NEW APPROACH TO EXPONENTIAL STABILITY ANALYSIS AND STABILIZATION FOR DELAYED T-S FUZZY MARKOVIAN JUMP SYSTEMS

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Abstract. This paper is concerned with delay-dependent exponential stability analysis and stabilization for continuous-time T-S fuzzy Markovian jump systems with mode-dependent time-varying delay. By constructing a novel Lyapunov-Krasovskii functional and utilizing some advanced techniques, less conservative conditions are presented to guarantee the closed-loop system is mean-square exponentially stable. Then, the stabilization conditions are derived and the fuzzy controller can be obtained by solving a set solutions of LMIs. The upper bound of time-delay that the system can be stabilized is given by using an optimal algorithm. Two examples are presented to illustrate the effectiveness and potential of our methods.

1. Introduction

As a special kind of hybrid systems, Markovian jump systems have been received much attention in the past years due to the fact that they are more powerful and appropriate to model varieties of systems mainly those with abrupt changes in their structure, see e.g. [11, 14, 20, 21, 23, 24, 25, 26], and the references therein. On the other hand, time-delays are frequently encountered in practical systems such as engineering and biological systems. Their existence may cause oscillation, divergence, or even instability of the system. Hence, many results on stability and control for time delay Markovian jump systems have reported in the literature, see e.g. [5, 7, 8, 9, 15, 18].

However, there have been few results reported on Markovian jump nonlinear systems because of the difficulties inherent in the analysis of nonlinear dynamics. Recently, the T-S fuzzy-model-based control technique has been proved to be an efficient method to deal with the complex nonlinear systems. The results related to T-S fuzzy systems with or without time delay can be found in [1, 3, 4, 10, 13, 17, 22, 27, 31], and the references therein. In the literature to date, some attention has been devoted to fuzzy Markovian jump systems (FMJS). For example, references [6, 12, 19] investigated the robust stability and stabilization for fuzzy Markovian jump systems in terms of LMI approach. When time delay appears, [29] studied the continuous-time Markovian jump systems with constant time delay. The problem of output feedback stabilization for discrete-time Markovian jump systems

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with time-varying delay was investigated [30]. [2] studied the stability and control for a class of nonlinear uncertain stochastic fuzzy systems with mode-dependent time delay and Markovian jump parameters. However, although the decay rate in [2, 6, 12, 19, 29, 30] can be computed, it is a fixed value that one cannot adjust to deduce if a larger decay rate is possible. In addition, it should be pointed out that the problems of stability analysis and stabilization for time-delay fuzzy Markovian jump systems have not been adequately investigated and there is still room for improvement in the handing of the delay term.

In this paper, we focus on the exponential stability and stabilization conditions for continuous-time T-S fuzzy Markovian jump systems with mode-dependent interval time-varying delays. Motivated by the discretized Lyapunov functional method proposed [9], a novel Lyapunov-Krasovskii functional is introduced by decomposing the lower bound of the time delay interval into multiple equidistant subintervals. By using the new functional and introducing some slack matrix variables, sufficient conditions are derived to guarantee the close-loop system is mean-square exponentially stable by a prescribed value. The state feedback fuzzy controller is then obtained by solving a set solutions of LMIs and the upper bound of time delay can be estimated by using an optimal algorithm. Two illustrative examples are presented to show the effectiveness of proposed results.

Notation: $\mathbf{R}^n$ represents the n-dimensional Euclidean space. For real symmetric matrices X and Y, the national $X \geq Y$ and $X > Y$ mean that the matrix $X - Y$ is positive-semidefinite and positive-definite, respectively. The superscript "T" represents the transpose. * is used as an ellipsis for terms that are induced by symmetry. $L_2\,[0, \infty)$ is the space of square-integrable vector functions over $[0, \infty)$. $\|\cdot\|$ stands for the Euclidean norm for matrices. $\mathbf{E}\{\cdot\}$ stands for the expectation operator with respect to some probability measure. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

2. **Problem Formulation**

Consider the following continuous-time T-S fuzzy Markovian jump systems with interval mode-dependent time delay:

Plant Rule i: IF $\theta_1(t)$ is M_{i1} and $\theta_2(t)$ is M_{i2} and $\cdots$ and $\theta_p(t)$ is M_{ip}, THEN

$$\begin{cases} \dot{x}(t) = A_i(r_t)x(t) + A_{\tau i}(r_t)x\left(t - \tau(t, r_t)\right) + B_i(r_t)u(t) \\ x(t) = \varphi(t), t \in [-\tau_2, 0] \end{cases} \tag{1}$$

where $i \in T = \{1, 2, \ldots r\}$ and r is the number of IF-THEN rules; M_{ij} is the fuzzy set; $x(t) \in \mathbf{R}^n$ is the state vector; $\theta_1(t)$, $\theta_2(t)$, $\ldots$, $\theta_p(t)$ are premise variables; $x(t) \in \mathbf{R}^n$ is the state vector; $u(t) \in \mathbf{R}^m$ is the input vector; $w(t) \in \mathbf{R}^p$ is the exogenous disturbance signal in $L_2\,[0, \infty)$; $\varphi(t)$ is the continuously differential initial condition; Let $\{r_t, t \geq 0\}$ be a continuous-time Markov process with a right continuous trajectory taking values in a finite set $\mathbf{S} = \{1, 2, \ldots, s\}$ with transition probability matrix $\Lambda = \{\pi_{kl}\}$ given by

$$\mathbf{P}\left[r_{t+\Delta t} = l | r_t = k\right] = \begin{cases} \pi_{kl}\Delta + o\left(\Delta\right), \ \text{if } l \neq k \\ 1 + \pi_{kl}\Delta + o\left(\Delta\right), \ \text{if } l = k \end{cases}$$

where $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$; $\pi_{kl} > 0,\ 1 \neq k$ and $\pi_{kk} = -\sum\limits_{l\neq k} \pi_{kl}$ for each $k \in \mathbf{S}$. $A_i(r_t)$, $A_{\tau i}(r_t)$ and $B_i(r_t)$ are known real-valued matrix functions of r_t.

For notational simplicity, in the sequel, for each possible $r_t = k,\ k \in \mathbf{S}$, a matix $M(r_t)$ will be denoted by M_k; for example, $\tau(t, r_t)$ is denoted by $\tau_k(t)$, $A_i(r_t)$ is denoted by $A_{i,k}$, $B_i(r_t)$ is denoted by $B_{i,k}$, and so on.

In this system, the mode-dependent time-varying delay satisfies the following condition:

$$0 < \tau_{1k} \leq \tau_k(t) \leq \tau_{2k} < \infty,\ \dot{\tau}_k(t) \leq \mu_k,\ \forall r_t = k,\ k \in \mathbf{S} \tag{2}$$

where $\tau_{1k},\ \tau_{2k}$ are lower and upper bounds of $\tau_k(t)$ and μ_k is the time derivative of $\tau_k(t)$ for each $k \in \mathbf{S}$. Define $\tau_1 = \min\{\tau_{1k},\ k \in \mathbf{S}\},\ \tau_2 = \max\{\tau_{2k},\ k \in \mathbf{S}\},\ \mu = \max\{\mu_k, k \in \mathbf{S}\}$.

Remark 2.1. : The interval mode-dependent time-varying delay is introduced into the model in our paper, which is more natural and general than the mode-independent constant/time-varying ones in T-S fuzzy Markovian jump systems [1, 3, 4, 12, 19]. [6] studied the fuzzy systems with mode-dependent time delay and Markovian jump parameters. However, it will turn to be inapplicable when the delay changes rapidly because of the constraint $u_k < 1,\ k \in \mathbf{S}$. Hence, our results are more general than the existing results about T-S fuzzy delayed Markovian jump systems.

