

EXTENDED FUZZY BCK -SUBALGEBRAS

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ABSTRACT. This paper extends the notion of fuzzy BCK -subalgebras to fuzzy hyper BCK -subalgebras and defines an extended fuzzy BCK -subalgebras. This study considers a type of fuzzy hyper BCK -ideals in this hyperstructure and describes the relationship between hyper BCK -ideals and fuzzy hyper BCK -ideals. In fact, it tries to introduce a strongly regular relation on hyper BCK -algebras. Moreover, by using the fuzzy hyper BCK -ideals, it defines a congruence relation on (weak commutative) hyper BCK -algebras that under some conditions is strongly regular and the quotient of any hyper BCK -algebra via this relation is a (hyper BCK -algebra) BCK -algebra.

1. Introduction

The theory of hyperstructures was first introduced by Marty in 1934 during the 8th Congress of the Scandinavian Mathematicians [15]. In classical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements, therefore it can be said that the notion of hyperstructures is a generalization of classical algebraic structures, from this point of view. Marty introduced the concept of hypergroups and used it in different contexts like algebraic functions, rational fractions and non commutative groups. In recent years, many researchers have been working on hyper algebraic structures and the series of hyperalgebras branches such as superior, hypergroups, hypermoduls and hyperlattices which have appeared subsequently. Hyperstructures have many applications to several sectors of both pure and applied sciences such as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, artificial intelligence, probabilities, chemistry and physics, especially in atomic physics and harmonic analysis [4, 6, 9]. Borzooei, et al. has applied the hyperstructures to BCK -algebras which were initiated by Imai and Iseki [10] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus and introduced the concepts of hyper BCK -ideals and weak hyper BCK -ideals and discovered the relations between them [11].

Based on the superiority of fuzzy sets [21] and through simplifying the complexity of algebraic structures, many scholars have combined hyperstructures with the fuzzy set theory, and started the research about the fuzzy theory on hyperstructures [8, 14, 18]. The concept of regular relations can provide an important means to discuss hyperquasigroups theory and its analogous role to congruences in semigroup theory.

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Zhan [22] has introduced the concept of regular relations on hyperquasigroups and investigated some of its related properties. The relationships between the fuzzy sets and the algebraic hyperstructures (structures) has already been considered by Xie and Zhan and others, to which the reader can refer to [19].

Regarding these points, the aim of this paper is to generalize the notion of fuzzy BCK -subalgebras by considering the notion of fuzzy hyper BCK -subalgebras and try to define the concept of extendable fuzzy BCK -subalgebras. Also, we want to establish the relationship between fuzzy BCK -algebras and fuzzy hyper BCK -algebras. In the paper, we consider a type of fuzzy hyper BCK -ideals in this hyperstructure and describe the relationship between (BCK -ideals) hyper BCK -ideals and fuzzy hyper BCK -ideals. We introduce a strongly regular relation on any hyper BCK -algebras by using the concept of fuzzy hyper BCK -ideals and obtain a quotient hyper BCK -algebras (BCK -algebras). Moreover, by using fuzzy hyper BCK -ideals, we define a congruence relation on (weak commutative) hyper BCK -algebras that the quotient of any hyper BCK -algebra via this relation is a (hyper BCK -algebra) BCK -algebra. An isomorphism theorem of fuzzy hyper BCK -ideals is obtained by using the one to one fuzzy hyper BCK -ideals.

2. Preliminaries

Definition 2.1. [10] Let X be a set with a binary operation “ $*$ ” and a constant “ 0 ”. Then $(X, *, 0)$ is called a BCK -algebra if it satisfies the following conditions: for all $x, y, z \in X$,

$$(BCI-1) ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCI-2) (x * (x * y)) * y = 0,$$

$$(BCI-3) x * x = 0,$$

$$(BCI-4) x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

$$(BCK-5) 0 * x = 0.$$

We define a binary relation “ \leq ” on X by $x \leq y$ if and only if $x * y = 0$. A BCK -algebra $(X, *, 0)$ is called *commutative* if for any $x, y \in X$, $x * (x * y) = y * (y * x)$.

Theorem 2.2. [10] Let $(X, *, 0)$ be a BCK -algebra. Then we have the following properties: for all $x, y, z \in X$,

$$(a) x \leq y \text{ implies } z * y \leq z * x,$$

$$(b) x \leq y \text{ implies } x * z \leq y * z,$$

$$(c) x \leq y \text{ and } y \leq z \text{ imply } x \leq z,$$

$$(d) (x * y) * z = (x * z) * y,$$

$$(e) x * y \leq z \text{ implies } x * z \leq y,$$

$$(f) (x * z) * (y * z) \leq x * y,$$

$$(g) x * y \leq x,$$

$$(h) x * 0 = x.$$

Definition 2.3. [5] Let X be a nonempty set and $P^*(X)$ be the family of all nonempty subsets of X . Functions $\circ_{i_x} : X \times X \rightarrow P^*(X)$, where $i \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$, are called binary hyperoperations. For all $x, y \in X$, $\circ_{i_x}(x, y)$ is called the hyperproduct of x and y . An algebraic system $(X, \circ_{1_x}, \circ_{2_x}, \dots, \circ_{n_x})$ is called

an n -algebraic hyperstructure, and the structure (X, \circ_X) endowed with only one hyperoperation is called a hypergroupoid. For any two nonempty subsets A and B of hypergroupoid X , we define $A \circ_X B = \bigcup_{a \in A, b \in B} a \circ_X b$

Definition 2.4. [2, 11] Let X be a non-empty set, endowed with a binary hyperoperation “ \circ ” and a constant “ 0 ”. Then, $(X, \circ, 0)$ is called a *hyper BCK-algebra* if satisfies the following axioms: for all $x, y, z \in X$

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(H3) $x \circ X \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$.

where $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ is defined by: for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

A hyper BCK-algebra X is called *weak commutative* if for any $x, y \in X$,

$$(x \circ (x \circ y)) \cap (y \circ (y \circ x)) \neq \emptyset.$$

Theorem 2.5. [11] In any hyper BCK-algebra X , the following statements hold: for all $x, y, z \in X$ and $A, B \subseteq X$,

(a1) $0 \circ 0 = \{0\}$,

(a2) $0 \ll x$,

(a3) $x \ll x$,

(a4) $A \ll A$,

(a5) $A \ll 0$ implies $A = \{0\}$,

(a6) $A \subseteq B$ implies $A \ll B$,

(a7) $0 \circ x = \{0\}$,

(a8) $x \circ y \ll x$,

(a9) $x \in x \circ 0$,

(a10) $y \ll z$ implies that $x \circ z \ll x \circ y$,

(a11) $A \circ B \ll A$.

Definition 2.6. [11] Let I be a nonempty subset of a hyper BCK-algebra X . Then I is called:

- (i) a hyper BCK-subalgebra of X , if for all $x, y \in I$, $x \circ y \subseteq I$;
- (ii) a weak hyper BCK-ideal of X if (1) $0 \in I$, (2) for all $x, y \in X$, $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$;
- (iii) a hyper BCK-ideal of X if (1) $0 \in I$, (2) for all $x, y \in X$, $x \circ y \ll I$ and $y \in I$ imply $x \in I$.

Definition 2.7. [2] Let $(X; \circ, 0)$ be a hyper BCK-algebra, R be an equivalence relation on X and $A, B \subseteq X$.

- (i) ARB means that there exist $a \in A$ and $b \in B$ such that aRb and $\overline{\overline{ARB}}$ means that for all $a \in A$ and $b \in B$, we have aRb ;
- (ii) R is called *strongly regular* if for all x, a, b of X , from aRb it follows that $(a \circ x) \overline{\overline{R}} (b \circ x)$ and $(x \circ a) \overline{\overline{R}} (x \circ b)$;

- (iii) $\mathcal{L}(a_1, a_2, \dots, a_n)$ will denote the set of all finite combinations of elements $\{a_1, a_2, \dots, a_n\}$ where $a_1, a_2, \dots, a_n \in X$;
 (iv) $\beta_1 = \{(x, x) \mid x \in X\}$ and for every integer $n \geq 1$, let β_n be defined as follows:

$$x\beta_n y \iff \exists (a_1, \dots, a_n) \in X^n \text{ and } \exists u \in \mathcal{L}(a_1, \dots, a_n) \text{ such that } \{x, y\} \subseteq u.$$

For any $n \geq 1$, the relations β_n are symmetric, the relation $\beta = \bigcup_{n \geq 1} \beta_n$ is a reflexive and symmetric relation. Let β^* be the transitive closure of β (the smallest transitive relation such that contains β).

