} \end{cases}$$

where  $x, t \in X$ . Then for any  $x, y, t' \in X$  we have:

(i)  $\bar{\mu}(\beta^*(0)) = \mu(0) \geq \bigwedge_{t' \beta^* x} \mu(t') = \bar{\mu}(\beta^*(x))$ .

(ii) Since  $x\beta^*y$  and  $\beta^*$  is transitive, we get that

$$\bar{\mu}(\beta^*(x)) = \bigwedge_{t \beta^* x} \mu(t) = \bigwedge_{t \beta^* y} \mu(t) = \bar{\mu}(\beta^*(y)).$$

(iii) Since  $x \beta^* x$  and  $y \beta^* y$ , we get that

$$\mu(x) \geq \bar{\mu}(\beta^*(x)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$$

and  $\mu(y) \geq \bar{\mu}(\beta^*(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$ . Hence

$$\min(\mu(x), \mu(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y))).$$

(iv)

$$\bar{\mu}(\beta^*(x) * \beta^*(y)) = \bar{\mu}(\beta^*(x \circ y)) = \bar{\mu}\{\beta^*(m) \mid m \in x \circ y\} = \bigwedge_{s \beta^* m \in x \circ y} \mu(s).$$

Now, since  $s\beta^*m$  and  $m \in x \circ y$ , then  $s \in x \circ y$ , and so there exists  $t \in x \circ y$  such that  $\mu(t) = \bigwedge_{s \beta^* m \in x \circ y} \mu(s)$ .  $\square$

**Theorem 3.8.** Let  $(X, \circ, 0)$  be a weak commutative hyper BCK-algebra and  $\mu$  be a fuzzy hyper BCK-subalgebra on  $X$ . Then there exists a fuzzy set  $\nu : X/\beta^* \rightarrow [0, 1]$  such that on BCK-algebra  $(X/\beta^*, *, \beta^*(0))$  is a fuzzy BCK-subalgebra. Moreover,  $\nu \circ \pi \leq \mu$  or the following diagram is quasi commutative:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & [0, 1] \\ \pi \downarrow & \nearrow \bar{\mu} & \\ X/\beta^* & & \end{array}$$

*Proof.* We consider  $\nu = \bar{\mu}$ . Then by Lemma 3.7, (i) for any  $x \in X, \nu(\beta^*(0)) \geq \nu(\beta^*(x))$ , (ii) for any  $x, y \in X$ , there exists  $t \in x \circ y$  such that  $\bar{\mu}(\beta^*(x \circ y)) = \mu(t)$  and so,

$$\begin{aligned} \nu(\beta^*(x) * \beta^*(y)) &= \nu(\beta^*(x \circ y)) \\ &= \mu(t) \geq \inf(\mu(x \circ y)) \\ &\geq \min(\mu(x), \mu(y)) \geq \min(\nu(\beta^*(x)), \nu(\beta^*(y))) \end{aligned}$$

Therefore  $\nu$  is a fuzzy BCK-subalgebra on  $X/\beta^*$  and clearly  $\nu \circ \pi \leq \mu$ . □

**Example 3.9.** Consider the BCK-algebras  $(\{1, 2, 3, 4\}, \star, 1)$  and  $(\{a, b\}, \star', a)$  as follows:

$$\begin{array}{c|cccc} \star & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 3 & 3 & 3 & 1 & 3 \\ 4 & 4 & 4 & 4 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} \star' & a & b \\ \hline a & a & a \\ b & b & a \end{array} .$$

For any  $x, y \in \{a, b\}$  and  $w \in \{1, 2, 3, 4\}$ , we set  $w_x = (w, x), \bar{w}_x = \{(w, x)\}, \bar{w}_{x,y} = \{(w, x), (w, y)\}$  and define a hyperoperation “ $\circ$ ” on  $\{1, 2, 3, 4\} \times \{a, b\}$  as follows:

$\circ$	$1_a$	$1_b$	$2_a$	$2_b$	$3_a$	$3_b$	$4_a$	$4_b$
$1_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$
$1_b$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{1}_a$	$\bar{1}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$
$2_a$	$\bar{2}_a$	$\bar{2}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{2}_a$	$\bar{2}_a$	$\bar{2}_a$	$\bar{2}_a$
$2_b$	$\bar{2}_b$	$\bar{2}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{2}_b$	$\bar{2}_{a,b}$	$\bar{2}_b$	$\bar{2}_{a,b}$
$3_a$	$\bar{3}_a$	$\bar{3}_a$	$\bar{3}_a$	$\bar{3}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{3}_a$	$\bar{3}_a$
$3_b$	$\bar{3}_b$	$\bar{3}_{a,b}$	$\bar{3}_b$	$\bar{3}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{3}_b$	$\bar{3}_{a,b}$
$4_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{1}_a$	$\bar{1}_a$
$4_b$	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$

and a fuzzy set  $\mu$  on  $X$  by

$$\mu \begin{array}{c|cccccccc} & 1_a & 1_b & 2_a & 2_b & 3_a & 3_b & 4_a & 4_b \\ \hline & 1 & \frac{7}{8} & \frac{6}{8} & \frac{5}{8} & \frac{4}{8} & \frac{3}{8} & \frac{2}{8} & \frac{1}{8} \end{array} .$$

Then it is easy to see that  $(\{1, 2, 3, 4\} \times \{a, b\}, \circ, 1_a)$  is a weak commutative hyper BCK-algebra and  $\mu$  is a fuzzy hyper BCK-algebra on  $\{1, 2, 3, 4\} \times \{a, b\}$ . We have  $\beta^*((1, a)) = \{(1, a), (1, b)\}, \beta^*((2, a)) = \{(2, a), (2, b)\}, \beta^*((3, a)) = \{(3, a), (3, b)\}$  and  $\beta^*((4, a)) = \{(4, a), (4, b)\}$ .

Hence  $\frac{Z}{\beta^*} = \{\beta^*((1, a)), \beta^*((2, a)), \beta^*((3, a)), \beta^*((4, a))\}$  and we can obtain the following tables:

*	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(3, a)$	$\beta^*(4, a)$
$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$
$\beta^*(2, a)$	$\beta^*(2, a)$	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(2, a)$
$\beta^*(3, a)$	$\beta^*(3, a)$	$\beta^*(3, a)$	$\beta^*(1, a)$	$\beta^*(3, a)$
$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(1, a)$

and

$\nu = \bar{\mu}$	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(3, a)$	$\beta^*(4, a)$
	1	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Clearly  $\nu$  is a fuzzy *BCK*-subalgebra on *BCK*-algebra  $(\frac{Z}{\beta^*}, *, \beta^*((0, a)))$  and for any  $(x, y) \in Z$ ,

$$(\nu \circ \pi)(x, y) = \nu(\beta^*(x, y)) \leq \mu(x, y).$$

**Definition 3.10.** (i) Let  $(X, *, 0)$  be a *BCK*-algebra and  $(Y, \circ, 0)$  be a hyper *BCK*-algebra. We say that the *BCK*-algebra  $X$  is derived from the hyper *BCK*-algebra  $Y$  if  $X$  is isomorphic to a nontrivial quotient of  $Y$ . ( $X \cong Y/\beta^*$ )

(ii) A fuzzy *BCK*-subalgebra  $\mu$  on the *BCK*-algebra  $X$  is called an *extendable* fuzzy *BCK*-subalgebra, if there exist a hyper *BCK*-algebra  $(Y, \circ, 0)$ , a fuzzy hyper *BCK*-subalgebra  $\nu$  on  $Y$  and  $n \in \mathbb{N}$  such that  $|(X, *, \mu)| = |(Y, \circ, \nu)| - n$ , and *BCK*-algebra  $X$  is derived from hyper *BCK*-algebra  $Y$ . If  $X = Y$  and almost everywhere  $\mu = \nu$  ( $\mu = \nu$  a.e that means  $|\{x; \mu(x) \neq \nu(x)\}| = 1$ ), we will say that it is an extended fuzzy *BCK*-subalgebra.