Given a pair $(x(t),\ u(t))$, we obtain the following final output of the fuzzy system in (1) for any $k \in \mathbf{S}$:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} h_i(\theta(t)) \{A_{i,k}x(t) + A_{\tau i,k}x(t - \tau_k(t)) + B_{i,k}u(t)\} \\ x(t) = \varphi(t),\ t \in [-\tau_2, 0] \end{cases} \tag{3}$$

where $h_i(\theta(t)) = v_i(\theta(t)) / \sum_{i=1}^{r} v_i(\theta(t))$, $v_i(\theta(t)) = \prod\limits_{j=1}^{p} M_{ij}(\theta_j(t))$ with $M_{ij}(\theta_j(t))$ representing the grade of membership of $\theta_j(t)$ in M_{ij}. Then, it can be seen that, for $k \in \mathbf{S}$ and all t, $\sum_{i=1}^{r} h_i(\theta(t)) = 1$, and $h_i(\theta(t)) \geq 0$.

The parallel distributed compensation strategy is utilized and the fuzzy state-feedback controller obeys the following rules:

Controller rule: IF $\theta_1(t)$ is M_{i1} and $\theta_2(t)$ is M_{i2} and $\cdots$ and is $\theta_p(t)$ is M_{ip}, THEN

$$u(t) = K_{i,k}x(t),\ i \in T,\ k \in \mathbf{S} \tag{4}$$

where $K_{i,k},\ i \in T,\ k \in \mathbf{S}$ are matrices to be determined. Thus, the overall state-feedback fuzzy controller is given by

$$u(t) = \sum_{i=1}^{r} h_i(\theta(t)) K_{i,k}x(t),\ i \in T,\ k \in \mathbf{S} \tag{5}$$

Combining (1) and (5), the closed-loop system is obtained as follows:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r} h_i(\theta(t)) h_j(\theta(t)) \{A_{ui,k}x(t) + A_{\tau i,k}x(t - \tau_k(t))\} \\ x(t) = \varphi(t),\ t \in [-\tau_2, 0] \end{cases} \tag{6}$$

where $A_{ui,k} = A_{i,k} + B_{i,k}K_{j,k}$.

Throughout this paper, we shall use the following concept.

Definition 2.2. The T-S fuzzy Markovian jump system (6) is said to be mean-square exponentially stable for any finite $\varphi(t) \in \mathbf{R}^n$ and initial mode $r_0 \in S$, if there exist constants $\sigma > 0$ and $\lambda > 0$ such that

$$\mathbf{E}\|x(t)\|^2 \leq \sigma e^{-\lambda t} \|\varphi(t)\|^2_{\tau_2} \tag{7}$$

where σ and λ are called the decay coefficient and decay rate, respectively.

3. **Delay-Dependent Stability Analysis**

In this section, some conditions are proposed in terms of LMI, which guarantee system (6) is mean-square exponentially stable.

Theorem 3.1. *Given a decay rate $\lambda > 0$, for any $\tau_k(t)$ satisfying (2), system (6) is mean square exponentially stable, if there exist matrices $P_k > 0$, $Q_{1mk} > 0$, $P_k > 0$, $Q_{1mk} > 0$, $Q_{2k} > 0$, $Q_{3k} > 0$, $R_{1m} > 0$, $R_2 > 0$, $Z_{1m} > 0$, $Z_2 > 0$, $m = 1, 2, 3, \cdots N$ such that the following LMIs hold:*

$$\Theta_{ii,k} < 0, \ i \in T, \ k \in \mathbf{S} \tag{8}$$

$$\Theta_{ij,k} + \Theta_{ji,k} < 0, \ 1 \leq i < j \leq r, \ k \in \mathbf{S} \tag{9}$$

$$e^{\lambda m\delta} \sum_{l=1}^{s} \pi_{kl} Q_{1ml} \leq Z_{1m}, \ m = 1, 2, 3, \cdots, N \tag{10}$$

$$\sum_{l=1, l\neq k}^{s} \pi_{kl} Q_{2l} + e^{\lambda\tau_2} \sum_{l=1}^{s} \pi_{kl} Q_{3l} \leq Z_2 \tag{11}$$

where

$$\Theta_{ij,k} =$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \cdots & A^T_{ui,k}M^T_{N,k} & A^T_{ui,k}M^T_{(N+1),k} & A^T_{ui,k}M^T_{(N+2),k} & \Xi_{1(N+3)} & \Xi_{1(N+4)} \\ * & \Xi_{22} & \cdots & 0 & 0 & 0 & M_{2,k}A_{\tau i,k} & -M_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \Xi_{NN} & R_{1N} & 0 & M_{N,k}A_{\tau i,k} & -M_{N,k} \\ * & * & \cdots & * & \Xi_{(N+1)(N+1)} & 0 & \Xi_{(N+1)(N+3)} & -M_{(N+1),k} \\ * & * & \cdots & * & * & \Xi_{(N+2)(N+2)} & \Xi_{(N+2)(N+3)} & -M_{(N+2),k} \\ * & * & \cdots & * & * & * & \Xi_{(N+3)(N+3)} & \Xi_{(N+3)(N+4)} \\ * & * & \cdots & * & * & * & * & \Xi_{(N+4)(N+4)} \end{bmatrix}$$

$$\Xi_{11} = \lambda P_k + \sum_{l=1}^{s} \pi_{kl} P_l + M_{1,k}A_{ui,k} + A^T_{ui,k}M^T_{1,k} + e^{\lambda\delta} Q_{11k}$$

$$+ \sum_{m=1}^{N} Z_{1m} \frac{e^{\lambda m\delta} - e^{\lambda(m-1)\delta}}{\lambda} + Z_2 \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda} - R_{11}$$

$$\Xi_{12} = R_{11} + A^T_{ui,k}M^T_{2,k}$$

$$\Xi_{1(N+3)} = M_{1,k}A_{\tau i,k} + A_{ui,k}^T M_{(N+3),k}^T$$

$$\Xi_{1(N+4)} = P_k^T - M_{1,k} + A_{ui,k}^T M_{(N+4),k}^T$$

$$\Xi_{22} = -Q_{11k} + e^{\lambda\delta}Q_{12k} - R_{11} - R_{12}$$

$$\Xi_{NN} = -Q_{1(N-1)k} + e^{\lambda\delta}Q_{1Nk} - R_{1(N-1)} - R_{1N}$$

$$\Xi_{(N+1)(N+1)} = -Q_{1Nk} + e^{\lambda(\tau_2-\tau_1)}Q_{2k} + e^{\lambda(\tau_2-\tau_1)}Q_{3k} - R_{1N} - R_2$$

$$\Xi_{(N+1)(N+3)} = R_2 + M_{(N+1),k}A_{\tau i,k}$$

$$\Xi_{(N+2)(N+2)} = -Q_{3k} - R_2$$

$$\Xi_{(N+2)(N+3)} = R_2 + M_{(N+2),k}A_{\tau i,k}$$

$$\Xi_{(N+3)(N+3)} = -Q_{2k} + \mu_k e^{\lambda(\tau_2-\tau_1)}Q_{2k} - 2R_2 + M_{(N+3),k}A_{\tau i,k} + A_{\tau i,k}^T M_{(N+3),k}^T$$

$$\Xi_{(N+3)(N+4)} = -M_{(N+3),k} + A_{\tau i,k}^T M_{(N+4),k}^T$$

$$\Xi_{(N+4)(N+4)} = \delta\sum_{m=1}^{N} R_{1m}\frac{e^{\lambda m\delta} - e^{\lambda(m-1)\delta}}{\lambda} + (\tau_2-\tau_1)R_2\frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda} - M_{(N+4),k} - M_{(N+4),k}^T$$