Theorem 2.8. [2] Let $(X; \circ, 0)$ be a weak commutative hyper BCK-algebra. Then β^* is a strongly regular relation on X and $(X/\beta^*; *, \bar{0})$ is a BCK-algebra.

Definition 2.9. [7] Let $(X, \circ, 0)$ be a hyper BCK-algebra. A fuzzy set $\mu : X \rightarrow [0, 1]$, is called a fuzzy hyper BCK-subalgebra if for all $x, y \in X$,

$$\inf(\mu(x \circ y)) \geq \min(\mu(x), \mu(y)).$$

Lemma 2.10. [7] Let $(X, \circ, 0)$ be a hyper BCK-algebra and μ be a fuzzy hyper BCK-subalgebra on X . Then for any $x \in X$, we have $\mu(x) \leq \mu(0)$.

Definition 2.11. [12] Let $(X, \circ, 0)$ be a hyper BCK-algebra. A fuzzy set ν on X is called a fuzzy hyper BCK-ideal, if for all $x, y \in X$ it satisfies the following properties:

$$(FH1) \ x \ll y \implies \nu(x) \geq \nu(y),$$

$$(FH2) \ \nu(y) \geq \min\{\nu(x), \inf(\nu(y \circ x))\}.$$

Proposition 2.12. [12] Let $(X, \circ, 0)$ be a hyper BCK-algebra and ν be a fuzzy hyper BCK-ideal of X . Then for any $x \in X$, $\nu(0) \geq \nu(x)$.

Definition 2.13. [17] Let X be a nonempty set. A fuzzy set $\mu : X \rightarrow [0, 1]$ is said to be normal if there exists $x \in X$ such that $\mu(x) = 1$.

3. Fuzzy Hyper BCK-subalgebras

In this section, the concept of fuzzy hyper BCK-subalgebras will be considered as a generalization of fuzzy BCK-subalgebras and some of its properties will be investigated. We will also prove that a fuzzy hyper BCK-subalgebra and fuzzy BCK-subalgebras of fuzzy hyper BCK-subalgebras can be constructed via the fundamental relation. We will define the concept of extendable fuzzy BCK-subalgebras and will show that any infinite set is an extended fuzzy BCK-subalgebra.

Theorem 3.1. Let X be a nonempty set and $0 \notin X$. Then there exist a hyperoperation “ \circ ” and a fuzzy set μ on $X' = X \cup \{0\}$ such that $(X', \circ, 0)$ is a hyper BCK-subalgebra and μ is a fuzzy hyper BCK-subalgebra on X' .

Proof. Let X be a nonempty set and $0 \notin X$. For any $x, y \in X'$, define the binary hyperoperation “ \circ ” on X' as follows:

$$x \circ y = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ x, & \text{otherwise.} \end{cases} \quad (1)$$

Clearly $(X', \circ, 0)$ is a hyper BCK-algebra. Now, it is easy to see that any fuzzy set $\mu : X' \rightarrow [0, 1]$ such that $\mu(0) = 1$, is a fuzzy hyper BCK-subalgebra on X' . \square

Corollary 3.2. *For any nonempty set, we can construct at least a hyper BCK-algebra.*

Theorem 3.3. *Let (X, \circ) be a hyper BCK-subalgebra which is defined in Theorem 3.1 and $\mathcal{Fh} = \{\mu \mid \mu \text{ is a fuzzy hyper BCK-subalgebra on } (X, \circ)\}$. If $|X| \geq 1$, then $|\mathcal{Fh}| = |\mathbb{R}|$.*

Proof. (i) Let $X = \{x\}$. Then (X, \circ, x) is a hyper BCK-algebra such that $x \circ x = \{x\}$. Then $\mu : X \rightarrow [0, 1]$ such that it is defined by $\mu(x) = \alpha$ is a fuzzy hyper BCK-subalgebra on X where $\alpha \in [0, 1]$.

(ii) Let $|X| \geq 2$. Then by Corollary 3.2, we can construct at least a hyper BCK-subalgebra on X . If $(X, \circ, 0)$ is one of the hyper BCK-algebras which are defined in Theorem 3.1, then for all $\alpha \in [0, 1]$ we define $\mu_\alpha : X \rightarrow [0, 1]$ by

$$\mu_\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ \alpha, & \text{if } x \neq 0. \end{cases}$$

Now for any $y \in X$ and by (1), we get that

$$\inf(\mu(x \circ y)) = \mu(y) \geq \min(\mu(y), \mu(x)).$$

Hence $\mu \in \mathcal{Fh}$ and so $|\mathcal{Fh}| = |[0, 1]|$. \square

Theorem 3.4. *Let $(X, \circ, 0)$ be a hyper BCK-algebra and μ be a fuzzy hyper BCK-subalgebra on X . Then for any $x \in X$, we have $\mu(x) = \inf(\mu(x \circ 0))$.*

Proof. Since $x \in x \circ 0$, by Lemma 2.10, we get that

$$\mu(x) = \min(\mu(0), \mu(x)) \leq \inf(\mu(x \circ 0)) \leq \mu(x).$$

It follows that $\mu(x) = \inf(\mu(x \circ 0))$. \square

Corollary 3.5. *Let $(X, \circ, 0)$ be a hyper BCK-algebra, μ be a fuzzy hyper BCK-subalgebra on X , $\alpha \in [0, 1]$ and $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$. Then*

- (i) $0 \in \mu_\alpha$,
- (ii) μ_α is a hyper BCK-subalgebra of $(X, \circ, 0)$,
- (iii) if $0 \leq \alpha \leq \alpha' \leq 1$, then $\mu_{\alpha'} \subseteq \mu_\alpha$.

Proof. (i) Let $x \in \mu_\alpha$. Since by Lemma 2.10, $\mu(0) \geq \mu(x) \geq \alpha$, we get that $0 \in \mu_\alpha$.

(ii) Let $x, y \in \mu_\alpha$. Then $\min(\mu(x), \mu(y)) \geq \alpha$. Now, for any $a \in x \circ y$, $\mu(a) \geq \inf(\mu(x \circ y)) \geq \min(\mu(x), \mu(y)) \geq \alpha$. Hence $a \in \mu_\alpha$ and so $x \circ y \subseteq \mu_\alpha$. Then μ_α is a hyper BCK-subalgebra of $(X, \circ, 0)$.

(iii) It is straightforward by definition. \square

Theorem 3.6. Let $(X, \circ, 0)$ be a hyper BCK-algebra and S be a hyper subalgebra of X . Define $\mu : X \rightarrow [0, 1]$ by,

$$\mu(x) = \begin{cases} \alpha', & \text{if } x \in S, \\ \alpha, & \text{if } x \notin S, \end{cases} \text{ where } 0 < \alpha < \alpha' < 1.$$

Then μ is a fuzzy hyper BCK-subalgebra on X .

Proof. Let $x, y \in X$, then $\inf(\mu(x \circ y)) = \alpha$ or α' . If $x, y \in S$, then $x \circ y \subseteq S$ and so $\inf(\mu(x \circ y)) = \alpha' = \min(\mu(x), \mu(y))$. If $x \in S$ and $y \notin S$, then $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha) = \min(\mu(x), \mu(y))$. If $x \notin S$ and $y \in S$, then $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha') = \min(\mu(x), \mu(y))$. If $x \notin S$ and $y \notin S$, then $\inf(\mu(x \circ y)) \geq \alpha = \min(\alpha, \alpha) = \min(\mu(x), \mu(y))$. It follows that μ is a fuzzy hyper BCK-subalgebra on X . \square

Lemma 3.7. If $(X, \circ, 0)$ is a weak commutative hyper BCK-subalgebra and μ is a fuzzy hyper BCK-algebra on X , then there exists a fuzzy set $\bar{\mu}$ on BCK-algebra $(X/\beta^*, *, \bar{0})$ such that: for any $x, y \in X$, (i) $\bar{\mu}(\beta^*(0)) \geq \bar{\mu}(\beta^*(x))$;

- (ii) if $x\beta^*y$, then $\bar{\mu}(\beta^*(x)) = \bar{\mu}(\beta^*(y))$;
- (iii) $\min(\mu(x), \mu(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$;
- (iv) there exists $t \in x \circ y$ such that $\bar{\mu}(\beta^*(x \circ y)) = \mu(t)$.