**Example 3.11.** (i) Let  $X = \{a, b, c, d\}$ . Then  $\mu$  is a fuzzy *BCK*-subalgebra on *BCK*-algebra  $(X, *, a)$  as follows:

*	$a$	$b$	$c$	$d$
$a$	$a$	$a$	$a$	$a$
$b$	$b$	$a$	$b$	$b$
$c$	$c$	$c$	$a$	$c$
$d$	$d$	$d$	$d$	$a$

and  $\frac{\mu}{\beta^*} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0.2 & 0.4 & 0.6 \end{array}$ .

Now, set  $Y = \{e, a, b, c, d\} = X \cup \{e\}$ . Then  $\nu$  is a fuzzy hyper *BCK*-subalgebra on hyper *BCK*-algebra  $(Y, \circ, e)$  as follows:

$\circ$	$e$	$a$	$b$	$c$	$d$
$e$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$
$a$	$\{a\}$	$\{e, a\}$	$\{e, a\}$	$\{e, a\}$	$\{e, a\}$
$b$	$\{b\}$	$\{b\}$	$\{e, a\}$	$\{b\}$	$\{b\}$
$c$	$\{c\}$	$\{c\}$	$\{c\}$	$\{e, a\}$	$\{c\}$
$d$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{e, a\}$

and  $\frac{\nu}{\beta^*} = \begin{array}{c|ccccc} & e & a & b & c & d \\ \hline e & 1 & 1 & 0.2 & 0.4 & 0.6 \end{array}$ .

Clearly  $Y/\beta^* \cong X$ ,  $|Y| = |X| + 1$  and so  $\mu$  is an extendable fuzzy *BCK*-subalgebra on  $(X, *, a)$ .



(ii)  $\mu$  is a fuzzy BCK-subalgebra on BCK-algebra  $(\mathbb{N}, *, 1)$  as follows:

$$x * y = \begin{cases} 1, & \text{if } x = y, \\ x, & \text{otherwise} \end{cases} \text{ and } \mu(x) = 1/x.$$

Now,  $\nu$  is a fuzzy hyper BCK-subalgebra on hyper BCK-algebra  $(\mathbb{N}, \circ, 1)$  as follows:

$$x \circ y = \begin{cases} 1, & \text{if } x = 1, \\ \{1, 2\}, & \text{if } x = y \text{ and } x \neq 1, \\ \{2\}, & \text{if } x = 2 \text{ and } y = 1, \\ \{1, 2\}, & \text{if } x = 2 \text{ and } y \neq 1, \\ x, & \text{otherwise} \end{cases} \text{ and } \nu(x) = \begin{cases} 1, & \text{if } x = 1 = 2, \\ 1/x, & \text{otherwise.} \end{cases}$$

Clearly ( $\mu = \nu$  a.e),  $\mathbb{N}/\beta^* \cong \mathbb{N}$  so  $\mu$  is an extended fuzzy BCK-subalgebra on  $(\mathbb{N}, *, 1)$ .

**Theorem 3.12.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra,  $\mu$  be a fuzzy hyper BCK-subalgebra on  $X$  and  $\bar{X} = \{\mu(x) \mid x \in X\}$ . If  $\mu$  is one to one, then:*

- (i) *there exists a hyperoperation  $\circ'$  on  $\bar{X}$  such that  $(\bar{X}, \circ', \mu(0))$  is a hyper BCK-algebra;*
- (ii) *there exists a fuzzy set  $\bar{\mu} : \bar{X} \rightarrow [0, 1]$  such that  $\bar{\mu}$  is a fuzzy hyper BCK-subalgebra on  $\bar{X}$ ;*
- (iii) *there exists an operation  $*$  on  $\bar{X}$  such that  $(\bar{X}, *, \mu(0))$  is a BCK-algebra.*

*Proof.* (i) For any  $x, y \in X$ , define a hyperoperation  $\circ'$  on  $\bar{X}$ , by  $\mu(x) \circ' \mu(y) = \mu(x \circ y)$ . It can be easily seen that  $\mu(x) \ll' \mu(y) \Leftrightarrow x \ll y$ . It is easy to see that  $(\bar{X}, \circ', \mu(0))$  is a hyper BCK-algebra.

(ii) For any  $x \in X$ , define  $\bar{\mu}(\mu(x)) = \mu(x)$ . Clearly  $\bar{\mu}$  is a fuzzy hyper BCK-subalgebra on  $(\bar{X}, \circ')$ .

(iii) For any  $x, y \in X$ , define an operation  $*$  on  $\bar{X}$  by

$$\mu(x) * \mu(y) = \begin{cases} \mu(x), & \text{if } y = 0, \\ \sup(\mu(x \circ y)), & \text{otherwise.} \end{cases}$$

We just prove BCI-4. Let  $x, y \in X$  and  $\mu(x) * \mu(y) = \mu(y) * \mu(x) = \mu(0)$ . It follows  $0 \in x \circ y$  and  $0 \in y \circ x$ , that is,  $\mu(x) = \mu(y)$ . It is easy to see that BCI-1, BCI-2, BCI-3 and BCK-5 are valid and so  $(\bar{X}, *, \mu(0))$  is a BCK-algebra.  $\square$

**Corollary 3.13.** *Let  $(\bar{X}, \circ, \mu(0))$  be a hyper BCK-algebra and  $\mu$  be a fuzzy hyper BCK-subalgebra on  $\bar{X}$ . Then there exists a binary operation  $*$  on  $\bar{X}$ , such that  $(\bar{X}, *, \mu(0))$  is a BCK-algebra and  $\mu$  is a fuzzy BCK-subalgebra on  $\bar{X}$ .*

**Theorem 3.14.** *Let  $X$  be a nonempty set,  $0 \notin X$  and  $X' = X \cup \{0\}$ . Then there exist a hyperoperation  $\circ$  on  $X'$ , a hyperoperation  $\circ'$  on  $\bar{X}'$ , a binary operation  $*$  on  $\bar{X}'$ , a fuzzy set  $\mu$  on  $X'$  and fuzzy set  $\xi$  on  $\bar{X}'$  such that:*

- (i)  *$(X', \circ, 0)$  is a hyper BCK-algebra and  $\mu$  is a fuzzy hyper BCK-subalgebra on  $X'$ ;*

- (ii)  $(\overline{X'}, \circ', \mu(0))$  is a hyper BCK-algebra and  $\mu$  is a fuzzy hyper BCK-subalgebra on  $\overline{X'}$ ;  
 (iii)  $(\overline{X'}, *, \mu(0))$  is a BCK-algebra and  $\xi$  is a fuzzy BCK-subalgebra on  $\overline{X'}$ ;  
 (iv)  $|X'| = |\overline{X'}| + 1$ .