Proof. Define a new process $\{(x_t, r_t),\ t \geq 0\}$ by $x_t(s) = x(t+s),\ -\tau_2 \leq s \leq 0$, then $\{(x_t, r_t),\ t \geq 0\}$ is a Markov process with initial state $(\varphi(\cdot),\ r_0)$. Now choose the stochastic Lyapunov-Krasovskii functional candidate for system (6):

$$V(x_t, r_t, t) = \sum_{j=1}^{4} V_j(x_t, r_t, t) \tag{12}$$

where

$$V_1(x_t, r_t, t) = e^{\lambda t}x^T(t)P(r_t)x(t)$$

$$\begin{aligned} V_2(x_t, r_t, t) = &\sum_{m=1}^{N}\int_{t-m\delta}^{t-(m-1)\delta} e^{\lambda(\alpha+m\delta)}x^T(\alpha)Q_{1m}(r_t)x(\alpha)d\alpha \\ &+ \int_{t-\tau_k(t)}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)}x^T(\alpha)Q_2(r_t)x(\alpha)d\alpha \\ &+ \int_{t-\tau_2}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)}x^T(\alpha)Q_3(r_t)x(\alpha)d\alpha \end{aligned}$$

$$\begin{aligned} V_3(x_t, r_t, t) = \delta &\sum_{m=1}^{N}\int_{-m\delta}^{-(m-1)\delta}\int_{t+\beta}^{t} e^{\lambda(\alpha-\beta)}\dot{x}^T(\alpha)R_{1m}\dot{x}(\alpha)d\alpha d\beta \\ &+ (\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} e^{\lambda(\alpha-\beta)}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)d\alpha d\beta \end{aligned}$$

$$V_4(x_t, r_t, t) = \sum_{m=1}^{N} \int_{-m\delta}^{-(m-1)\delta} \int_{t+\beta}^{t} e^{\lambda(\alpha-\beta)} x^T(\alpha) Z_{1m} x(\alpha) d\alpha d\beta + \int_{-\tau_2}^{-\tau_1} \int_{t+\beta}^{t} e^{\lambda(\alpha-\beta)} x^T(\alpha) Z_2 x(\alpha) d\alpha d\beta$$

$\delta = \tau_1/N$ and N is the number of divisions of the interval $[-\tau_1,\ 0]$.

The weak infinitesimal operator L is defined as

$$LV(x_t, r_t, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \mathbf{E}\left[V\left(x(t+\Delta),\ r(t+\Delta),\ t+\Delta\right)\right] - V(x_t, r_t, t) \right\} \tag{13}$$

Then we have

$$LV_1(x_t, r_t, t) = \lambda e^{\lambda t} x^T(t) P_k x(t) + 2e^{\lambda t} x^T(t) P_k \dot{x}(t) + e^{\lambda t} x^T(t) \sum_{l=1}^{s} \pi_{kl} P_l x(t)$$

$$\begin{aligned} LV_2(x_t, r_t, t) \leq\ & e^{\lambda(t+\delta)} \sum_{m=1}^{N} x^T\left(t-(m-1)\delta\right) Q_{1mk} x\left(t-(m-1)\delta\right) \\ & - e^{\lambda t} \sum_{m=1}^{N} x^T(t-m\delta) Q_{1mk} x(t-m\delta) \\ & + \sum_{m=1}^{N} \int_{t-m\delta}^{t-(m-1)\delta} e^{\lambda(\alpha+m\delta)} x^T(\alpha) \left(\sum_{l=1}^{s} \pi_{kl} Q_{1ml} \right) x(\alpha) d\alpha \\ & + e^{\lambda(t+\tau_2-\tau_1)} x^T(t-\tau_1) Q_{2k} x(t-\tau_1) \\ & - e^{\lambda t} x^T\left(t-\tau_k(t)\right) Q_{2k} x\left(t-\tau_k(t)\right) \\ & + \mu_k e^{\lambda(t+\tau_2-\tau_1)} x^T\left(t-\tau_k(t)\right) Q_{2k} x\left(t-\tau_k(t)\right) \\ & + \sum_{l=1}^{s} \pi_{kl} \int_{t-\tau_l(t)}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)} x^T(\alpha) Q_{2l} x(\alpha) d\alpha \\ & + e^{\lambda(t+\tau_2-\tau_1)} x^T(t-\tau_1) Q_{3k} x(t-\tau_1) \\ & - e^{\lambda t} x^T(t-\tau_2) Q_{3k} x(t-\tau_2) \\ & + \int_{t-\tau_2}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)} x^T(\alpha) \left(\sum_{l=1}^{s} \pi_{kl} Q_{3l} \right) x(\alpha) d\alpha \end{aligned}$$

$$\begin{aligned} LV_3(x_t, r_t, t) =\ & \delta e^{\lambda t} \sum_{m=1}^{N} \dot{x}^T(t) R_{1m} \dot{x}(t) \frac{e^{\lambda m\delta} - e^{\lambda(m-1)\delta}}{\lambda} \\ & - \delta e^{\lambda t} \sum_{m=1}^{N} \int_{t-m\delta}^{t-(m-1)\delta} \dot{x}^T(\alpha) R_{1m} \dot{x}(\alpha) d\alpha \\ & + (\tau_2-\tau_1) e^{\lambda t} \dot{x}^T(t) R_2 \dot{x}(t) \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda} \\ & - (\tau_2-\tau_1) e^{\lambda t} \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(\alpha) R_2 \dot{x}(\alpha) d\alpha \end{aligned}$$

$$
\begin{aligned}
LV_4(x_t,r_t,t) = {} & e^{\lambda t}\sum_{m=1}^{N} x^T(t)Z_{1m}x(t)\frac{e^{\lambda m\delta}-e^{\lambda(m-1)\delta}}{\lambda} \\
& - e^{\lambda t}\sum_{m=1}^{N}\int_{t-m\delta}^{t-(m-1)\delta} x^T(\alpha)Z_{1m}x(\alpha)d\alpha \\
& + e^{\lambda t}x^T(t)Z_2x(t)\frac{e^{\lambda\tau_2}-e^{\lambda\tau_1}}{\lambda} \\
& - e^{\lambda t}\int_{t-\tau_2}^{t-\tau_1} x^T(\alpha)Z_2x(\alpha)d\alpha
\end{aligned}
$$

By Jensen' s inequality, we have

$$
\begin{aligned}
& -(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)d\alpha \\
& \le -(\tau_k(t)-\tau_1)\int_{t-\tau_k(t)}^{t-\tau_1}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)d\alpha-(\tau_2-\tau_k(t))\int_{t-\tau_2}^{t-\tau_k(t)}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)d\alpha \\
& \le \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \\ x\left(t-\tau_k(t)\right)\end{bmatrix}^T \begin{bmatrix} -R_2 & 0 & R_2 \\ * & -R_2 & R_2 \\ * & * & -2R_2\end{bmatrix}\begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \\ x\left(t-\tau_k(t)\right)\end{bmatrix}
\end{aligned} \tag{14}
$$

Similarly, we also get

$$
\begin{aligned}
& -\delta\int_{t-m\delta}^{t-(m-1)\delta}\dot{x}^T(\alpha)R_{1m}\dot{x}(\alpha)d\alpha \\
& \le \begin{bmatrix} x\left(t-(m-1)\delta\right) \\ x(t-m\delta)\end{bmatrix}^T\begin{bmatrix} -R_{1m} & R_{1m} \\ * & -R_{1m}\end{bmatrix}\begin{bmatrix} x\left(t-(m-1)\delta\right) \\ x(t-m\delta)\end{bmatrix}
\end{aligned} \tag{15}
$$

By Newton-Leibniz formula, for any appropriately dimensioned matrices $M_{1,k}$, ... , $M_{(N+4),k}$, $k \in \mathbf{S}$, we obtain

$$
\begin{aligned}
& 2\left[x^T(t)M_{1,k}+x^T(t-\delta)M_{2,k}+\cdots+x^T(t-N\delta)M_{(N+1),k}\right. \\
& \left.+x^T(t-\tau_2)M_{(N+2),k}+x^T\left(t-\tau_k(t)\right)M_{(N+3),k}+\dot{x}^T(t)M_{(N+4),k}\right] \\
& \times\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(\theta(t)\right)h_j\left(\theta(t)\right)\left[A_{ui,k}x(t)+A_{\tau i,k}x\left(t-\tau_k(t)\right)-\dot{x}(t)\right]=0
\end{aligned} \tag{16}
$$