Proof. By Theorem 2.8, $(X/\beta^*, *, \bar{0})$ is a BCK-algebra. We define a map $\bar{\mu} : X/\beta^* \rightarrow [0, 1]$ by

$$\bar{\mu}(\beta^*(t)) = \begin{cases} \mu(0), & \text{if } 0 \in \beta^*(x), \\ \bigwedge_{t\beta^*x} \mu(x), & \text{otherwise,} \end{cases}$$

where $x, t \in X$. Then for any $x, y, t' \in X$ we have:

(i) $\bar{\mu}(\beta^*(0)) = \mu(0) \geq \bigwedge_{t' \beta^* x} \mu(t') = \bar{\mu}(\beta^*(x))$.

(ii) Since $x\beta^*y$ and β^* is transitive, we get that

$$\bar{\mu}(\beta^*(x)) = \bigwedge_{t \beta^* x} \mu(t) = \bigwedge_{t \beta^* y} \mu(t) = \bar{\mu}(\beta^*(y)).$$

(iii) Since $x \beta^* x$ and $y \beta^* y$, we get that

$$\mu(x) \geq \bar{\mu}(\beta^*(x)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$$

and $\mu(y) \geq \bar{\mu}(\beta^*(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y)))$. Hence

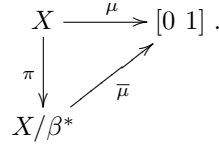
$$\min(\mu(x), \mu(y)) \geq \min(\bar{\mu}(\beta^*(x)), \bar{\mu}(\beta^*(y))).$$

(iv)

$$\bar{\mu}(\beta^*(x) * \beta^*(y)) = \bar{\mu}(\beta^*(x \circ y)) = \bar{\mu}\{\beta^*(m) \mid m \in x \circ y\} = \bigwedge_{s \beta^* m \in x \circ y} \mu(s).$$

Now, since $s\beta^*m$ and $m \in x \circ y$, then $s \in x \circ y$, and so there exists $t \in x \circ y$ such that $\mu(t) = \bigwedge_{s \beta^* m \in x \circ y} \mu(s)$. \square

Theorem 3.8. Let $(X, \circ, 0)$ be a weak commutative hyper BCK-algebra and μ be a fuzzy hyper BCK-subalgebra on X . Then there exists a fuzzy set $\nu : X/\beta^* \rightarrow [0, 1]$ such that on BCK-algebra $(X/\beta^*, *, \beta^*(0))$ is a fuzzy BCK-subalgebra. Moreover, $\nu \circ \pi \leq \mu$ or the following diagram is quasi commutative:



Proof. We consider $\nu = \bar{\mu}$. Then by Lemma 3.7, (i) for any $x \in X, \nu(\beta^*(0)) \geq \nu(\beta^*(x))$, (ii) for any $x, y \in X$, there exists $t \in x \circ y$ such that $\bar{\mu}(\beta^*(x \circ y)) = \mu(t)$ and so,

$$\begin{aligned}
 \nu(\beta^*(x) * \beta^*(y)) &= \nu(\beta^*(x \circ y)) \\
 &= \mu(t) \geq \inf(\mu(x \circ y)) \\
 &\geq \min(\mu(x), \mu(y)) \geq \min(\nu(\beta^*(x)), \nu(\beta^*(y)))
 \end{aligned}$$

Therefore ν is a fuzzy BCK-subalgebra on X/β^* and clearly $\nu \circ \pi \leq \mu$. □

Example 3.9. Consider the BCK-algebras $(\{1, 2, 3, 4\}, \star, 1)$ and $(\{a, b\}, \star', a)$ as follows:

\star	1	2	3	4
1	1	1	1	1
2	2	1	2	2
3	3	3	1	3
4	4	4	4	1

and

\star'	a	b
a	a	a
b	b	a

For any $x, y \in \{a, b\}$ and $w \in \{1, 2, 3, 4\}$, we set $w_x = (w, x), \bar{w}_x = \{(w, x)\}, \bar{w}_{x,y} = \{(w, x), (w, y)\}$ and define a hyperoperation “ \circ ” on $\{1, 2, 3, 4\} \times \{a, b\}$ as follows:

\circ	1_a	1_b	2_a	2_b	3_a	3_b	4_a	4_b
1_a	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{1}_a$
1_b	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{1}_a$	$\bar{1}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$
2_a	$\bar{2}_a$	$\bar{2}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{2}_a$	$\bar{2}_a$	$\bar{2}_a$	$\bar{2}_a$
2_b	$\bar{2}_b$	$\bar{2}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{2}_b$	$\bar{2}_{a,b}$	$\bar{2}_b$	$\bar{2}_{a,b}$
3_a	$\bar{3}_a$	$\bar{3}_a$	$\bar{3}_a$	$\bar{3}_a$	$\bar{1}_a$	$\bar{1}_a$	$\bar{3}_a$	$\bar{3}_a$
3_b	$\bar{3}_b$	$\bar{3}_{a,b}$	$\bar{3}_b$	$\bar{3}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$	$\bar{3}_b$	$\bar{3}_{a,b}$
4_a	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{4}_a$	$\bar{1}_a$	$\bar{1}_a$
4_b	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{4}_b$	$\bar{4}_{a,b}$	$\bar{1}_b$	$\bar{1}_{a,b}$

and a fuzzy set μ on X by

μ	1_a	1_b	2_a	2_b	3_a	3_b	4_a	4_b
	1	$\frac{7}{8}$	$\frac{6}{8}$	$\frac{5}{8}$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

Then it is easy to see that $(\{1, 2, 3, 4\} \times \{a, b\}, \circ, 1_a)$ is a weak commutative hyper BCK-algebra and μ is a fuzzy hyper BCK-algebra on $\{1, 2, 3, 4\} \times \{a, b\}$. We have $\beta^*((1, a)) = \{(1, a), (1, b)\}, \beta^*((2, a)) = \{(2, a), (2, b)\}, \beta^*((3, a)) = \{(3, a), (3, b)\}$ and $\beta^*((4, a)) = \{(4, a), (4, b)\}$.

Hence $\frac{Z}{\beta^*} = \{\beta^*((1, a)), \beta^*((2, a)), \beta^*((3, a)), \beta^*((4, a))\}$ and we can obtain the following tables:

*	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(3, a)$	$\beta^*(4, a)$
$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$	$\beta^*(1, a)$
$\beta^*(2, a)$	$\beta^*(2, a)$	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(2, a)$
$\beta^*(3, a)$	$\beta^*(3, a)$	$\beta^*(3, a)$	$\beta^*(1, a)$	$\beta^*(3, a)$
$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(4, a)$	$\beta^*(1, a)$

and

$\nu = \bar{\mu}$	$\beta^*(1, a)$	$\beta^*(2, a)$	$\beta^*(3, a)$	$\beta^*(4, a)$
	1	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Clearly ν is a fuzzy *BCK*-subalgebra on *BCK*-algebra $(\frac{Z}{\beta^*}, *, \beta^*((0, a)))$ and for any $(x, y) \in Z$,

$$(\nu \circ \pi)(x, y) = \nu(\beta^*(x, y)) \leq \mu(x, y).$$

Definition 3.10. (i) Let $(X, *, 0)$ be a *BCK*-algebra and $(Y, \circ, 0)$ be a hyper *BCK*-algebra. We say that the *BCK*-algebra X is derived from the hyper *BCK*-algebra Y if X is isomorphic to a nontrivial quotient of Y . ($X \cong Y/\beta^*$)

(ii) A fuzzy *BCK*-subalgebra μ on the *BCK*-algebra X is called an *extendable* fuzzy *BCK*-subalgebra, if there exist a hyper *BCK*-algebra $(Y, \circ, 0)$, a fuzzy hyper *BCK*-subalgebra ν on Y and $n \in \mathbb{N}$ such that $|(X, *, \mu)| = |(Y, \circ, \nu)| - n$, and *BCK*-algebra X is derived from hyper *BCK*-algebra Y . If $X = Y$ and almost everywhere $\mu = \nu$ ($\mu = \nu$ a.e that means $|\{x; \mu(x) \neq \nu(x)\}| = 1$), we will say that it is an extended fuzzy *BCK*-subalgebra.