*Proof.* Let  $|X| \geq 2$  and  $b \in X$  be fixed. For any  $x, y \in X'$ , define a binary hyperoperation  $\circ$  on  $X'$  as follows,

$$x \circ y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, b\}, & \text{if } x = y \text{ and } x \neq 0, \\ \{b\}, & \text{if } x = b \text{ and } y = 0, \\ \{0, b\}, & \text{if } x = b \text{ and } y \neq 0, \\ x, & \text{otherwise.} \end{cases} \quad (2)$$

Now, we show that  $(X', \circ, 0)$  is a hyper BCK-algebra. We just check that conditions (H1) and (H2) are valid.

(H1): Let  $x, y, z \in X'$ . If  $x = 0$ , then  $(x \circ z) \circ (y \circ z) = \{0\} \circ (y \circ z) = \{0\} \ll x \circ y$ . If  $x = b$ , then  $(x \circ z) \circ (y \circ z) \subseteq \{0, b\} \circ (y \circ z) \subseteq \{0, b\} \ll x \circ y$ . If  $x \notin \{0, b\}$ , we consider the following cases:

Case 1:  $x = y \neq z$ . Then  $(x \circ z) \circ (y \circ z) = x \circ y = x \circ x = \{0, b\} \ll \{0, b\} = x \circ y$ .

Case 2:  $x = z \neq y$ . Then  $(x \circ z) \circ (y \circ z) = \{0, b\} \circ (y \circ z) = \{0, b\} \ll x = x \circ y$ .

Case 3:  $y = z \neq x$ . Then  $(x \circ z) \circ (y \circ z) \subseteq x \circ \{0, b\} = \{0, b\} \ll x = x \circ y$ .

Case 4:  $x \neq y \neq z$ . Then  $(x \circ z) \circ (y \circ z) = x \circ y = x \ll x = x \circ y$ .

Case 5:  $x = y = z$ . Then  $(x \circ z) \circ (y \circ z) = \{0, b\} \ll \{0, b\} = x \circ y$ .

(H2): Let  $x, y, z \in X$ . The proof of  $(x \circ y) \circ z = (x \circ z) \circ y$  is similar to that of (H1), and then it is easy to see that  $(X', \circ, 0)$  is a hyper BCK-algebra. Consider a fuzzy set  $\mu$  on  $X'$  such that  $\mu(0) = \mu(b) = 1$ , by (2), we get that

$$\inf(\mu(x \circ y)) = \mu(x) \geq \min(\mu(x), \mu(y))$$

and so  $\mu$  is a fuzzy hyper BCK-subalgebra on  $(X', \circ, 0)$ . Now for any  $x, y \in X$ , define a hyperoperation  $\circ'$  on  $\overline{X'}$  by  $\mu(x) \circ' \mu(y) = \mu(x \circ y)$ ,  $\xi : \overline{X'} \rightarrow [0, 1]$  by  $\xi(\mu(x)) = \mu(x)$  and an operation  $*$  on  $\overline{X'}$  as follows:

$$\mu(x) * \mu(y) = \sup(\mu(x) \circ' \mu(y)).$$

It can be easily seen that  $\mu(x) \ll' \mu(y) \Leftrightarrow x \ll y$ ,  $(\overline{X'}, \circ', 0)$  is a hyper BCK-algebra,  $\mu$  is a fuzzy hyper BCK-subalgebra on  $\overline{X'}$ ,  $(\overline{X'}, *, 0)$  is a BCK-algebra,  $\xi$  is a fuzzy BCK-subalgebra on  $\overline{X'}$  and since  $\mu(0) = \mu(b)$ , we get that  $|X'| = |\overline{X'}| + 1$ .  $\square$

**Corollary 3.15.** *On any nonempty set we can construct an extendable fuzzy BCK-subalgebra.*

**Example 3.16.** Let  $X = \{0, a, b, c, d\}$ . Then  $\mu$  is a fuzzy hyper BCK-subalgebra on hyper BCK-algebra  $(X, \circ, 0)$  as follows:

$\circ$	0	$a$	$b$	$c$	$d$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$a$	$\{a\}$	$\{0, b\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$\{b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$
$c$	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, b\}$	$\{c\}$
$d$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, b\}$

$\mu$	0	$a$	$b$	$c$	$d$
	1	0.2	1	0.3	0.4

Clearly  $\bar{X} = \{\mu(0), \mu(a), \mu(c), \mu(d)\}$  is a BCK-algebra as follows:

$*$	$\mu(0)$	$\mu(a)$	$\mu(c)$	$\mu(d)$
$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(0)$
$\mu(a)$	$\mu(a)$	$\mu(0)$	$\mu(a)$	$\mu(a)$
$\mu(c)$	$\mu(c)$	$\mu(c)$	$\mu(0)$	$\mu(c)$
$\mu(d)$	$\mu(d)$	$\mu(d)$	$\mu(d)$	$\mu(0)$

#### 4. Fuzzy Hyper BCK-ideals in Hyper BCK-algebras

In this section, the notion of fuzzy hyper BCK-ideal in hyper BCK-algebras was applied and some significant hyper BCK-ideals were obtained. Moreover, we defined a regular equivalence relation on any hyper BCK-algebras via the concept of a fuzzy hyper BCK-ideal.

**Notation.** A fuzzy hyper BCK-ideal  $\nu$  on a hyper BCK-algebra  $(X, \circ)$  is called *commutative*, if for all  $x, y \in X$ ,  $\sup(\nu(x \circ y)) = \sup(\nu(y \circ x))$ .

**Theorem 4.1.** *Let  $X$  be a nonempty set and  $0 \in X$ . Then there exists a hyperoperation  $\circ$  on  $X$  such that  $(X, \circ)$  is a hyper BCK-algebra and any normal fuzzy set on  $(X, \circ)$  is a fuzzy hyper BCK-ideal.*

*Proof.* For any  $x, y \in X$ , we define a binary hyperoperation  $\circ$  on  $X$  as follows:

$$x \circ y = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ \{x\}, & \text{otherwise.} \end{cases}$$

Clearly,  $(X, \circ, 0)$  is a hyper BCK-algebra. Let  $\nu : X \rightarrow [0, 1]$  be a normal fuzzy set and  $x, y \in X$ , then  $\nu(0) = 1 \geq \nu(y)$ . If  $x \neq y$ , then

$$\nu(x) \geq \min(\nu(y), \nu(x)) = \min(\nu(y), \inf(\nu(x \circ y))).$$

If  $0 \neq x = y$ , then

$$\nu(x) \geq \min(\nu(y), \nu(x)) = \min(\nu(y), \inf(\nu(x \circ y))).$$

Therefore,  $\nu$  is a fuzzy hyper BCK-ideal. □

**Corollary 4.2.** *Let  $(X, \circ)$  be a hyper BCK-algebra which is defined in Theorem 4.1.  $\mathcal{Fhi} = \{\mu \mid \mu \text{ is a fuzzy hyper BCK-ideal on } (X, \circ)\}$ . If  $|X| \geq 1$ , then  $|\mathcal{Fhi}| = |\mathbb{R}|$ .*