Noting $\pi_{ij} > 0$ for $j \neq i$ and $\pi_{ii} < 0$, we have

$$
\begin{aligned}
& \sum_{l=1}^{s}\pi_{kl}\int_{t-\tau_l(t)}^{t-\tau_1}e^{\lambda(\alpha+\tau_2)}x^T(\alpha)Q_{2l}x(\alpha)d\alpha \\
& \le \int_{t-\tau_2}^{t-\tau_1}e^{\lambda(\alpha+\tau_2)}x^T(\alpha)\left(\sum_{l=1,l\neq k}^{s}\pi_{kl}Q_{2l}\right)x(\alpha)d\alpha
\end{aligned} \tag{17}
$$

From (13)-(17), we can obtain

$$\begin{aligned}
LV(x_t, r_t, t) \leq{}& e^{\lambda t}\zeta^T(t)\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(\theta(t)\right) h_j\left(\theta(t)\right)\Theta_{ij,k}\zeta(t) \\
&+ \sum_{m=1}^{N}\int_{t-m\delta}^{t-(m-1)\delta} e^{\lambda(\alpha+m\delta)}x^T(\alpha)\left(\sum_{l=1}^{s}\pi_{kl}Q_{1ml}\right)x(\alpha)d\alpha \\
&+ \int_{t-\tau_2}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)}x^T(\alpha)\left(\sum_{l=1,l\neq k}^{s}\pi_{kl}Q_{2l}\right)x(\alpha)d\alpha \\
&+ \int_{t-\tau_2}^{t-\tau_1} e^{\lambda(\alpha+\tau_2)}x^T(\alpha)\left(\sum_{l=1}^{s}\pi_{kl}Q_{3l}\right)x(\alpha)d\alpha \\
&- e^{\lambda t}\sum_{m=1}^{N}\int_{t-m\delta}^{t-(m-1)\delta} x^T(\alpha)Z_{1m}x(\alpha)d\alpha \\
&- e^{\lambda t}\int_{t-\tau_2}^{t-\tau_1} x^T(\alpha)Z_2x(\alpha)d\alpha
\end{aligned}$$

where

$$\begin{aligned}
\zeta(t) = {}& \begin{bmatrix} x^T(t) & x^T(t-\delta) & \cdots & x^T\left(t-(N-1)\delta\right) & x^T(t-N\delta) \end{bmatrix.} \\
& \left. x^T(t-\tau_2) \quad x^T\left(t-\tau_k(t)\right) \quad \dot{x}^T(t)\right]^T
\end{aligned}$$

It is easy to see that if (10) and (11) hold,

$$LV(x_t, r_t, t) \leq e^{\lambda t}\zeta^T(t)\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(\theta(t)\right) h_j\left(\theta(t)\right)\Theta_{ij,k}\zeta(t) \tag{18}$$

From (8)-(9) and by Schur complement lemma, we get $LV(x_t, r_t, t) < 0$.

Let $d_1 = \max_{i\in T,k\in s}\{\|A_{ui,k}\|\}$, $d_2 = \max_{i\in T,k\in s}\{\|A_{\tau i,k}\|\}$ and $d_3 = \max_{i\in T,k\in s}\{\|B_{1i,k}\|\}$, when $t > 0$, by Dynkins formula, we have

$$\mathbf{E}V(x_t, r_t, t) = \mathbf{E}V(x_0, r_0, 0) + \mathbf{E}\int_0^t LV(x_\alpha, r_\alpha, \alpha)d\alpha \leq \Lambda\|\varphi\|_{\tau_2}^2 \tag{19}$$

where

$$\begin{aligned}
\Lambda ={}& \max_{k\in s}\{\|P_k\|\} + \frac{e^{\lambda\delta}-1}{\lambda}\sum_{m=1}^{N}\left(\max_{k\in s}\{\lambda_{\max}(Q_{1mk})\}\right) \\
&+ \frac{e^{\lambda(\tau_2-\tau_1)}-1}{\lambda}\max_{k\in s}\{\|Q_{2k}\|\} + \frac{e^{\lambda(\tau_2-\tau_1)}-1}{\lambda}\max_{k\in s}\{\|Q_{3k}\|\} \\
&+ 3\delta(d_1^2+d_2^2+d_3^2d^2)\sum_{m=1}^{N}\left(\frac{e^{\lambda m\delta}(1-e^{-\lambda\delta})-\lambda\delta}{\lambda^2}\|R_{1m}\|\right) \\
&+ 3(\tau_2-\tau_1)(d_1^2+d_2^2+d_3^2d^2)\frac{e^{\lambda\tau_2}-e^{\lambda\tau_1}-\lambda(\tau_2-\tau_1)}{\lambda^2}\|R_2\| \\
&+ \sum_{m=1}^{N}\left(\frac{e^{\lambda m\delta}(1-e^{-\lambda\delta})-\lambda\delta}{\lambda^2}\|Z_{1m}\|\right) + \frac{e^{\lambda\tau_2}-e^{\lambda\tau_1}-\lambda\tau_2}{\lambda^2}\|Z_2\|
\end{aligned}$$

On the other hand

$$\mathbf{E}V(x_t, r_t, t) \geq \mathbf{E}\left\{e^{\lambda t} x^T(t) P_k x(t)\right\} \geq e^{\lambda t} \frac{1}{\max\limits_{k \in s} \left\| P_k^{-1} \right\|} \mathbf{E}\|x(t)\|^2 \tag{20}$$

Therefore, we have

$$\mathbf{E}\|x(t)\|^2 \leq \max_{k \in s} \left\| P_k^{-1} \right\| \Lambda \left\|\varphi\right\|_{\tau_2}^2 e^{-\lambda t} \tag{21}$$

From Definition 2.2, system (6) is mean-square exponentially stable with the decay rate . This completes the proof. □

Remark 3.2. Theorem 3.1 provides some new stability conditions for time delay T-S fuzzy Markovian jump systems. Unlike the existing results in [2, 6, 12, 19, 29, 30], the decay rate in Theorem 3.1 is a free value. It equals to a prescribed constant that can be selected according to different practical conditions, which is benefited from the LKF in (12). This will introduce more flexibility in the analysis of systems. In addition, compared with the LKF proposed in [2, 6, 12, 19, 29, 30], the LKF constructed in (12) is based on the idea of delay partition proposed in [9], which is expected to yield less conservative delay-dependent stability criterion.

Remark 3.3. By decomposing the delay interval $[-\tau_1,\ 0]$ into N equidistant subintervals, the functional in (12) allows to take different weighing matrices on different subintervals. The conservatism will be reduced with N increasing, and will be illustrated via examples in the next section.

Remark 3.4. The conditions are formulated in terms of LMI, which can be easily solved by standard software. In addition, it should be pointed out that, for simplicity only, we do not consider uncertainties in our models. The proposed method can be easily extended to cases with norm-bounded uncertainties.

With the case $\tau_1 = \tau_2 = \tau$, the mode-dependent time delay $\tau_k(t)$ becomes a constant one. Taking $\lambda \to 0^+$, we obtain the following proposition.