Example 3.11. (i) Let $X = \{a, b, c, d\}$. Then μ is a fuzzy *BCK*-subalgebra on *BCK*-algebra $(X, *, a)$ as follows:

*	a	b	c	d
a	a	a	a	a
b	b	a	b	b
c	c	c	a	c
d	d	d	d	a

and $\frac{\mu}{} \left| \begin{array}{cccc} a & b & c & d \\ 1 & 0.2 & 0.4 & 0.6 \end{array} \right.$

Now, set $Y = \{e, a, b, c, d\} = X \cup \{e\}$. Then ν is a fuzzy hyper *BCK*-subalgebra on hyper *BCK*-algebra (Y, \circ, e) as follows:

\circ	e	a	b	c	d
e	{e}	{e}	{e}	{e}	{e}
a	{a}	{e, a}	{e, a}	{e, a}	{e, a}
b	{b}	{b}	{e, a}	{b}	{b}
c	{c}	{c}	{c}	{e, a}	{c}
d	{d}	{d}	{d}	{d}	{e, a}

and $\frac{\nu}{} \left| \begin{array}{ccccc} e & a & b & c & d \\ 1 & 1 & 0.2 & 0.4 & 0.6 \end{array} \right.$

Clearly $Y/\beta^* \cong X$, $|Y| = |X| + 1$ and so μ is an extendable fuzzy *BCK*-subalgebra on $(X, *, a)$.

(ii) μ is a fuzzy BCK-subalgebra on BCK-algebra $(\mathbb{N}, *, 1)$ as follows:

$$x * y = \begin{cases} 1, & \text{if } x = y, \\ x, & \text{otherwise} \end{cases} \text{ and } \mu(x) = 1/x.$$

Now, ν is a fuzzy hyper BCK-subalgebra on hyper BCK-algebra $(\mathbb{N}, \circ, 1)$ as follows:

$$x \circ y = \begin{cases} 1, & \text{if } x = 1, \\ \{1, 2\}, & \text{if } x = y \text{ and } x \neq 1, \\ \{2\}, & \text{if } x = 2 \text{ and } y = 1, \\ \{1, 2\}, & \text{if } x = 2 \text{ and } y \neq 1, \\ x, & \text{otherwise} \end{cases} \text{ and } \nu(x) = \begin{cases} 1, & \text{if } x = 1 = 2, \\ 1/x, & \text{otherwise.} \end{cases}$$

Clearly ($\mu = \nu$ a.e), $\mathbb{N}/\beta^* \cong \mathbb{N}$ so μ is an extended fuzzy BCK-subalgebra on $(\mathbb{N}, *, 1)$.

Theorem 3.12. *Let $(X, \circ, 0)$ be a hyper BCK-algebra, μ be a fuzzy hyper BCK-subalgebra on X and $\bar{X} = \{\mu(x) \mid x \in X\}$. If μ is one to one, then:*

- (i) *there exists a hyperoperation \circ' on \bar{X} such that $(\bar{X}, \circ', \mu(0))$ is a hyper BCK-algebra;*
- (ii) *there exists a fuzzy set $\bar{\mu} : \bar{X} \rightarrow [0, 1]$ such that $\bar{\mu}$ is a fuzzy hyper BCK-subalgebra on \bar{X} ;*
- (iii) *there exists an operation $*$ on \bar{X} such that $(\bar{X}, *, \mu(0))$ is a BCK-algebra.*

Proof. (i) For any $x, y \in X$, define a hyperoperation \circ' on \bar{X} , by $\mu(x) \circ' \mu(y) = \mu(x \circ y)$. It can be easily seen that $\mu(x) \ll' \mu(y) \Leftrightarrow x \ll y$. It is easy to see that $(\bar{X}, \circ', \mu(0))$ is a hyper BCK-algebra.

(ii) For any $x \in X$, define $\bar{\mu}(\mu(x)) = \mu(x)$. Clearly $\bar{\mu}$ is a fuzzy hyper BCK-subalgebra on (\bar{X}, \circ') .

(iii) For any $x, y \in X$, define an operation $*$ on \bar{X} by

$$\mu(x) * \mu(y) = \begin{cases} \mu(x), & \text{if } y = 0, \\ \sup(\mu(x \circ y)), & \text{otherwise.} \end{cases}$$

We just prove BCI-4. Let $x, y \in X$ and $\mu(x) * \mu(y) = \mu(y) * \mu(x) = \mu(0)$. It follows $0 \in x \circ y$ and $0 \in y \circ x$, that is, $\mu(x) = \mu(y)$. It is easy to see that BCI-1, BCI-2, BCI-3 and BCK-5 are valid and so $(\bar{X}, *, \mu(0))$ is a BCK-algebra. \square

Corollary 3.13. *Let $(\bar{X}, \circ, \mu(0))$ be a hyper BCK-algebra and μ be a fuzzy hyper BCK-subalgebra on \bar{X} . Then there exists a binary operation $*$ on \bar{X} , such that $(\bar{X}, *, \mu(0))$ is a BCK-algebra and μ is a fuzzy BCK-subalgebra on \bar{X} .*

Theorem 3.14. *Let X be a nonempty set, $0 \notin X$ and $X' = X \cup \{0\}$. Then there exist a hyperoperation \circ on X' , a hyperoperation \circ' on \bar{X}' , a binary operation $*$ on \bar{X}' , a fuzzy set μ on X' and fuzzy set ξ on \bar{X}' such that:*

- (i) *$(X', \circ, 0)$ is a hyper BCK-algebra and μ is a fuzzy hyper BCK-subalgebra on X' ;*

- (ii) $(\overline{X'}, \circ', \mu(0))$ is a hyper BCK-algebra and μ is a fuzzy hyper BCK-subalgebra on $\overline{X'}$;
 (iii) $(\overline{X'}, *, \mu(0))$ is a BCK-algebra and ξ is a fuzzy BCK-subalgebra on $\overline{X'}$;
 (iv) $|X'| = |\overline{X'}| + 1$.

Proof. Let $|X| \geq 2$ and $b \in X$ be fixed. For any $x, y \in X'$, define a binary hyperoperation \circ on X' as follows,

$$x \circ y = \begin{cases} 0, & \text{if } x = 0, \\ \{0, b\}, & \text{if } x = y \text{ and } x \neq 0, \\ \{b\}, & \text{if } x = b \text{ and } y = 0, \\ \{0, b\}, & \text{if } x = b \text{ and } y \neq 0, \\ x, & \text{otherwise.} \end{cases} \quad (2)$$

Now, we show that $(X', \circ, 0)$ is a hyper BCK-algebra. We just check that conditions (H1) and (H2) are valid.

(H1): Let $x, y, z \in X'$. If $x = 0$, then $(x \circ z) \circ (y \circ z) = \{0\} \circ (y \circ z) = \{0\} \ll x \circ y$. If $x = b$, then $(x \circ z) \circ (y \circ z) \subseteq \{0, b\} \circ (y \circ z) \subseteq \{0, b\} \ll x \circ y$. If $x \notin \{0, b\}$, we consider the following cases:

Case 1: $x = y \neq z$. Then $(x \circ z) \circ (y \circ z) = x \circ y = x \circ x = \{0, b\} \ll \{0, b\} = x \circ y$.

Case 2: $x = z \neq y$. Then $(x \circ z) \circ (y \circ z) = \{0, b\} \circ (y \circ z) = \{0, b\} \ll x = x \circ y$.

Case 3: $y = z \neq x$. Then $(x \circ z) \circ (y \circ z) \subseteq x \circ \{0, b\} = \{0, b\} \ll x = x \circ y$.

Case 4: $x \neq y \neq z$. Then $(x \circ z) \circ (y \circ z) = x \circ y = x \ll x = x \circ y$.

Case 5: $x = y = z$. Then $(x \circ z) \circ (y \circ z) = \{0, b\} \ll \{0, b\} = x \circ y$.