**Example 4.3.** Let  $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$ . Then  $(\mathbb{N}_5, \circ, 1)$  is a hyper *BCK*-algebra as follows:

$\circ$	1	2	3	4	5
1	{1}	{1}	{1}	{1}	{1}
2	{2}	{1, 2}	{2}	{2}	{2}
3	{3}	{3}	{1, 3}	{3}	{3}
4	{4}	{4}	{4}	{1, 4}	{4}
5	{5}	{5}	{5}	{5}	{1, 5}

Now, define  $\nu : \mathbb{N}_5 \rightarrow [0, 1]$  by  $\nu(x) = \frac{1}{x}$ . It is easy to see that  $\nu$  is a fuzzy hyper *BCK*-ideal.

**Theorem 4.4.** Let  $(X, \circ, 0)$  be a hyper *BCK*-algebra and  $\nu$  be a fuzzy hyper *BCK*-ideal of  $X$ . Then for any  $x, y \in X$  and  $A, B \subset X$ :

- (i) if  $A \ll B$ , then there exist  $b \in B$  such that  $\sup(\nu(A)) \geq \nu(b)$ ;
- (ii) if  $A \ll B$ , then  $\inf(\nu(A)) \geq \inf(\nu(B))$ ;
- (iii)  $\nu(x) \leq \inf(\nu(x \circ y))$  and  $\inf(\nu(A \circ B)) \geq \inf(\nu(A))$ ;
- (iv) if  $\nu$  is commutative, then  $\nu(x) \leq \inf(\nu(y \circ x))$ .

*Proof.* (i) Since  $A \ll B$ , then for any  $a \in A$  there exists  $b \in B$  such that  $a \ll b$ . Hence

$$\sup(\nu(A)) \geq \nu(a) \geq \nu(b),$$

therefore  $\sup(\nu(A)) \geq \nu(b)$ .

(ii) Let  $A \ll B$ . Then for any  $a \in A$  there exists  $b \in B$  such that  $a \ll b$  and so  $\nu(a) \geq \nu(b)$ . Hence we have  $\inf(\nu(A)) \geq \nu(b) \geq \inf(\nu(B))$ .

(iii) By Theorem 2.5,  $x \circ y \ll x$ . Then by (iii), we get that

$$\nu(x) \leq \inf(\nu(x \circ y)).$$

(iv) Since  $\nu$  is commutative, then by (iv) we get that  $\sup(\nu(y \circ x)) \geq \nu(x)$ .  $\square$

**Example 4.5.** Let  $(\mathbb{N}_5, \circ, 1)$  be the hyper *BCK*-algebra that is defined in Example 4.3,  $A = \{1, 2\}$  and  $B = \{1, 2, 5\}$ . Clearly  $A \ll B$  and

$$\inf(\nu(A)) = \inf\{\nu(1), \nu(2)\} = \frac{1}{2} \geq \frac{1}{5} \geq \inf\{\nu(1), \nu(2), \nu(5)\} = \inf(\nu(B))$$

and  $A \circ B = A$ , therefore  $\inf(\nu(A \circ B)) = \inf(\nu(A))$ .

**Corollary 4.6.** Let  $(X, \circ, 0)$  be a hyper *BCK*-algebra,  $\alpha \in [0, 1]$  and  $\nu$  be a fuzzy hyper *BCK*-ideal of  $X$  such that  $\nu_\alpha \neq \emptyset$ . Then for all  $x, y, z \in X$ :

- (i)  $0 \in \nu_\alpha$ ;
- (ii) if  $y \in \nu_\alpha$  and  $x \ll y$ , then  $x \in \nu_\alpha$ ;
- (iii)  $(y \circ z) \ll x$  implies that  $\nu(y) \geq \min(\nu(z), \nu(x))$ ;
- (iv)  $\nu_\alpha$  is a hyper *BCK*-ideal of  $X$ .

*Proof.* (i) There exists  $x \in \nu_\alpha$  such that  $\nu(x) \geq \alpha$ . Since by definition  $\nu(0) \geq \nu(x)$ , we get that  $0 \in \nu_\alpha$ .

(ii) Since  $x \ll y$ , by definition, we get that  $\nu(x) \geq \nu(y)$ . Now,  $y \in \nu_\alpha$  then  $x \in \nu_\alpha$ .

(iii)  $(y \circ z) \ll x$  implies that  $0 \in (y \circ z) \circ x$ , then by Theorem 4.4, we get that  $\nu(x) \leq \inf(\nu(y \circ z))$ . Now,  $\nu$  is a fuzzy hyper BCK-ideal so

$$\nu(y) \geq \min(\nu(z), \inf(\nu(y \circ z))) \geq \min(\nu(z), \nu(x)).$$

(iv) Let  $x, y \in X, x \circ y \ll \nu_\alpha$  and  $y \in \nu_\alpha$ . Then  $\nu(y) \geq \alpha$  and by Theorem 4.4,  $\inf(\nu(x \circ y)) \geq \alpha$ . Hence

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) \geq \min(\alpha, \alpha) = \alpha.$$

Therefore,  $x \in \nu_\alpha$  and so  $\nu_\alpha$  is a hyper BCK-ideal. □

**Corollary 4.7.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $\nu$  be a fuzzy hyper BCK-ideal on  $X$ . If  $\nu$  is an increasing map, then  $\nu$  is a commutative fuzzy hyper BCK-ideal.*

*Proof.* Let  $x \in X$ . Then by Proposition 2.12,  $\nu(x) \leq \nu(0)$ . Since  $0 \ll x$  and  $\nu$  is increasing, we get that  $\nu(0) \leq \nu(x)$ . Hence  $\nu(x) = \nu(0)$  and so for any  $x, y \in X, \sup(\nu(x \circ y)) = \sup(\nu(y \circ x)) = \nu(0)$ . □

**Theorem 4.8.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $f : X \rightarrow X$  be an onto homomorphism. Then the fuzzy set  $\nu : X \rightarrow [0, 1]$ , is a fuzzy hyper BCK-ideal on  $X$  if and only if  $\nu_f : X \rightarrow [0, 1]$  which is defined by  $\nu_f(x) = \nu(f(x))$  is a fuzzy hyper BCK-ideal on  $X$ .*

*Proof.* Let  $\nu$  be a fuzzy hyper BCK-ideal on  $X$  and  $x \in X$ . Then,

$$\nu_f(0) = \nu(f(0)) = \nu(0) \geq \nu(f(x)) = \nu_f(x)$$

and for any  $x, y \in X$ ,

$$\begin{aligned} \nu_f(y) = \nu(f(y)) &\geq \min(\nu(f(x)), \inf(\nu(f(y) \circ f(x)))) \\ &= \min(\nu(f(x)), \inf(\nu(f(y \circ x)))) \\ &= \min(\nu_f(x), \inf(\nu_f(y \circ x))) \end{aligned}$$

Hence  $\nu_f$  is a fuzzy hyper BCK-ideal on  $X$ .