Proposition 3.5. *For constant time delay τ, system (6) is mean square exponentially stable, if there exist matrices $P_k > 0$, $Q_{1mk} > 0$, $R_{1m} > 0$, $Z_{1m} > 0$, $m = 1, 2, 3, \cdots N$ such that the following LMIs hold:*

$$\Theta_{ii,k} < 0,\ i \in T,\ k \in \mathbf{S} \tag{22}$$

$$\Theta_{ij,k} + \Theta_{ji,k} < 0,\ 1 \leq i < j \leq r,\ k \in \mathbf{S} \tag{23}$$

$$\sum_{l=1}^{s} \pi_{kl} Q_{1ml} \leq Z_{1m},\ m = 1, 2, 3, \cdots N \tag{24}$$

where

$$\Theta_{ij,k} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \cdots & A_{ui,k}^T M_{N,k}^T & A_{ui,k}^T M_{(N+1),k}^T & \Xi_{1(N+2)} \\ * & \Xi_{22} & \cdots & 0 & M_{2,k} A_{\tau i,k} & -M_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & \Xi_{NN} & R_{1N} & -M_{N,k} \\ * & * & \cdots & * & \Xi_{(N+1)(N+1)} & \Xi_{(N+1)(N+2)} \\ * & * & \cdots & * & * & \Xi_{(N+2)(N+2)} \end{bmatrix}$$

$$\Xi_{11} = \sum_{l=1}^{s} \pi_{kl} P_l + M_{1,k} A_{ui,k} + A_{ui,k}^T M_{1,k}^T + Q_{11k} + \delta \sum_{m=1}^{N} Z_{1m} - R_{11}$$

$$\Xi_{12} = R_{11} + A_{ui,k}^T M_{2,k}^T$$

$$\Xi_{1(N+2)} = P_k^T - M_{1,k} + A_{ui,k}^T M_{(N+2),k}^T$$

$$\Xi_{22} = -Q_{11k} + Q_{12k} - R_{11} - R_{12}$$

$$\Xi_{NN} = -Q_{1(N-1)k} + Q_{1Nk} - R_{1(N-1)} - R_{1N}$$

$$\Xi_{(N+1)(N+1)} = -Q_{1Nk} - R_{1N} + M_{(N+1),k} A_{\tau i,k} + A_{\tau i,k}^T M_{(N+1),k}^T$$

$$\Xi_{(N+1)(N+2)} = -M_{(N+1),k} + A_{\tau i,k}^T M_{(N+2),k}^T$$

$$\Xi_{(N+2)(N+2)} = \delta^2 \sum_{m=1}^{N} R_{1m} - M_{(N+2),k} - M_{(N+2),k}^T$$

Proof. The Lyapunov-Krasovskii functional is chosen as follows:

$$V(x_t, r_t, t) = \sum_{j=1}^{4} V_j(x_t, r_t, t)$$

where

$$V_1(x_t, r_t, t) = x^T(t) P(r_t) x(t)$$

$$V_2(x_t, r_t, t) = \sum_{m=1}^{N} \int_{t-m\delta}^{t-(m-1)\delta} x^T(\alpha) Q_{1m}(r_t) x(\alpha) d\alpha$$

$$V_3(x_t, r_t, t) = \delta \sum_{m=1}^{N} \int_{-m\delta}^{-(m-1)\delta} \int_{t+\beta}^{t} \dot{x}^T(\alpha) R_{1m} \dot{x}(\alpha) d\alpha d\beta$$

$$V_4(x_t, r_t, t) = \sum_{m=1}^{N} \int_{-m\delta}^{-(m-1)\delta} \int_{t+\beta}^{t} x^T(\alpha) Z_{1m} x(\alpha) d\alpha d\beta$$

$\delta = \tau/N$ and N is the number of divisions of the interval $[-\tau,\ 0]$.

By using the same method presented in Theorem 3.1, we can obtain the results. □

Remark 3.6. By decomposing the delay interval $[-\tau,\ 0]$ into N equidistant subintervals and introduce some slack matrices, sufficient conditions for the exponential stability of T-S fuzzy Markovian jump systems with constant time delay are proposed in Proposition 3.5. It should be pointed out that our results still have less conservatism than the results in [29] and [2] even with the case $N = 1$. The reduced conservatism benefits from $V_2(x_t, r_t, t)$ and $V_3(x_t, r_t, t)$, which are selected to be mode-dependent in our paper.

4. **Delay-dependent Stabilization**

In this section, we consider the stabilization problem for the closed-loop fuzzy Markovian jump system (6). Based on Theorem 3.1, the fuzzy state-feedback controllers in (5) will be designed such that the system is mean-square exponentially stable.

Theorem 4.1. *Given a decay rate $\lambda > 0$ and scalars $\rho_{j,k}$, $\rho_{(N+4),k} \neq 0$, $j = 2, 3, \cdots, N+4$, $k \in \mathbf{S}$, for any time delay $\tau_k(t)$ satisfying (2), there exists a fuzzy state-feedback controller in the form of (5) such that the closed-loop system in (6) is mean square exponentially stable, if there exist matrices $\tilde{P}_k > 0$, $\tilde{Q}_{1mk} > 0$, $\tilde{Q}_{2k} > 0$, $\tilde{Q}_{3k} > 0$, $\tilde{R}_{1m} > 0$, $\tilde{R}_2 > 0$, $\tilde{Z}_{1m} > 0$, $\tilde{Z}_2 > 0$, $m = 1, 2, 3, \cdots, N$ such that the following LMIs hold:*

$$\tilde{\Theta}_{ii,k} < 0, \ i \in T, \ k \in \mathbf{S} \tag{25}$$

$$\tilde{\Theta}_{ij,k} + \tilde{\Theta}_{ji,k} < 0, \ 1 \le i < j \le r, \ k \in \mathbf{S} \tag{26}$$

$$e^{\lambda m\delta} \sum_{l=1}^{s} \pi_{kl} \tilde{Q}_{1ml} \le \tilde{Z}_{1m}, \ m = 1, 2, 3, \cdots, N \tag{27}$$

$$\sum_{l=1, l\neq k}^{s} \pi_{kl} \tilde{Q}_{2l} + e^{\lambda\tau_2} \sum_{l=1}^{s} \pi_{kl} \tilde{Q}_{3l} \le \tilde{Z}_2 \tag{28}$$

where

$$\tilde{\Theta}_{ij,k} =$$

$$\begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \cdots & \tilde{\Xi}_{1N} & \tilde{\Xi}_{1(N+1)} & \tilde{\Xi}_{1(N+2)} & \tilde{\Xi}_{1(N+3)} & \tilde{\Xi}_{1(N+4)} \\ * & \tilde{\Xi}_{22} & \cdots & 0 & 0 & 0 & \rho_{2,k} A_{\tau i,k} X_k^T & -\rho_{2,k} X_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \tilde{\Xi}_{NN} & \tilde{R}_{1N} & 0 & \rho_{N,k} A_{\tau i,k} X_k^T & -\rho_{N,k} X_k^T \\ * & * & \cdots & * & \tilde{\Xi}_{(N+1)(N+1)} & 0 & \tilde{\Xi}_{(N+1)(N+3)} & -\rho_{(N+1),k} X_k^T \\ * & * & \cdots & * & * & \tilde{\Xi}_{(N+2)(N+2)} & \tilde{\Xi}_{(N+2)(N+3)} & -\rho_{(N+2),k} X_k^T \\ * & * & \cdots & * & * & * & \tilde{\Xi}_{(N+3)(N+3)} & \tilde{\Xi}_{(N+3)(N+4)} \\ * & * & \cdots & * & * & * & * & \tilde{\Xi}_{(N+4)(N+4)} \end{bmatrix}$$

$$\tilde{\Xi}_{11} = \lambda \tilde{P}_k + \sum_{l=1}^{s} \pi_{kl} \tilde{P}_l + A_{i,k} X_k^T + X_k A_{i,k}^T + B_{i,k} Y_{j,k} + Y_{j,k}^T B_{i,k}^T$$
$$+ e^{\lambda\delta} \tilde{Q}_{11k} + \sum_{m=1}^{N} \tilde{Z}_{1m} \frac{e^{\lambda m\delta} - e^{\lambda(m-1)\delta}}{\lambda} + \tilde{Z}_2 \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda} - \tilde{R}_{11}$$

$$\tilde{\Xi}_{12} = \tilde{R}_{11} + \rho_{2,k} X_k A_{i,k}^T + \rho_{2,k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{1N} = \rho_{N,k} X_k A_{i,k}^T + \rho_{N,k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{1(N+1)} = \rho_{(N+1),k} X_k A_{i,k}^T + \rho_{(N+1),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{1(N+2)} = \rho_{(N+2),k} X_k A_{i,k}^T + \rho_{(N+2),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{1(N+3)} = A_{\tau i,k} X_k^T + \rho_{(N+3),k} X_k A_{i,k}^T + \rho_{(N+3),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{1(N+4)} = \tilde{P}_k^T - X_k^T + \rho_{(N+4),k} X_k A_{i,k}^T + \rho_{(N+4),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\Xi}_{22} = -\tilde{Q}_{11k} + e^{\lambda\delta} \tilde{Q}_{12k} - \tilde{R}_{11} - \tilde{R}_{12}$$