(H2): Let $x, y, z \in X$. The proof of $(x \circ y) \circ z = (x \circ z) \circ y$ is similar to that of (H1), and then it is easy to see that $(X', \circ, 0)$ is a hyper BCK-algebra. Consider a fuzzy set μ on X' such that $\mu(0) = \mu(b) = 1$, by (2), we get that

$$\inf(\mu(x \circ y)) = \mu(x) \geq \min(\mu(x), \mu(y))$$

and so μ is a fuzzy hyper BCK-subalgebra on $(X', \circ, 0)$. Now for any $x, y \in X$, define a hyperoperation \circ' on $\overline{X'}$ by $\mu(x) \circ' \mu(y) = \mu(x \circ y)$, $\xi : \overline{X'} \rightarrow [0, 1]$ by $\xi(\mu(x)) = \mu(x)$ and an operation $*$ on $\overline{X'}$ as follows:

$$\mu(x) * \mu(y) = \sup(\mu(x) \circ' \mu(y)).$$

It can be easily seen that $\mu(x) \ll' \mu(y) \Leftrightarrow x \ll y$, $(\overline{X'}, \circ', 0)$ is a hyper BCK-algebra, μ is a fuzzy hyper BCK-subalgebra on $\overline{X'}$, $(\overline{X'}, *, 0)$ is a BCK-algebra, ξ is a fuzzy BCK-subalgebra on $\overline{X'}$ and since $\mu(0) = \mu(b)$, we get that $|X'| = |\overline{X'}| + 1$. \square

Corollary 3.15. *On any nonempty set we can construct an extendable fuzzy BCK-subalgebra.*

Example 3.16. Let $X = \{0, a, b, c, d\}$. Then μ is a fuzzy hyper BCK-subalgebra on hyper BCK-algebra $(X, \circ, 0)$ as follows:

\circ	0	a	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, b\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, b\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, b\}$

μ	0	a	b	c	d
	1	0.2	1	0.3	0.4

Clearly $\bar{X} = \{\mu(0), \mu(a), \mu(c), \mu(d)\}$ is a BCK-algebra as follows:

$*$	$\mu(0)$	$\mu(a)$	$\mu(c)$	$\mu(d)$
$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(0)$
$\mu(a)$	$\mu(a)$	$\mu(0)$	$\mu(a)$	$\mu(a)$
$\mu(c)$	$\mu(c)$	$\mu(c)$	$\mu(0)$	$\mu(c)$
$\mu(d)$	$\mu(d)$	$\mu(d)$	$\mu(d)$	$\mu(0)$

4. Fuzzy Hyper BCK-ideals in Hyper BCK-algebras

In this section, the notion of fuzzy hyper BCK-ideal in hyper BCK-algebras was applied and some significant hyper BCK-ideals were obtained. Moreover, we defined a regular equivalence relation on any hyper BCK-algebras via the concept of a fuzzy hyper BCK-ideal.

Notation. A fuzzy hyper BCK-ideal ν on a hyper BCK-algebra (X, \circ) is called *commutative*, if for all $x, y \in X$, $\sup(\nu(x \circ y)) = \sup(\nu(y \circ x))$.

Theorem 4.1. *Let X be a nonempty set and $0 \in X$. Then there exists a hyperoperation \circ on X such that (X, \circ) is a hyper BCK-algebra and any normal fuzzy set on (X, \circ) is a fuzzy hyper BCK-ideal.*

Proof. For any $x, y \in X$, we define a binary hyperoperation \circ on X as follows:

$$x \circ y = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{0, x\}, & \text{if } x = y, x \neq 0, \\ \{x\}, & \text{otherwise.} \end{cases}$$

Clearly, $(X, \circ, 0)$ is a hyper BCK-algebra. Let $\nu : X \rightarrow [0, 1]$ be a normal fuzzy set and $x, y \in X$, then $\nu(0) = 1 \geq \nu(y)$. If $x \neq y$, then

$$\nu(x) \geq \min(\nu(y), \nu(x)) = \min(\nu(y), \inf(\nu(x \circ y))).$$

If $0 \neq x = y$, then

$$\nu(x) \geq \min(\nu(y), \nu(x)) = \min(\nu(y), \inf(\nu(x \circ y))).$$

Therefore, ν is a fuzzy hyper BCK-ideal. □

Corollary 4.2. *Let (X, \circ) be a hyper BCK-algebra which is defined in Theorem 4.1. $\mathcal{Fhi} = \{\mu \mid \mu \text{ is a fuzzy hyper BCK-ideal on } (X, \circ)\}$. If $|X| \geq 1$, then $|\mathcal{Fhi}| = |\mathbb{R}|$.*

Example 4.3. Let $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$. Then $(\mathbb{N}_5, \circ, 1)$ is a hyper *BCK*-algebra as follows:

\circ	1	2	3	4	5
1	{1}	{1}	{1}	{1}	{1}
2	{2}	{1, 2}	{2}	{2}	{2}
3	{3}	{3}	{1, 3}	{3}	{3}
4	{4}	{4}	{4}	{1, 4}	{4}
5	{5}	{5}	{5}	{5}	{1, 5}

Now, define $\nu : \mathbb{N}_5 \rightarrow [0, 1]$ by $\nu(x) = \frac{1}{x}$. It is easy to see that ν is a fuzzy hyper *BCK*-ideal.

Theorem 4.4. Let $(X, \circ, 0)$ be a hyper *BCK*-algebra and ν be a fuzzy hyper *BCK*-ideal of X . Then for any $x, y \in X$ and $A, B \subset X$:

- (i) if $A \ll B$, then there exist $b \in B$ such that $\sup(\nu(A)) \geq \nu(b)$;
- (ii) if $A \ll B$, then $\inf(\nu(A)) \geq \inf(\nu(B))$;
- (iii) $\nu(x) \leq \inf(\nu(x \circ y))$ and $\inf(\nu(A \circ B)) \geq \inf(\nu(A))$;
- (iv) if ν is commutative, then $\nu(x) \leq \inf(\nu(y \circ x))$.

Proof. (i) Since $A \ll B$, then for any $a \in A$ there exists $b \in B$ such that $a \ll b$. Hence

$$\sup(\nu(A)) \geq \nu(a) \geq \nu(b),$$

therefore $\sup(\nu(A)) \geq \nu(b)$.

(ii) Let $A \ll B$. Then for any $a \in A$ there exists $b \in B$ such that $a \ll b$ and so $\nu(a) \geq \nu(b)$. Hence we have $\inf(\nu(A)) \geq \nu(b) \geq \inf(\nu(B))$.

(iii) By Theorem 2.5, $x \circ y \ll x$. Then by (iii), we get that

$$\nu(x) \leq \inf(\nu(x \circ y)).$$

(iv) Since ν is commutative, then by (iv) we get that $\sup(\nu(y \circ x)) \geq \nu(x)$. \square

Example 4.5. Let $(\mathbb{N}_5, \circ, 1)$ be the hyper *BCK*-algebra that is defined in Example 4.3, $A = \{1, 2\}$ and $B = \{1, 2, 5\}$. Clearly $A \ll B$ and

$$\inf(\nu(A)) = \inf\{\nu(1), \nu(2)\} = \frac{1}{2} \geq \frac{1}{5} \geq \inf\{\nu(1), \nu(2), \nu(5)\} = \inf(\nu(B))$$

and $A \circ B = A$, therefore $\inf(\nu(A \circ B)) = \inf(\nu(A))$.

Corollary 4.6. Let $(X, \circ, 0)$ be a hyper *BCK*-algebra, $\alpha \in [0, 1]$ and ν be a fuzzy hyper *BCK*-ideal of X such that $\nu_\alpha \neq \emptyset$. Then for all $x, y, z \in X$:

- (i) $0 \in \nu_\alpha$;
- (ii) if $y \in \nu_\alpha$ and $x \ll y$, then $x \in \nu_\alpha$;
- (iii) $(y \circ z) \ll x$ implies that $\nu(y) \geq \min(\nu(z), \nu(x))$;
- (iv) ν_α is a hyper *BCK*-ideal of X .

Proof. (i) There exists $x \in \nu_\alpha$ such that $\nu(x) \geq \alpha$. Since by definition $\nu(0) \geq \nu(x)$, we get that $0 \in \nu_\alpha$.

(ii) Since $x \ll y$, by definition, we get that $\nu(x) \geq \nu(y)$. Now, $y \in \nu_\alpha$ then $x \in \nu_\alpha$.