Conversely, assume that  $\nu_f$  is a fuzzy hyper BCK-ideal on  $X$  and  $y \in X$ . Since  $f$  is onto, there exists  $x \in X$  such that  $f(x) = y$ . Then

$$\nu(0) = \nu(f(0)) = \nu_f(0) \geq \nu_f(x) = \nu(y)$$

Let  $x, y \in X$ . Then there exists  $a, b \in X$  such that  $f(a) = x$  and  $f(b) = y$ . Hence we get that

$$\begin{aligned} \nu(y) = \nu(f(b)) &= \nu_f(b) \\ &\geq \min(\nu_f(a), \inf(\nu_f(b \circ a))) \\ &= \min(\nu(f(a)), \inf(\nu(f(b \circ a)))) \\ &= \min(\nu(f(a)), \inf(\nu(f(b) \circ f(a)))) \\ &= \min(\nu(x), \inf(\nu(y \circ x))) \end{aligned}$$

Therefore  $\nu$  is a fuzzy hyper BCK-ideal on  $X$ . □

**Proposition 4.9.** Let  $(X, \circ, 0)$  be a hyper BCK-algebra,  $\nu : X \rightarrow [0, 1]$  be a fuzzy hyper BCK-ideal on  $X$  and  $f : X \rightarrow X$  be a homomorphism,

- (i) if  $x \in \ker(f)$ , then for any  $y \in X$ ,  $\nu_f(x) \geq \nu(y)$ .
- (ii) if  $\nu$  is one to one, then  $\ker(f)$  is a hyper BCK-ideal.
- (iii) if there exists  $x \in X$  such that  $\nu(x) = 1$ , then  $\nu_1 = \{x \in X \mid \nu(x) = 1\}$  is a hyper BCK-ideal in  $X$ .
- (iv)  $\nu_0$  is a fuzzy hyper BCK-ideal in  $X$ .

*Proof.* (i) Let  $x \in \ker(f)$ . Then,  $\nu_f(x) = \nu(f(x)) = \nu(0)$  and so for any  $y \in X$ ,  $\nu_f(x) \geq \nu(y)$ .

(ii) Clearly  $0 \in \ker(f)$ . Let  $y \in \ker(f)$  and  $x \circ y \ll \ker(f)$ , where  $x, y \in X$ . Then  $\nu_f(y) = \nu(0)$ ,  $\inf(\nu_f(x \circ y)) = \nu(0)$  and so

$$\nu_f(x) \geq \min(\nu_f(y), \inf(\nu_f(x \circ y))) = \min(\nu(0), \nu(0)) = \nu(0).$$

Hence  $\nu_f(x) = \nu(0)$  and then  $x \in \ker(f)$ .

(iii) Since there exists  $x \in X$  such that  $\nu(x) = 1$ , we get that  $1 = \nu(x) \leq \nu(0)$ . Hence  $\nu(0) = 1$  and so  $0 \in \nu_1$ . Now, let  $y \in \nu_1$  and  $x \circ y \ll \nu_1$ , where  $x, y \in X$ . Then,  $\nu(y) = 1$ ,  $\inf(\nu(x \circ y)) = 1$  and so

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) = \min(1, 1) = 1.$$

Hence  $\nu(x) = 1$  and  $x \in \nu_1$ .

(iv) Since  $\nu_0 = X$ , then the proof is clear. □

**Theorem 4.10.** Let  $(X, \circ, 0)$  be a hyper BCK-algebra,  $I$  be a hyper BCK-ideal and  $\nu, \nu'$  be fuzzy hyper BCK-ideals on  $X$ . Then

- (i)  $X_\nu = \{x \in X \mid \nu(x) = \nu(0)\}$  is a hyper BCK-ideal in  $X$ ;
- (ii) if  $\nu'(0) = \nu(0)$ , then  $X_{\nu'} \circ X_\nu = \bigcup_{a' \in X_{\nu'}, a \in X_\nu} (a' \circ a)$  is a hyper BCK-ideal;
- (iii)  $X_\nu$  is a hyper BCK-ideal of  $X_\nu \circ X_{\nu'}$ ;
- (iv) if  $\nu$  is restricted to  $I$ , then  $\nu$  is a fuzzy hyper BCK-ideal on  $I$ .

*Proof.* (i) Let  $x, y \in X$  such that  $x \circ y \ll X_\nu$  and  $y \in X_\nu$ . Then  $\nu(y) = \nu(0)$ ,  $\inf(\nu(x \circ y)) = \nu(0)$  and

$$\nu(x) \geq \min\{\nu(y), \inf(\nu(x \circ y))\} = \nu(0)$$

and so  $\nu(x) = \nu(0)$ . Hence  $x \in X_\nu$  and  $X_\nu$  is a hyper BCK-ideal.

(ii) Clearly  $0 \in X_{\nu'} \circ X_\nu$ . Let  $t, t' \in X$  such that  $t' \circ t \ll X_{\nu'} \circ X_\nu$  and  $t \in X_{\nu'} \circ X_\nu$ . Then there exist  $a' \in X_{\nu'}$  and  $a \in X_\nu$  such that  $t \in a' \circ a$  so by Theorem 4.4,

$$\nu'(t) \geq \inf(\nu'(a' \circ a)) \geq \nu'(a') = \nu'(0)$$

and so

$$\nu'(t) \geq \min(\nu'(t), \inf(\nu'(t' \circ t))) \geq \min(\nu'(t), \nu'(0)).$$

Hence  $t' \in X_{\nu'}$  and so  $t' \in t' \circ 0 \subseteq X_{\nu'} \circ X_\nu$ . Therefore  $X_{\nu'} \circ X_\nu$  is a hyper BCK-ideal in  $X$ .

(iii) Let  $x \in X_\nu$ . Since  $x \in x \circ 0$ , we get that  $x \in X_\nu \subseteq X_\nu \circ X_{\nu'}$  and by (i),  $X_\nu$  is a hyper BCK-ideal of  $X_\nu \circ X_{\nu'}$ .

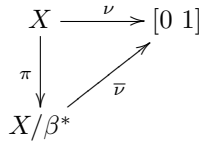
(iv) The proof is clear. □

**Theorem 4.11.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra. If  $\nu : X \rightarrow [0, 1]$  is a fuzzy hyper BCK-ideal on  $X$ , then there exists a fuzzy hyper BCK-ideal  $\bar{\nu}$  on  $(X/\beta^*, *, \bar{0})$  such that for any  $x, y \in X$ ,*

- (i)  $\bar{\nu}(\bar{0}) \geq \bar{\nu}(\beta^*(x))$ ;
- (ii)  $\bar{\nu}(\beta^*(y)) \geq \min(\bar{\nu}(\beta^*(x)), \inf(\bar{\nu}(\beta^*(y) * \beta^*(x)))$ .

*Proof.* (i) We define  $\bar{\nu} : X/\beta^* \rightarrow [0, 1]$  by  $\bar{\nu}(\beta^*(t)) = \bigvee_{t \beta^* x} \nu(x)$  where  $x, t \in X$ .