$$\tilde{\Xi}_{NN} = -\tilde{Q}_{1(N-1)k} + e^{\lambda\delta} \tilde{Q}_{1Nk} - \tilde{R}_{1(N-1)} - \tilde{R}_{1N}$$

$$\tilde{\Xi}_{(N+1)(N+1)} = -\tilde{Q}_{1Nk} + e^{\lambda(\tau_2-\tau_1)} \tilde{Q}_{2k} + e^{\lambda(\tau_2-\tau_1)} \tilde{Q}_{3k} - \tilde{R}_{1N} - \tilde{R}_2$$

$$\tilde{\Xi}_{(N+1)(N+3)} = \tilde{R}_2 + \rho_{(N+1),k} A_{\tau i,k} X_k^T$$

$$\tilde{\Xi}_{(N+2)(N+2)} = -\tilde{Q}_{3k} - \tilde{R}_2$$

$$\tilde{\Xi}_{(N+2)(N+3)} = \tilde{R}_2 + \rho_{(N+2),k} A_{\tau i,k} X_k^T$$

$$\begin{aligned}\tilde{\Xi}_{(N+3)(N+3)} &= -\tilde{Q}_{2k} + \mu_k e^{\lambda(\tau_2-\tau_1)} \tilde{Q}_{2k} - 2\tilde{R}_2 + \rho_{(N+3),k} A_{\tau i,k} X_k^T \\ &\quad + \rho_{(N+3),k} X_k A_{\tau i,k}^T\end{aligned}$$

$$\tilde{\Xi}_{(N+3)(N+4)} = -\rho_{(N+3),k} X_k^T + \rho_{(N+4),k} X_k A_{\tau i,k}^T$$

$$\begin{aligned}\tilde{\Xi}_{(N+4)(N+4)} &= \delta \sum_{m=1}^{N} \tilde{R}_{1m} \frac{e^{\lambda m\delta} - e^{\lambda(m-1)\delta}}{\lambda} + (\tau_2 - \tau_1) \tilde{R}_2 \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda} \\ &\quad - \rho_{(N+4),k} X_k^T - \rho_{(N+4),k} X_k\end{aligned}$$

In this case, the desired state-feedback controller can be obtained by

$$K_{j,k} = Y_{j,k} X_k^{-T}, \ j \in T, \ k \in \mathbf{S} \tag{29}$$

Proof. Denote $M_{1,k} = X_k^{-1}$, $M_{2,k} = \rho_{2,k} X_k^{-1}$, $M_{3,k} = \rho_{3,k} X_k^{-1}$, $\cdots$, $M_{(N+3),k} = \rho_{(N+3),k} X_k^{-1}$ and $M_{(N+4),k} = \rho_{(N+4),k} X_k^{-1}$, where $\rho_{(N+4),k} \neq 0$ and X_k, $k \in \mathbf{S}$ are nonsingular matrices. Define $J = diag\{X, X, \cdots, X, X, X, X, X\}$.

Pre- and post-multiplying (8)-(11) by J, J, X, X, and their transposes, respectively, together with the change of matrix variables defined by $\tilde{P}_k = XP_kP$, $\tilde{Q}_{1mk} = XQ_{1mk}X$, $\tilde{Q}_{2k} = XQ_{2k}X$, $\tilde{Q}_{3k} = XQ_{3k}X$, $\tilde{R}_{1m} = XR_{1m}X$, $\tilde{R}_2 = XR_2X$, $\tilde{Z}_{1m} = XZ_{1m}X$, $\tilde{Z}_2 = XZ_2X$, $m = 1, 2, 3, \cdots, N$, $k \in \mathbf{S}$, we can obtain (25)-(28). This completes the proof. □

Remark 4.2. Theorem 4.1 provides some sufficient exponential stabilization conditions for T-S fuzzy Markovian jump systems (6). For given decay rate $\lambda > 0$, Algorithm 1 in [16] provided a method to achieve the suboptimal value of $\rho_{j,k}$, $j = 2, 3, \cdots, N+4$, $k \in \mathbf{S}$.

With the case $\tau_1 = \tau_2 = \tau$ and taking $\lambda \to 0^+$, we obtain the following corollary based on the methods of Proposition 3.5 and Theorem 4.1.

Corollary 4.3. *For constant time delay τ, and scalars $\rho_{j,k}$, $\rho_{(N+2),k} \neq 0$, $j = 2, 3, \cdots, N+2$, $k \in \mathbf{S}$, there exists a fuzzy state-feedback controller in the form of (5) such that the closed-loop system in (6) is mean square exponentially stable, if there exist matrices $\tilde{P}_k > 0$, $\tilde{Q}_{1mk} > 0$, $\tilde{R}_{1m} > 0$, $\tilde{Z}_{1m} > 0$, $m = 1, 2, 3, \cdots, N$ such that the following LMIs hold:*

$$\tilde{\Theta}_{ii,k} < 0, \ i \in T, \ k \in \mathbf{S} \tag{30}$$

$$\tilde{\Theta}_{ij,k} + \tilde{\Theta}_{ji,k} < 0 \tag{31}$$

$$\sum_{l=1}^{s} \pi_{kl} \tilde{Q}_{1ml} \leq \tilde{Z}_{1m}, \ m = 1, 2, 3, \cdots, N \tag{32}$$

where

$$\tilde{\Theta}_{ij,k} = \begin{bmatrix} \tilde{\tilde{\Xi}}_{11} & \tilde{\tilde{\Xi}}_{12} & \cdots & \tilde{\tilde{\Xi}}_{1N} & \tilde{\tilde{\Xi}}_{1(N+1)} & \tilde{\tilde{\Xi}}_{1(N+2)} \\ * & \tilde{\tilde{\Xi}}_{22} & \cdots & 0 & \rho_{2,k} A_{\tau i,k} X_k^T & -\rho_{2,k} X_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & \tilde{\tilde{\Xi}}_{NN} & \tilde{R}_{1N} & -\rho_{N,k} X_k^T \\ * & * & \cdots & * & \tilde{\tilde{\Xi}}_{(N+1)(N+1)} & \tilde{\tilde{\Xi}}_{(N+1)(N+2)} \\ * & * & \cdots & * & * & \tilde{\tilde{\Xi}}_{(N+2)(N+2)} \end{bmatrix}$$

$$\tilde{\tilde{\Xi}}_{11} = \sum_{l=1}^{s} \pi_{kl} \tilde{P}_l + A_{i,k} X_k^T + X_k A_{i,k}^T + B_{i,k} Y_{j,k} + Y_{j,k}^T B_{i,k}^T + \tilde{Q}_{11k} + \delta \sum_{m=1}^{N} \tilde{Z}_{1m} - \tilde{R}_{11}$$

$$\tilde{\tilde{\Xi}}_{12} = \tilde{R}_{11} + \rho_{2,k} X_k A_{i,k}^T + \rho_{2,k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\tilde{\Xi}}_{1N} = \rho_{N,k} X_k A_{i,k}^T + \rho_{N,k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\tilde{\Xi}}_{1(N+1)} = \rho_{(N+1),k} X_k A_{i,k}^T + \rho_{(N+1),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\tilde{\Xi}}_{1(N+2)} = \tilde{P}_k^T - X_k^T + \rho_{(N+2),k} X_k A_{i,k}^T + \rho_{(N+2),k} Y_{j,k}^T B_{j,k}$$

$$\tilde{\tilde{\Xi}}_{22} = -\tilde{Q}_{11k} + \tilde{Q}_{12k} - \tilde{R}_{11} - \tilde{R}_{12}$$

$$\tilde{\tilde{\Xi}}_{NN} = -\tilde{Q}_{1(N-1)k} + \tilde{Q}_{1Nk} - \tilde{R}_{1(N-1)} - \tilde{R}_{1N}$$

$$\tilde{\tilde{\Xi}}_{(N+1)(N+1)} = -\tilde{Q}_{1Nk} - \tilde{R}_{1N} + \rho_{(N+1),k} A_{\tau i,k} X_k^T + \rho_{(N+1),k} X_k A_{\tau i,k}^T$$

$$\tilde{\Xi}_{(N+1)(N+2)} = -\rho_{(N+1),k} X_k^T + \rho_{(N+2),k} X_k A_{\tau i,k}^T$$
$$\tilde{\Xi}_{(N+2)(N+2)} = \delta^2 \sum_{m=1}^{N} \tilde{R}_{1m} - \rho_{(N+2),k} X_k^T - \rho_{(N+2),k} X_k$$

Remark 4.4. Similarly, we can use the method of Algorithm 1 in [16] to get the maximum time delay upper bound τ.