(iii) $(y \circ z) \ll x$ implies that $0 \in (y \circ z) \circ x$, then by Theorem 4.4, we get that $\nu(x) \leq \inf(\nu(y \circ z))$. Now, ν is a fuzzy hyper BCK-ideal so

$$\nu(y) \geq \min(\nu(z), \inf(\nu(y \circ z))) \geq \min(\nu(z), \nu(x)).$$

(iv) Let $x, y \in X, x \circ y \ll \nu_\alpha$ and $y \in \nu_\alpha$. Then $\nu(y) \geq \alpha$ and by Theorem 4.4, $\inf(\nu(x \circ y)) \geq \alpha$. Hence

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) \geq \min(\alpha, \alpha) = \alpha.$$

Therefore, $x \in \nu_\alpha$ and so ν_α is a hyper BCK-ideal. □

Corollary 4.7. *Let $(X, \circ, 0)$ be a hyper BCK-algebra and ν be a fuzzy hyper BCK-ideal on X . If ν is an increasing map, then ν is a commutative fuzzy hyper BCK-ideal.*

Proof. Let $x \in X$. Then by Proposition 2.12, $\nu(x) \leq \nu(0)$. Since $0 \ll x$ and ν is increasing, we get that $\nu(0) \leq \nu(x)$. Hence $\nu(x) = \nu(0)$ and so for any $x, y \in X, \sup(\nu(x \circ y)) = \sup(\nu(y \circ x)) = \nu(0)$. □

Theorem 4.8. *Let $(X, \circ, 0)$ be a hyper BCK-algebra and $f : X \rightarrow X$ be an onto homomorphism. Then the fuzzy set $\nu : X \rightarrow [0, 1]$, is a fuzzy hyper BCK-ideal on X if and only if $\nu_f : X \rightarrow [0, 1]$ which is defined by $\nu_f(x) = \nu(f(x))$ is a fuzzy hyper BCK-ideal on X .*

Proof. Let ν be a fuzzy hyper BCK-ideal on X and $x \in X$. Then,

$$\nu_f(0) = \nu(f(0)) = \nu(0) \geq \nu(f(x)) = \nu_f(x)$$

and for any $x, y \in X$,

$$\begin{aligned} \nu_f(y) = \nu(f(y)) &\geq \min(\nu(f(x)), \inf(\nu(f(y) \circ f(x)))) \\ &= \min(\nu(f(x)), \inf(\nu(f(y \circ x)))) \\ &= \min(\nu_f(x), \inf(\nu_f(y \circ x))) \end{aligned}$$

Hence ν_f is a fuzzy hyper BCK-ideal on X .

Conversely, assume that ν_f is a fuzzy hyper BCK-ideal on X and $y \in X$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Then

$$\nu(0) = \nu(f(0)) = \nu_f(0) \geq \nu_f(x) = \nu(y)$$

Let $x, y \in X$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we get that

$$\begin{aligned} \nu(y) = \nu(f(b)) &= \nu_f(b) \\ &\geq \min(\nu_f(a), \inf(\nu_f(b \circ a))) \\ &= \min(\nu(f(a)), \inf(\nu(f(b \circ a)))) \\ &= \min(\nu(f(a)), \inf(\nu(f(b) \circ f(a)))) \\ &= \min(\nu(x), \inf(\nu(y \circ x))) \end{aligned}$$

Therefore ν is a fuzzy hyper BCK-ideal on X . □

Proposition 4.9. Let $(X, \circ, 0)$ be a hyper BCK-algebra, $\nu : X \rightarrow [0, 1]$ be a fuzzy hyper BCK-ideal on X and $f : X \rightarrow X$ be a homomorphism,

- (i) if $x \in \ker(f)$, then for any $y \in X$, $\nu_f(x) \geq \nu(y)$.
- (ii) if ν is one to one, then $\ker(f)$ is a hyper BCK-ideal.
- (iii) if there exists $x \in X$ such that $\nu(x) = 1$, then $\nu_1 = \{x \in X \mid \nu(x) = 1\}$ is a hyper BCK-ideal in X .
- (iv) ν_0 is a fuzzy hyper BCK-ideal in X .

Proof. (i) Let $x \in \ker(f)$. Then, $\nu_f(x) = \nu(f(x)) = \nu(0)$ and so for any $y \in X$, $\nu_f(x) \geq \nu(y)$.

(ii) Clearly $0 \in \ker(f)$. Let $y \in \ker(f)$ and $x \circ y \ll \ker(f)$, where $x, y \in X$. Then $\nu_f(y) = \nu(0)$, $\inf(\nu_f(x \circ y)) = \nu(0)$ and so

$$\nu_f(x) \geq \min(\nu_f(y), \inf(\nu_f(x \circ y))) = \min(\nu(0), \nu(0)) = \nu(0).$$

Hence $\nu_f(x) = \nu(0)$ and then $x \in \ker(f)$.

(iii) Since there exists $x \in X$ such that $\nu(x) = 1$, we get that $1 = \nu(x) \leq \nu(0)$. Hence $\nu(0) = 1$ and so $0 \in \nu_1$. Now, let $y \in \nu_1$ and $x \circ y \ll \nu_1$, where $x, y \in X$. Then, $\nu(y) = 1$, $\inf(\nu(x \circ y)) = 1$ and so

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) = \min(1, 1) = 1.$$

Hence $\nu(x) = 1$ and $x \in \nu_1$.

(iv) Since $\nu_0 = X$, then the proof is clear. □

Theorem 4.10. Let $(X, \circ, 0)$ be a hyper BCK-algebra, I be a hyper BCK-ideal and ν, ν' be fuzzy hyper BCK-ideals on X . Then

- (i) $X_\nu = \{x \in X \mid \nu(x) = \nu(0)\}$ is a hyper BCK-ideal in X ;
- (ii) if $\nu'(0) = \nu(0)$, then $X_{\nu'} \circ X_\nu = \bigcup_{a' \in X_{\nu'}, a \in X_\nu} (a' \circ a)$ is a hyper BCK-ideal;
- (iii) X_ν is a hyper BCK-ideal of $X_\nu \circ X_{\nu'}$;
- (iv) if ν is restricted to I , then ν is a fuzzy hyper BCK-ideal on I .

Proof. (i) Let $x, y \in X$ such that $x \circ y \ll X_\nu$ and $y \in X_\nu$. Then $\nu(y) = \nu(0)$, $\inf(\nu(x \circ y)) = \nu(0)$ and

$$\nu(x) \geq \min\{\nu(y), \inf(\nu(x \circ y))\} = \nu(0)$$

and so $\nu(x) = \nu(0)$. Hence $x \in X_\nu$ and X_ν is a hyper BCK-ideal.

(ii) Clearly $0 \in X_{\nu'} \circ X_\nu$. Let $t, t' \in X$ such that $t' \circ t \ll X_{\nu'} \circ X_\nu$ and $t \in X_{\nu'} \circ X_\nu$. Then there exist $a' \in X_{\nu'}$ and $a \in X_\nu$ such that $t \in a' \circ a$ so by Theorem 4.4,

$$\nu'(t) \geq \inf(\nu'(a' \circ a)) \geq \nu'(a') = \nu'(0)$$

and so

$$\nu'(t') \geq \min(\nu'(t), \inf(\nu'(t' \circ t))) \geq \min(\nu'(t), \nu'(0)).$$

Hence $t' \in X_{\nu'}$ and so $t' \in t' \circ 0 \subseteq X_{\nu'} \circ X_\nu$. Therefore $X_{\nu'} \circ X_\nu$ is a hyper BCK-ideal in X .

(iii) Let $x \in X_\nu$. Since $x \in x \circ 0$, we get that $x \in X_\nu \subseteq X_\nu \circ X_{\nu'}$ and by (i), X_ν is a hyper BCK-ideal of $X_\nu \circ X_{\nu'}$.

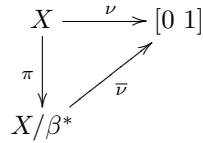
(iv) The proof is clear. □

Theorem 4.11. *Let $(X, \circ, 0)$ be a hyper BCK-algebra. If $\nu : X \rightarrow [0, 1]$ is a fuzzy hyper BCK-ideal on X , then there exists a fuzzy hyper BCK-ideal $\bar{\nu}$ on $(X/\beta^*, *, \bar{0})$ such that for any $x, y \in X$,*

- (i) $\bar{\nu}(\bar{0}) \geq \bar{\nu}(\beta^*(x))$;
- (ii) $\bar{\nu}(\beta^*(y)) \geq \min(\bar{\nu}(\beta^*(x)), \inf(\bar{\nu}(\beta^*(y) * \beta^*(x))))$.