Consider the following quasi commutative diagram:



firstly we show that  $\bar{\nu}$  is well-defined. Let  $t, t', x \in X$  and  $\beta^*(t) = \beta^*(t')$ . Then  $t \beta^* t'$  and

$$\bar{\nu}(\beta^*(t)) = \bigvee_{x \beta^* t} \nu(x) = \bigvee_{x \beta^* t'} \nu(x) = \bar{\nu}(\beta^*(t'))$$

then for any  $x, t \in X$ , we get that

$$\bar{\nu}(\beta^*(0)) = \bigvee_{t \beta^* 0} \nu(t) = \nu(0) \geq \bigvee_{t \beta^* x} \nu(t) = \bar{\nu}(\beta^*(x))$$

(ii) Let  $x, y \in X$ . For any  $t \in \beta^*(y), t' \in \beta^*(x)$ ,

$$\bigvee_{t \beta^* y} \nu(t) \geq \nu(t) \geq \min(\nu(t'), \inf(\nu(t \circ t')))$$

we get that

$$\begin{aligned}
 \bar{\nu}(\beta^*(y)) &= \bigvee_{t \beta^* y} \nu(t) \\
 &\geq \bigvee_{t' \in \beta^*(x)} (\min(\nu(t'), \inf(\nu(t \circ t')))) \\
 &\geq \min(\bigvee_{t' \in \beta^*(x)} \nu(t'), \bigvee_{t' \in \beta^*(x)} \bigwedge_{t \in \beta^*(y)} (\nu(t \circ t'))) \\
 &\geq \min(\bigvee_{t' \in \beta^*(x)} \nu(t'), \bigwedge_{m \in \beta^*(y) * \beta^*(x)} \bigvee (\nu(m))) \\
 &\geq \min(\bar{\nu}(\beta^*(x)), \inf(\bar{\nu}(\beta^*(y) * \beta^*(x)))
 \end{aligned}$$

□

**Corollary 4.12.** *Let  $(X, \circ, 0)$  be a weak commutative hyper BCK-algebra. If  $\nu : X \rightarrow [0, 1]$  is a fuzzy hyper BCK-ideal on  $X$ , then there exists a fuzzy BCK-ideal  $\xi$  on  $(X/\beta^*, *, \bar{0})$  such that  $\xi \circ \pi \geq \nu$ .*

*Proof.* By Theorem 4.11, we consider  $\xi = \bar{\nu}$ . For any  $x \in X$ , since  $x \beta^* x$ , we get that

$$(\xi \circ \pi)(x) = \xi(\beta^*(x)) = \bigvee_{t \beta^* x} \nu(t) \geq \nu(x).$$

□

**Example 4.13.** [1] Let  $X = \{0, b, c, d\}$ . Then  $\nu$  is a fuzzy hyper  $BCK$ -subalgebra on hyper  $BCK$ -algebra  $(X, \circ, 0)$  as follows:

$\circ$	0	$b$	$c$	$d$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$b$	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
$c$	$\{c\}$	$\{c\}$	$\{0\}$	$\{0\}$
$d$	$\{d\}$	$\{d\}$	$\{c\}$	$\{0, c\}$

and  $\frac{\nu}{\quad} \left| \begin{array}{cccc} 0 & b & c & d \\ 1 & 0.9 & 0.3 & 0.3 \end{array} \right.$ .

Clearly  $(X, \circ, \nu)$  is not weak commutative and  $\nu$  is a fuzzy hyper  $BCK$ -ideal. Now we get that  $X/\beta^* = \{\beta^*(0) = \{0, c\}, \beta^*(b) = \{b\}, \beta^*(d) = \{d\}\}$ ,

$*$	$\beta^*(0)$	$\beta^*(b)$	$\beta^*(d)$
$\beta^*(0)$	$\beta^*(0)$	$\beta^*(0)$	$\beta^*(0)$
$\beta^*(b)$	$\beta^*(b)$	$\beta^*(0)$	$\beta^*(0)$
$\beta^*(d)$	$\beta^*(d)$	$\beta^*(d)$	$\beta^*(0)$

and  $\frac{\bar{\nu}}{\quad} \left| \begin{array}{ccc} \beta^*(0) & \beta^*(b) & \beta^*(d) \\ 1 & 0.9 & 0.3 \end{array} \right.$ .

It is easy to see that  $(X/\beta^*, *, \beta^*(0), \bar{\nu})$  is a hyper  $BCK$ -algebra.

**Definition 4.14.** Let  $(X, \circ, 0)$  be a hyper  $BCK$ -algebra and  $\nu$  be a fuzzy hyper  $BCK$ -ideal on  $X$ . For any  $x, y \in X$ , we define a binary relation  $\rho$  on  $X$  by

$$x\rho y \text{ if and only if } \nu(x) \leq \nu(y) \text{ and } \inf(\nu(x \circ y)) \geq \nu(y).$$

**Example 4.15.** Let  $X = \{0, b, c, d\}$ . We consider the following table:

$\circ$	0	$b$	$c$	$d$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$b$	$\{b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$
$c$	$\{c\}$	$\{c\}$	$\{0, b\}$	$\{c\}$
$d$	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, b\}$

It is easy to see that  $(X, \circ, 0)$  is a hyper  $BCK$ -algebra. Define  $\nu : X \rightarrow [0, 1]$  by  $\nu(b) = 0, \nu(c) = 1/3, \nu(d) = 2/3, \nu(0) = 1$ . Clearly,  $\nu$  is a fuzzy hyper  $BCK$ -ideal and  $\rho = \{(0, 0), (b, b), (c, c), (d, d)\}$ .

**Theorem 4.16.** Let  $(X, \circ, 0)$  be a hyper  $BCK$ -algebra,  $\nu$  be a fuzzy hyper  $BCK$ -ideal of  $X$  and  $x, y \in X$ .

- (i)  $\rho$  is an equivalence relation on  $X$ .
- (ii) if  $\nu$  is one to one and  $x\rho y$ , then for any  $z \in X$  we have  $(x \circ z)\rho(y \circ z)$  and  $(z \circ x)\rho(z \circ y)$ .
- (iii) if  $\nu$  is one to one,  $x\rho y$  and  $u\rho w$  then  $(x \circ u)\rho(y \circ w)$  for all  $u, w \in X$ .

*Proof.* (i) By Theorem 4.4,  $\inf(\nu(x \circ x)) \geq \nu(x)$  and so  $\rho$  is reflexive. Let  $x, y \in X$  such that  $x\rho y$ . Then  $\nu(x) \leq \nu(y)$  and  $\inf(\nu(x \circ y)) \geq \nu(y)$ , since

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) \geq \min(\nu(y), \nu(y)) = \nu(y),$$

we get that  $\nu(x) = \nu(y)$ . Now we have  $\inf(\nu(y \circ x)) \geq \nu(y) = \nu(x)$  and so  $\rho$  is symmetric. Let  $x\rho y$  and  $y\rho z$ . Then  $\nu(x) = \nu(y) = \nu(z)$  and clearly  $\rho$  is transitive.