5. Illustrative Examples

In this section, two examples are provided to demonstrate the effectiveness of proposed approaches. The reduced conservatism of our methods becomes apparent by comparing with the results in the literature. The first example is to show the advantage of the proposed stability conditions and obtained controller design methods. The second example is to show the application of proposed controller design methods.

Example 5.1. Consider the following T-S fuzzy Markovian jump system with constant time delay:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(\theta(t)) \{A_{i,k} x(t) + A_{\tau i,k} x(t-\tau) + B_{i,k} u(t)\}$$

where

$$A_{1,1} = \begin{bmatrix} -8 & 0 \\ 0 & -9 \end{bmatrix},\ A_{1,2} = \begin{bmatrix} -15 & 0.1 \\ 0.1 & -12 \end{bmatrix},\ A_{2,1} = \begin{bmatrix} -4 & 0.2 \\ 0.1 & -3 \end{bmatrix},\ A_{2,2} = \begin{bmatrix} -8 & 0 \\ 0 & -7 \end{bmatrix}$$

$$A_{\tau 1,1} = \begin{bmatrix} -0.9 & 0.5 \\ -1 & -1 \end{bmatrix},\ A_{\tau 1,2} = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 2 \end{bmatrix},\ A_{\tau 2,1} = \begin{bmatrix} -0.2 & 0.8 \\ 1 & 0.2 \end{bmatrix},\ A_{\tau 2,2} = \begin{bmatrix} 2 & 0.2 \\ 1 & 0.1 \end{bmatrix}$$

$$B_{1,1} = \begin{bmatrix} -2 \\ 2 \end{bmatrix},\ B_{1,2} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},\ B_{2,1} = \begin{bmatrix} -0.7 \\ 0.6 \end{bmatrix},\ B_{2,2} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

We first assume $u(t) = 0$. By using the Matlab LMI Toolbox, Table 1 gives the comparisons of maximum allowed time delay τ obtained by the methods in [2, 29] and that proposed by us. It can be seen from Table 1 that our methods are clearly less conservative than the existing results even with the case $N = 1$. In addition, with N increasing, the conservatism of the obtained results will be reduced.

When $u(t) \neq 0$, letting $N = 1$ and based on Corollary 4.3, the maximum allowed τ can be obtained as 0.5236. The comparison results of maximum allowed time delay τ via different methods are recorded in Table 2. From Table 2, we can see that the proposed stabilization results in this paper are less conservative than those in [2, 29]. Letting $\tau = 0.3236$ and using Corollary 4.3, we can obtain the following state feedback gain matrices:

$$K_{1,1} = \begin{bmatrix} 7.6201 & -4.2135 \end{bmatrix},\ K_{1,2} = \begin{bmatrix} 5.7384 & -7.0710 \end{bmatrix}$$
$$K_{2,1} = \begin{bmatrix} 8.4931 & -5.9056 \end{bmatrix},\ K_{2,2} = \begin{bmatrix} 6.7812 & -5.0976 \end{bmatrix}$$

Example 5.2. Consider the following delayed computer simulated single link robot arm formulated in [19, 29].

$$\ddot{\theta}(t) = -\frac{MgL}{J}\sin(\theta(t)) - \frac{D(t)}{J}\dot{\theta}(t) + \frac{1}{J}u(t)$$

$$\begin{cases} \dot{x}_1(t) = \mu x_2(t) + (1-\mu)x_2(t-\tau_k(t)) \\ \dot{x}_2(t) = -\frac{M_k gL}{J_k}\sin(x_1(t)) - \frac{\mu D}{J_k}x_2(t) \end{cases} - \frac{(1-\mu)D}{J_k}x_2(t-\tau_k(t)) + \frac{1}{J_k}u(t) \qquad (33)$$

where $x_1(t)$, $x_2(t)$ and $u(t)$ are the angle of the arm and the control input respectively. $\mu = 0.7$ is the retarded coefficient, $L = 0.5$ is the length of the arm, $g = 9.81$ is the acceleration of gravity, and $D = 2$ is the coefficient of vicious friction. The Mass M_k and the inertia J_k have three modes: $M_1 = J_1 = 1$, $M_2 = J_2 = 5$, $M_3 = J_3 = 10$. The transition rate of the operation modes is given by

$$\Pi = \begin{bmatrix} -0.3 & 0.25 & 0.05 \\ 0.1 & -0.2 & 0.1 \\ 0.03 & 0.07 & -0.1 \end{bmatrix}$$

The time-varying delay $\tau_k(t),\ k = 1, 2, 3$ in this example is dependent on the system mode, which is more natural and general than the time delay in [12].

Similar to [12], the fuzzy basis functions are chosen as

$$h_1(x_1(t)) = \begin{cases} \frac{\sin(x_1(t)) - \beta x_1(t)}{x_1(t)(1-\beta)}, & x_1(t) \neq 0 \\ 1, & x_1(t) \neq 0 \end{cases}$$

Methods	τ=0.3
[27]	0.5713
[29]	0.2154
Proposition 3.5, N=1	0.8139
Proposition 3.5, N=2	0.9747
Proposition 3.5, N=3	1.0518

Table 1. Maximum Time Delay τ via Different Delay-Dependent Stability Criteria

Methods	Maximum allowed τ
[27]	0.3073
[29]	0.1097
Corollary 4.3	0.5236

Table 2. Maximum Time Delay τ via Different Stability Methods

$$h_2\left(x_1(t)\right) = \begin{cases} \dfrac{\sin\left(x_1(t)\right) - \beta x_1(t)}{x_1(t)(1-\beta)}, & x_1(t) \neq 0 \\ 1, \ x_1(t) \neq 0 \end{cases}$$

where $\beta = 10^{-2}/\pi$. Then, the nonlinear Markovian jump system in (33) can be exactly represented by the following T-S fuzzy model.