Proof. (i) We define $\bar{\nu} : X/\beta^* \rightarrow [0, 1]$ by $\bar{\nu}(\beta^*(t)) = \bigvee_{t \beta^* x} \nu(x)$ where $x, t \in X$.

Consider the following quasi commutative diagram:



firstly we show that $\bar{\nu}$ is well-defined. Let $t, t', x \in X$ and $\beta^*(t) = \beta^*(t')$. Then $t \beta^* t'$ and

$$\bar{\nu}(\beta^*(t)) = \bigvee_{x \beta^* t} \nu(x) = \bigvee_{x \beta^* t'} \nu(x) = \bar{\nu}(\beta^*(t'))$$

then for any $x, t \in X$, we get that

$$\bar{\nu}(\beta^*(0)) = \bigvee_{t \beta^* 0} \nu(t) = \nu(0) \geq \bigvee_{t \beta^* x} \nu(t) = \bar{\nu}(\beta^*(x))$$

(ii) Let $x, y \in X$. For any $t \in \beta^*(y), t' \in \beta^*(x)$,

$$\bigvee_{t \beta^* y} \nu(t) \geq \nu(t) \geq \min(\nu(t'), \inf(\nu(t \circ t')))$$

we get that

$$\begin{aligned}
 \bar{\nu}(\beta^*(y)) &= \bigvee_{t \beta^* y} \nu(t) \\
 &\geq \bigvee_{t' \in \beta^*(x)} (\min(\nu(t'), \inf(\nu(t \circ t')))) \\
 &\geq \min(\bigvee_{t' \in \beta^*(x)} \nu(t'), \bigvee_{t' \in \beta^*(x)} \bigwedge_{t \in \beta^*(y)} (\nu(t \circ t'))) \\
 &\geq \min(\bigvee_{t' \in \beta^*(x)} \nu(t'), \bigwedge_{m \in \beta^*(y) * \beta^*(x)} \bigvee (\nu(m))) \\
 &\geq \min(\bar{\nu}(\beta^*(x)), \inf(\bar{\nu}(\beta^*(y) * \beta^*(x))))
 \end{aligned}$$

□

Corollary 4.12. *Let $(X, \circ, 0)$ be a weak commutative hyper BCK-algebra. If $\nu : X \rightarrow [0, 1]$ is a fuzzy hyper BCK-ideal on X , then there exists a fuzzy BCK-ideal ξ on $(X/\beta^*, *, \bar{0})$ such that $\xi \circ \pi \geq \nu$.*

Proof. By Theorem 4.11, we consider $\xi = \bar{\nu}$. For any $x \in X$, since $x \beta^* x$, we get that

$$(\xi \circ \pi)(x) = \xi(\beta^*(x)) = \bigvee_{t \beta^* x} \nu(t) \geq \nu(x).$$

□

Example 4.13. [1] Let $X = \{0, b, c, d\}$. Then ν is a fuzzy hyper BCK -subalgebra on hyper BCK -algebra $(X, \circ, 0)$ as follows:

\circ	0	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{0\}$	$\{0\}$
d	$\{d\}$	$\{d\}$	$\{c\}$	$\{0, c\}$

and

ν	0	b	c	d
	1	0.9	0.3	0.3

Clearly (X, \circ, ν) is not weak commutative and ν is a fuzzy hyper BCK -ideal. Now we get that $X/\beta^* = \{\beta^*(0) = \{0, c\}, \beta^*(b) = \{b\}, \beta^*(d) = \{d\}\}$,

$*$	$\beta^*(0)$	$\beta^*(b)$	$\beta^*(d)$
$\beta^*(0)$	$\beta^*(0)$	$\beta^*(0)$	$\beta^*(0)$
$\beta^*(b)$	$\beta^*(b)$	$\beta^*(0)$	$\beta^*(0)$
$\beta^*(d)$	$\beta^*(d)$	$\beta^*(d)$	$\beta^*(0)$

and

$\bar{\nu}$	$\beta^*(0)$	$\beta^*(b)$	$\beta^*(d)$
	1	0.9	0.3

It is easy to see that $(X/\beta^*, *, \beta^*(0), \bar{\nu})$ is a hyper BCK -algebra.

Definition 4.14. Let $(X, \circ, 0)$ be a hyper BCK -algebra and ν be a fuzzy hyper BCK -ideal on X . For any $x, y \in X$, we define a binary relation ρ on X by

$$x\rho y \text{ if and only if } \nu(x) \leq \nu(y) \text{ and } \inf(\nu(x \circ y)) \geq \nu(y).$$

Example 4.15. Let $X = \{0, b, c, d\}$. We consider the following table:

\circ	0	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{0, b\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, b\}$

It is easy to see that $(X, \circ, 0)$ is a hyper BCK -algebra. Define $\nu : X \rightarrow [0, 1]$ by $\nu(b) = 0, \nu(c) = 1/3, \nu(d) = 2/3, \nu(0) = 1$. Clearly, ν is a fuzzy hyper BCK -ideal and $\rho = \{(0, 0), (b, b), (c, c), (d, d)\}$.

Theorem 4.16. Let $(X, \circ, 0)$ be a hyper BCK -algebra, ν be a fuzzy hyper BCK -ideal of X and $x, y \in X$.

- (i) ρ is an equivalence relation on X .
- (ii) if ν is one to one and $x\rho y$, then for any $z \in X$ we have $(x \circ z)\rho(y \circ z)$ and $(z \circ x)\rho(z \circ y)$.
- (iii) if ν is one to one, $x\rho y$ and $u\rho w$ then $(x \circ u)\rho(y \circ w)$ for all $u, w \in X$.

Proof. (i) By Theorem 4.4, $\inf(\nu(x \circ x)) \geq \nu(x)$ and so ρ is reflexive. Let $x, y \in X$ such that $x\rho y$. Then $\nu(x) \leq \nu(y)$ and $\inf(\nu(x \circ y)) \geq \nu(y)$, since

$$\nu(x) \geq \min(\nu(y), \inf(\nu(x \circ y))) \geq \min(\nu(y), \nu(y)) = \nu(y),$$

we get that $\nu(x) = \nu(y)$. Now we have $\inf(\nu(y \circ x)) \geq \nu(y) = \nu(x)$ and so ρ is symmetric. Let $x\rho y$ and $y\rho z$. Then $\nu(x) = \nu(y) = \nu(z)$ and clearly ρ is transitive.

(ii) Let $x\rho y$ and $z \in X$. Then by (i), $\nu(x) = \nu(y)$ and since ν is one to one we have $x = y$. Hence there exist $a \in x \circ z$ and $y \in y \circ z$ such that $\nu(a) \leq \nu(b)$ and $\inf(\nu(a \circ b)) \geq \nu(b)$. Therefore $(x \circ z)\rho(y \circ z)$ and in a similar way we get that $(z \circ x)\rho(z \circ y)$.

(iii) Let $x\rho y$ and $u\rho w$. Then by (ii), $(x \circ u)\rho(y \circ u)$ and $(y \circ u)\rho(y \circ w)$. Using the transitivity of ρ , we get that $(x \circ u)\rho(y \circ w)$. \square

Corollary 4.17. *Let $(X, \circ, 0)$ be a hyper BCK-algebra and ν be a fuzzy hyper BCK-ideal on X and $x, y \in X$.*

- (i) *if ν is one to one, then ρ is a congruence relation on X ;*
- (ii) *$\rho(0) = X_\nu$ and if ν is one to one, then $\rho(0) = \{0\}$;*
- (iii) *if ν is one to one, then ρ is a strongly regular relation on X .*

Proof. (i) By Theorem 4.16, the proof is obvious.

(ii) Let $x \in \rho(0)$. Then by Theorem 4.16, $\nu(x) = \nu(0)$ and so $\rho(0) = X_\nu$. Since ν is one to one, we get that $X_\nu = \{x \mid \nu(x) = \nu(0)\} = \{x \mid x = 0\} = \{0\}$.