(ii) Let  $x\rho y$  and  $z \in X$ . Then by (i),  $\nu(x) = \nu(y)$  and since  $\nu$  is one to one we have  $x = y$ . Hence there exist  $a \in x \circ z$  and  $y \in y \circ z$  such that  $\nu(a) \leq \nu(b)$  and  $\inf(\nu(a \circ b)) \geq \nu(b)$ . Therefore  $(x \circ z)\rho(y \circ z)$  and in a similar way we get that  $(z \circ x)\rho(z \circ y)$ .

(iii) Let  $x\rho y$  and  $u\rho w$ . Then by (ii),  $(x \circ u)\rho(y \circ u)$  and  $(y \circ u)\rho(y \circ w)$ . Using the transitivity of  $\rho$ , we get that  $(x \circ u)\rho(y \circ w)$ .  $\square$

**Corollary 4.17.** *Let  $(X, \circ, 0)$  be a hyper BCK-algebra and  $\nu$  be a fuzzy hyper BCK-ideal on  $X$  and  $x, y \in X$ .*

- (i) *if  $\nu$  is one to one, then  $\rho$  is a congruence relation on  $X$ ;*
- (ii)  *$\rho(0) = X_\nu$  and if  $\nu$  is one to one, then  $\rho(0) = \{0\}$ ;*
- (iii) *if  $\nu$  is one to one, then  $\rho$  is a strongly regular relation on  $X$ .*

*Proof.* (i) By Theorem 4.16, the proof is obvious.

(ii) Let  $x \in \rho(0)$ . Then by Theorem 4.16,  $\nu(x) = \nu(0)$  and so  $\rho(0) = X_\nu$ . Since  $\nu$  is one to one, we get that  $X_\nu = \{x \mid \nu(x) = \nu(0)\} = \{x \mid x = 0\} = \{0\}$ .

(iii) Let  $x, y, z \in X$  and  $x\rho y$ . Then  $x = y$  and so  $x \circ z = y \circ z$ . Therefore  $(x \circ z)\bar{\rho}(y \circ z), (z \circ x)\bar{\rho}(z \circ y)$  and so  $\rho$  is a strongly regular relation.  $\square$

**Theorem 4.18.** *Let  $(X, \circ, 0)$  be a (weak commutative) hyper BCK-algebra and  $\nu$  be a one to one fuzzy hyper BCK-ideal of  $X$ . Then,  $(X/\rho, \circ', \rho(0))$  is a (BCK-algebra) hyper BCK-algebra such that for any  $x, y \in X, \rho(x) \circ' \rho(y) = \rho(x \circ y)$ .*

*Proof.* By Corollary 4.17,  $\circ'$  is well-defined and the proof is straightforward.  $\square$

**Example 4.19.** (i) Let  $(X, \circ, 0)$  be the hyper BCK-algebra that is defined in Example 4.15. Clearly,  $\nu$  is one to one and we have  $X/\rho = \{\rho(0), \rho(c), \rho(d)\}$  and by the following table

*	$\rho(0)$	$\rho(c)$	$\rho(d)$
$\rho(0)$	$\rho(0)$	$\rho(0)$	$\rho(0)$
$\rho(c)$	$\rho(c)$	$\rho(0)$	$\rho(c)$
$\rho(d)$	$\rho(d)$	$\rho(d)$	$\rho(0)$

it is easy to see that  $(X/\rho, *, \rho(0))$  is a BCK-algebra.

(ii) Let  $(X, \circ, 0)$  be the hyper BCK-algebra that is defined in Example 4.13. Clearly we obtain  $\rho = \{(0, 0), (b, b), (c, c), (d, d), (c, d), (d, c)\}$ . Now,  $\nu$  is not one to one and we have  $c \rho d$  but  $(c \circ c, d \circ c) \notin \rho$ . Hence  $\rho$  is not a congruence relation.

**Theorem 4.20.** *Let  $(X, \circ_1, 0)$  and  $(Y, \circ_2, 0')$  be (weak commutative) hyper BCK-algebras and  $\nu$  be a one to one fuzzy hyper BCK-ideal of  $Y$ . If  $f : X \rightarrow Y$  is an epimorphism, then*

- (i)  *$\nu_f$  is a fuzzy hyper BCK-ideal of  $X$ ;*
- (ii)  *$X/\rho_f \cong Y/\rho$  such that  $x\rho_f x' \iff \nu(f(x)) \leq \nu(f(x'))$  and  $\inf(\nu(f(x \circ_1 x'))) \geq \nu(f(x'))$ , where  $x, x' \in X$ .*

*Proof.* (i) Clearly for any  $x \in X, \nu_f(0') = \nu(f(0)) \geq \nu(f(x)) = \nu_f(x)$ . Let  $x, x' \in X$ . Since  $\nu$  is a fuzzy hyper BCK-ideal of  $Y$ , we get that

$$\begin{aligned}
\nu_f(x) = \nu(f(x)) &\geq \min\{\nu(f(x')), \inf(\nu(f(x) \circ_2 \nu(f(x'))))\} \\
&= \min\{\nu(f(x')), \inf(\nu(f(x \circ_1 x')))\} \\
&= \min\{\nu_f(x'), \inf(\nu_f(x \circ_1 x'))\}.
\end{aligned}$$

(ii) Since  $\nu$  and  $\nu_f$  are fuzzy hyper  $BCK$ -ideals on  $X$ , then by Theorem 4.18,  $(X/\rho_f, \circ', \rho_f(0))$  and  $(Y/\rho, \circ', \rho(0'))$  are ( $BCK$ -algebras) hyper  $BCK$ -algebras. Now, define a map  $\varphi : X/\rho_f \rightarrow Y/\rho$  by  $\varphi(\rho_f(x)) = \rho(f(x))$ . Let  $x, x' \in X$ . Then

$$\begin{aligned}
\varphi(\rho_f(x)) &= \varphi(\rho_f(x')) \\
\iff f(x)\rho_f(x') & \\
\iff \nu(f(x)) \leq \nu(f(x')) \text{ and } \inf(\nu(f(x) \circ_2 f(x'))) \geq \nu(f(x')) & \\
\iff \nu_f(x) \leq \nu_f(x') \text{ and } \inf(\nu(f(x \circ_1 x'))) \geq \nu(f(x')) & \\
\iff \nu_f(x) \leq \nu_f(x') \text{ and } \inf(\nu_f(x \circ_1 x')) \geq \nu_f(x') & \\
\iff \rho_f(x) = \rho_f(x'). &
\end{aligned}$$

Hence  $\varphi$  is well-defined and one to one. Clearly  $\varphi$  is an epimorphism, and so it is an isomorphism.  $\square$

**Corollary 4.21.** (*Isomorphism Theorem*) Let  $(X, \circ, 0)$  be a hyper  $BCK$ -algebra and  $\nu, \nu'$  be one to one fuzzy hyper  $BCK$ -ideals on  $X$  such that  $\nu(0) = \nu'(0)$ . Then

- (i)  $\nu' \cap \nu$  is a fuzzy hyper  $BCK$ -ideal of  $X$ ;
- (ii)  $(X_\nu \circ X_{\nu'})/\rho_\nu \cong X_\nu/\rho_{\nu' \cap \nu}$ .

*Proof.* (i) Let  $x \in X$ . Then

$$(\nu' \cap \nu)(0) = \min(\nu'(0), \nu(0)) \geq \min(\nu'(x), \nu(x)) = (\nu' \cap \nu)(x).$$

Let  $x, y \in X$ . Then

$$\begin{aligned}
(\nu' \cap \nu)(x) &= \min(\nu'(x), \nu(x)) \\
&\geq \min[\min[\nu'(y), \inf(\nu'(x \circ y))], \min[\nu(y), \inf(\nu(x \circ y))]] \\
&= \min[\min[\nu'(y), \nu(y)], \min[\inf(\nu'(x \circ y)), \inf(\nu(x \circ y))]] \\
&= \min[(\nu' \cap \nu)(y), \inf((\nu' \cap \nu)(x \circ y))]
\end{aligned}$$

(ii) By Theorem 4.10,  $\nu' \cap \nu$  is a fuzzy hyper  $BCK$ -ideal of  $X_\nu$ , then we define  $\varphi : X_\nu/\rho_{\nu' \cap \nu} \rightarrow (X_\nu \circ X_{\nu'})/\rho_\nu$  by  $\varphi(\rho_{\nu' \cap \nu}(x)) = \rho_\nu(x)$ . Let  $x, x' \in X_\nu$  and  $\rho_{\nu' \cap \nu}(x) = \rho_{\nu' \cap \nu}(x')$ . Then  $\nu' \cap \nu(x) = \nu' \cap \nu(x')$  and since  $\nu' \cap \nu$  is one to one, we get that  $x = x'$ . Hence  $\rho_\nu(x) = \rho_\nu(x')$ . Moreover,

$$\varphi(\rho_{\nu' \cap \nu}(x) \circ' \rho_{\nu' \cap \nu}(x')) = \varphi(\rho_{\nu' \cap \nu}(x \circ x')) = \rho_\nu(x \circ x') = \rho_\nu(x) \circ' \rho_\nu(x')$$

and so  $\varphi$  is a homomorphism. Clearly  $\varphi$  is bijection and so is an isomorphism.  $\square$

## 5. Conclusion

To conclude, the current paper has considered the notion of fuzzy hyper  $BCK$ -subalgebras and investigated some of its new useful properties. We defined the concept of the extended fuzzy  $BCK$ -subalgebras, applied the concept of fuzzy hyper  $BCK$ -ideal and showed that:

- (i) For any  $\alpha \in [0, 1]$  and a fuzzy set  $\mu$  of a fuzzy hyper  $BCK$ -subalgebra, then  $\mu$  is a fuzzy hyper  $BCK$ -subalgebra if and only if  $\mu_\alpha$  is a hyper  $BCK$ -subalgebra.
- (ii) By using the concept of fuzzy hyper  $BCK$ -ideal in any hyper  $BCK$ -algebra, we obtain some hyper  $BCK$ -ideals.
- (iii) Through the concept of fundamental relation  $\beta^*$ , we have generated the corresponding structure in hyperstructures such as fuzzy  $BCK$ -subalgebras of fuzzy hyper  $BCK$ -subalgebras and fuzzy  $BCK$ -ideals of fuzzy hyper  $BCK$ -ideals.
- (iv) With respect to the concept of hyper  $BCK$ -ideals on any hyper  $BCK$ -algebra, we introduced an equivalence relation such that it is a congruence and strongly regular relation; then, could we obtain a quotient  $BCK$ -algebra.
- (v) On any nonempty set, we have constructed an extendable fuzzy  $BCK$ -subalgebra and have showed that on any infinite set there exists an extended fuzzy  $BCK$ -subalgebra.

We hope that these results are helpful to further studies in fuzzy structures. In our future studies, we hope to obtain more results regarding fuzzy hyper  $BCK$ -subalgebras, fuzzy  $BCK$ -subalgebras, and their applications.

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#### REFERENCES

- [1] M. Bakhshi, M. M. Zahdi and R. A. Borzooei, *Fuzzy (positive, weak) implicative hyper BCK-ideals*, Iranian Journal of Fuzzy Systems, **1(2)** (2004), 63-79.
- [2] R. A. Borzooei, R. Ameri and M. Hamidi, *Fundamental relation on hyper BCK-algebras*, An. Univ. oradea, fasc. Mat., **21(1)** (2014), 123-136.
- [3] G. R. Biyogmam, O. A. Heubo-Kwegna and J. B. Nganou, *Super implicative hyper BCK-algebras*, Int. J. Pure Appl. Math., **76(2)** (2012), 267-275.
- [4] J. Chvalina, S. Hoskova-Mayerova and A. D. Nezhad, *General actions of hyperstructures and some applications*, An. St. Univ. Ovidius Constanta, **21(1)** (2013), 59-82.
- [5] P. Corsini, *Prolegomena of hypergroup theory*, Second Edition, Aviani Editor, 1993.
- [6] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, 2002.
- [7] J. Dongho, *Category of fuzzy hyper BCK-algebras*, arXiv:1101.2471.
- [8] P. F. He and X. L. Xin, *Fuzzy hyperlattices*, Comput. Math. Appl., **62** (2011), 4682-4690.
- [9] S. Hoskova, *Topological hypergroupoids*, Comput. Math. Appl., **64** (2012), 2845-2849.
- [10] Y. Imai and K. Iseki, *On axiom systems of propositional calculi*, XIV, Proc. Japan Acad. Ser. A Math. Sci., **42** (1966), 19-22.
- [11] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei, *On hyper BCK-algebras*, Ital. J. Pure Appl. Math., **10** (2000), 127-136.
- [12] Y. B. Jun and X. L. Xin, *Fuzzy hyper BCK-ideals of hyper BCK-algebras*, Sci. Math. Jpn., **53(2)** (2001), 353-360.
- [13] Y. B. Jun, M. S. Kang and S. Z. Song, *Several types of bipolar fuzzy hyper BCK-ideals in hyper BCK-algebras*, Honam Math. J., **34(2)** (2012), 145-159.
- [14] V. Leoreanu-Fotea, *Fuzzy hypermodules*, Comput. Math. Appl., **57** (2009), 466-475.

- [15] F. Marty, *Sur une Generalization de la notion de groupe*, 8th Congres Math. Scandinaves, Stockholm., (1934), 45–49.
- [16] J. Meng and Y. B. Jun, *BCK-algebra*, Kyung Moonsa, Seoul, 1994.
- [17] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, (Third Edition), Chapman and Hall/CRC. Boca Raton, 2005.
- [18] M. K. Sen, R. Ameri and G. Chowdhury, *Fuzzy hypersemigroups*, Soft Comput., **12** (2008), 891–900.
- [19] X. Xie, *Fuzzy ideals extensions in semigroups*, Kyungpook Math. J., **42(1)** (2002), 39–49.
- [20] X. L. Xin and P. Wang, *States and measures on hyper BCK-algebras*, J. Appl. Math., (2014), 1–7.
- [21] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338–353.
- [22] J. Zhan, *Fuzzy regular relations on hyperquasigroups*, J. Math. Res. Exposition, Nov., **30(6)** (2010), 1083–1090.

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