(iii) Let $x, y, z \in X$ and $x\rho y$. Then $x = y$ and so $x \circ z = y \circ z$. Therefore $(x \circ z)\bar{\rho}(y \circ z), (z \circ x)\bar{\rho}(z \circ y)$ and so ρ is a strongly regular relation. \square

Theorem 4.18. *Let $(X, \circ, 0)$ be a (weak commutative) hyper BCK-algebra and ν be a one to one fuzzy hyper BCK-ideal of X . Then, $(X/\rho, \circ', \rho(0))$ is a (BCK-algebra) hyper BCK-algebra such that for any $x, y \in X, \rho(x) \circ' \rho(y) = \rho(x \circ y)$.*

Proof. By Corollary 4.17, \circ' is well-defined and the proof is straightforward. \square

Example 4.19. (i) Let $(X, \circ, 0)$ be the hyper BCK-algebra that is defined in Example 4.15. Clearly, ν is one to one and we have $X/\rho = \{\rho(0), \rho(c), \rho(d)\}$ and by the following table

*	$\rho(0)$	$\rho(c)$	$\rho(d)$
$\rho(0)$	$\rho(0)$	$\rho(0)$	$\rho(0)$
$\rho(c)$	$\rho(c)$	$\rho(0)$	$\rho(c)$
$\rho(d)$	$\rho(d)$	$\rho(d)$	$\rho(0)$

it is easy to see that $(X/\rho, *, \rho(0))$ is a BCK-algebra.

(ii) Let $(X, \circ, 0)$ be the hyper BCK-algebra that is defined in Example 4.13. Clearly we obtain $\rho = \{(0, 0), (b, b), (c, c), (d, d), (c, d), (d, c)\}$. Now, ν is not one to one and we have $c \rho d$ but $(c \circ c, d \circ c) \notin \rho$. Hence ρ is not a congruence relation.

Theorem 4.20. *Let $(X, \circ_1, 0)$ and $(Y, \circ_2, 0')$ be (weak commutative) hyper BCK-algebras and ν be a one to one fuzzy hyper BCK-ideal of Y . If $f : X \rightarrow Y$ is an epimorphism, then*

- (i) *ν_f is a fuzzy hyper BCK-ideal of X ;*
- (ii) *$X/\rho_f \cong Y/\rho$ such that $x\rho_f x' \iff \nu(f(x)) \leq \nu(f(x'))$ and $\inf(\nu(f(x \circ_1 x'))) \geq \nu(f(x'))$, where $x, x' \in X$.*

Proof. (i) Clearly for any $x \in X, \nu_f(0') = \nu(f(0)) \geq \nu(f(x)) = \nu_f(x)$. Let $x, x' \in X$. Since ν is a fuzzy hyper BCK-ideal of Y , we get that

$$\begin{aligned}
\nu_f(x) = \nu(f(x)) &\geq \min\{\nu(f(x')), \inf(\nu(f(x) \circ_2 \nu(f(x'))))\} \\
&= \min\{\nu(f(x')), \inf(\nu(f(x \circ_1 x')))\} \\
&= \min\{\nu_f(x'), \inf(\nu_f(x \circ_1 x'))\}.
\end{aligned}$$

(ii) Since ν and ν_f are fuzzy hyper BCK -ideals on X , then by Theorem 4.18, $(X/\rho_f, \circ', \rho_f(0))$ and $(Y/\rho, \circ', \rho(0'))$ are (BCK -algebras) hyper BCK -algebras. Now, define a map $\varphi : X/\rho_f \rightarrow Y/\rho$ by $\varphi(\rho_f(x)) = \rho(f(x))$. Let $x, x' \in X$. Then

$$\begin{aligned}
\varphi(\rho_f(x)) &= \varphi(\rho_f(x')) \\
\iff f(x)\rho_f(x') & \\
\iff \nu(f(x)) \leq \nu(f(x')) \text{ and } \inf(\nu(f(x) \circ_2 f(x'))) \geq \nu(f(x')) & \\
\iff \nu_f(x) \leq \nu_f(x') \text{ and } \inf(\nu(f(x \circ_1 x'))) \geq \nu(f(x')) & \\
\iff \nu_f(x) \leq \nu_f(x') \text{ and } \inf(\nu_f(x \circ_1 x')) \geq \nu_f(x') & \\
\iff \rho_f(x) = \rho_f(x'). &
\end{aligned}$$

Hence φ is well-defined and one to one. Clearly φ is an epimorphism, and so it is an isomorphism. \square

Corollary 4.21. (*Isomorphism Theorem*) Let $(X, \circ, 0)$ be a hyper BCK -algebra and ν, ν' be one to one fuzzy hyper BCK -ideals on X such that $\nu(0) = \nu'(0)$. Then

- (i) $\nu' \cap \nu$ is a fuzzy hyper BCK -ideal of X ;
- (ii) $(X_\nu \circ X_{\nu'})/\rho_\nu \cong X_\nu/\rho_{\nu' \cap \nu}$.

Proof. (i) Let $x \in X$. Then

$$(\nu' \cap \nu)(0) = \min(\nu'(0), \nu(0)) \geq \min(\nu'(x), \nu(x)) = (\nu' \cap \nu)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}
(\nu' \cap \nu)(x) &= \min(\nu'(x), \nu(x)) \\
&\geq \min[\min[\nu'(y), \inf(\nu'(x \circ y))], \min[\nu(y), \inf(\nu(x \circ y))]] \\
&= \min[\min[\nu'(y), \nu(y)], \min[\inf(\nu'(x \circ y)), \inf(\nu(x \circ y))]] \\
&= \min[(\nu' \cap \nu)(y), \inf((\nu' \cap \nu)(x \circ y))]
\end{aligned}$$

(ii) By Theorem 4.10, $\nu' \cap \nu$ is a fuzzy hyper BCK -ideal of X_ν , then we define $\varphi : X_\nu/\rho_{\nu' \cap \nu} \rightarrow (X_\nu \circ X_{\nu'})/\rho_\nu$ by $\varphi(\rho_{\nu' \cap \nu}(x)) = \rho_\nu(x)$. Let $x, x' \in X_\nu$ and $\rho_{\nu' \cap \nu}(x) = \rho_{\nu' \cap \nu}(x')$. Then $\nu' \cap \nu(x) = \nu' \cap \nu(x')$ and since $\nu' \cap \nu$ is one to one, we get that $x = x'$. Hence $\rho_\nu(x) = \rho_\nu(x')$. Moreover,

$$\varphi(\rho_{\nu' \cap \nu}(x) \circ' \rho_{\nu' \cap \nu}(x')) = \varphi(\rho_{\nu' \cap \nu}(x \circ x')) = \rho_\nu(x \circ x') = \rho_\nu(x) \circ' \rho_\nu(x')$$

and so φ is a homomorphism. Clearly φ is bijection and so is an isomorphism. \square

5. Conclusion

To conclude, the current paper has considered the notion of fuzzy hyper BCK -subalgebras and investigated some of its new useful properties. We defined the concept of the extended fuzzy BCK -subalgebras, applied the concept of fuzzy hyper BCK -ideal and showed that:

- (i) For any $\alpha \in [0, 1]$ and a fuzzy set μ of a fuzzy hyper BCK -subalgebra, then μ is a fuzzy hyper BCK -subalgebra if and only if μ_α is a hyper BCK -subalgebra.
- (ii) By using the concept of fuzzy hyper BCK -ideal in any hyper BCK -algebra, we obtain some hyper BCK -ideals.
- (iii) Through the concept of fundamental relation β^* , we have generated the corresponding structure in hyperstructures such as fuzzy BCK -subalgebras of fuzzy hyper BCK -subalgebras and fuzzy BCK -ideals of fuzzy hyper BCK -ideals.
- (iv) With respect to the concept of hyper BCK -ideals on any hyper BCK -algebra, we introduced an equivalence relation such that it is a congruence and strongly regular relation; then, could we obtain a quotient BCK -algebra.
- (v) On any nonempty set, we have constructed an extendable fuzzy BCK -subalgebra and have showed that on any infinite set there exists an extended fuzzy BCK -subalgebra.

We hope that these results are helpful to further studies in fuzzy structures. In our future studies, we hope to obtain more results regarding fuzzy hyper BCK -subalgebras, fuzzy BCK -subalgebras, and their applications.

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