Plant Rule i: IF $x_1(t)$ is M_{i1}, THEN

$$\dot{x}(t) = A_{i,k}x(t) + A_{\tau i,k}x\left(t - \tau_k(t)\right) + B_{i,k}u(t)$$

Theorem 4.1	λ=0.3	λ=0.6
Proposition 3.5, N=1	0.3829	0.3174
Proposition 3.5, N=2	0.4037	0.3299
Proposition 3.5, N=3	0.4211	0.3405

TABLE 3. Maximum Time Delay Bound via Different λ

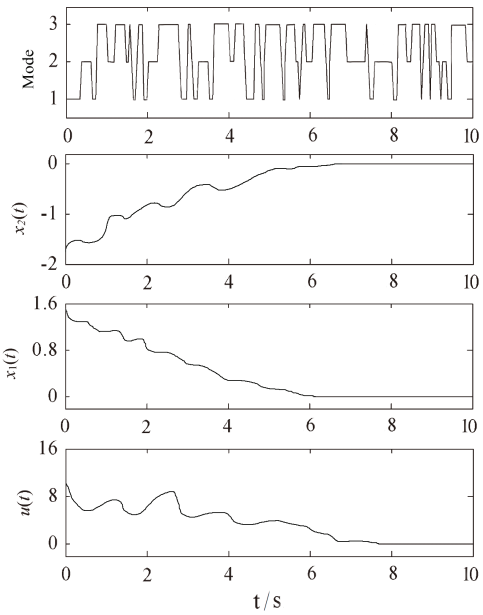

FIGURE 1. Operation Mode and Control Results

where M_{11} is about 0 rand, and M_{21} is about π rad or $-\pi$ rad. The system parameters are as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$A_{1,1}=\begin{bmatrix}0 & \mu\\ -gL & -2\mu\end{bmatrix},\ A_{1,2}=\begin{bmatrix}0 & \mu\\ -gL & -0.4\mu\end{bmatrix},\ A_{1,3}=\begin{bmatrix}0 & \mu\\ -gL & -0.2\mu\end{bmatrix}$$

$$A_{2,1}=\begin{bmatrix}0 & \mu\\ -\beta gL & -2\mu\end{bmatrix},\ A_{2,2}=\begin{bmatrix}0 & \mu\\ -\beta gL & -0.4\mu\end{bmatrix},\ A_{2,3}=\begin{bmatrix}0 & \mu\\ -\beta gL & -0.2\mu\end{bmatrix}$$

$$A_{\tau 1,1}=A_{\tau 2,1}=\begin{bmatrix}0 & 1-\mu\\ 0 & -2(1-\mu)\end{bmatrix}$$

$$A_{\tau 1,2}=A_{\tau 2,2}=\begin{bmatrix}0 & 1-\mu\\ 0 & -0.4(1-\mu)\end{bmatrix}$$

$$A_{\tau 1,3}=A_{\tau 2,3}=\begin{bmatrix}0 & 1-\mu\\ 0 & -0.2(1-\mu)\end{bmatrix}$$

$$B_{1,1}=B_{2,1}=\begin{bmatrix}0\\ 1\end{bmatrix},\ B_{1,2}=B_{2,2}=\begin{bmatrix}0\\ 0.2\end{bmatrix},\ B_{1,3}=B_{2,3}=\begin{bmatrix}0\\ 0.1\end{bmatrix}$$

For given $\tau_1 = 0.1$, and by using Theorem 4.1, we record the upper bounds τ_2 of the time delays via different decay rate λ in Table 3. It can be seen from Table 3 that the conservatism will be reduced with N increasing.

With the choice of $\lambda = 0.6$ and $\tau_2 = 1$, we obtain the following state-feedback gain matrices:

$$K_{1,1}=\begin{bmatrix}-2.9473 & 2.8248\end{bmatrix},\ K_{1,2}=\begin{bmatrix}-13.359 & 17.5371\end{bmatrix},$$
$$K_{1,3}=\begin{bmatrix}-20.1752 & 23.9174\end{bmatrix}$$

$$K_{2,1}=\begin{bmatrix}5.1079 & 3.7031\end{bmatrix},\ K_{2,2}=\begin{bmatrix}20.0047 & 19.0894\end{bmatrix},\ K_{2,3}=\begin{bmatrix}27.2649 & 30.2423\end{bmatrix}$$

Set the initial condition $\varphi(t) = \begin{bmatrix}0.5\pi & -2\end{bmatrix}^T$, $t \in [-1, 0]$, and apply the fuzzy controller in form of (5) to system (6). The simulation results are shown in Figure.1. The results show that the designed fuzzy controller can effectively stabilize the system (6) with the decay rate $\lambda = 0.6$.

6. **Conclusion**

In this paper, we have investigated the delay-dependent exponential stability analysis and stabilization for continuous-time T-S fuzzy Markovian jump systems with mode-dependent time-varying delay. To obtain less conservative stability and stabilization conditions, a new Lyapunov-Krasovskii functional has been constructed. In addition, the decay rate in our paper is a free value and can be adjusted according to the practical condition. Some comparisons have been made with the existing results to verify the less conservatism of our methods.

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References

[1] P. Balasubramaniam and R. Sathy, *Robust asymptotic stability of fuzzy Markovian jumping genetic regulatory networks with time-varying delays by delay decomposition approach*, Comm. Nonlinear Science Numerical Simul., **16(2)** (2011), 928–939.

[2] P. Balasubramaniam and T. Senthilkumar, *Delay-range-dependent robust stabilization and H_∞ control for nonlinear uncertain stochastic fuzzy systems with mode-dependent time delays and Markovian jump parameters*, Int. J. Syst. Sci.,(2012), 1–14.

[3] L. J. Banu and P. Balasubramaniam, *Admissibility analysis for discrete-time singular systems with time-varying delays by adopting the state-space Takagi-Sugeno fuzzy model*, Int. J. Fuzzy Syst., **12(3)** (2015), 1–16.

[4] L. J. Banu and P. Balasubramaniam, *Robust stability and stabilization analysis for discrete-time randomly switched fuzzy systems with known sojourn probabilities*, Nonlinear Anal. :Hybrid Syst., **17** (2015), 128–143.

[5] E. K. Boukas, Z. K. Liu and G. X. Liu, *Delay-dependent robust stability and H_∞ control of jump linear systems with time-delay*, Int. J. Control, **74(4)** (2001), 329–340.

[6] J. X. Dong and G. Yang, *Fuzzy controller design for Markovian jump nonlinear systems*, Int. J. Control, Autom., Syst., **5(6)** (2007), 712–717.

[7] Z. Y. Fei, H. J. Gao and P. Shi, *New results on stabilization of Markovian jump systems with time delays*, Automatica, **45(10)** (2010), 2300–2306.

[8] H. J. Gao, Z. Y. Fei, J. Lam and B. Du, *Further results on exponential estimates of Markovian jump systems with mode-depdendent time-varying delays*, IEEE Trans. Autom. Control, **56(1)** (2011), 223–229.

[9] K. Q. Gu, *A further refinement of discretized Lyapunov functional method for the stability of time-delay systems*, International Journal of Control, **74(1)** (2001), 737–744.

[10] D. W. C. Ho and Y. G. Niu, *Robust fuzzy design for nonlinear uncertain stochastic systems via sliding-mode control*, IEEE Trans. Fuzzy Syst., **15(3)** (2007), 350–358.

[11] M. Liu, P. Shi, L. X. Zhang and X. D. Zhao, *Fault-tolerant control for nonlinear Markovian jump systems via proportional and derivative sliding mode observer technique*, IEEE Trans. Circuits. Syst. I. Regul., **58(5)** (2011), 1109–1118.

[12] S. Nguang, W. Assawinchaichote, P. Shi and Y. Shi, *Robust H_∞ control design for uncertain fuzzy systems with Markovian jumps: an LMI approach*, In: American Control Conference, (2005), 1805–1810.

[13] S. K. Nguang, P. Shi and S. Ding, *Fault detection for uncertain fuzzy systems: an LMI approach*, IEEE Trans. Fuzzy Syst., **15(6)** (2007), 1251–1262.

[14] P. Shi, Y. Q. Xia, G. P. Liu and D. Rees, *On designing of sliding-mode control for stochastic jump systems*, IEEE Trans. Autom. Control, **51(1)** (2006), 97–103.

[15] Z. Shu, J. Lam and S. Y. Xu, *Robust stabilization of Markovian jump delay systems with delay-dependent exponential estimates*, Automatica, **42(11)** (2006), 2001–2008.

[16] E. G. Tian and C. Peng, *Delay-dependent stability analysis and synthesis of uncertain T-S fuzzy systems with time-varying delay*, Fuzzy Sets, Syst., **157** (2006), 544–559.

[17] S. C. Tong, W. Wang and L. J. Qu, *Decentralized robust control for uncertain T-S fuzzy large-scale systems with time-dela*, Int. J.Innovative Comput., **3(3)** (2007), 657–672.

[18] Z. D. Wang, Y. R. Liu and X. H. Liu, *Exponential stabilization of a class of stochastic system with Markovian jump parameters and mode-dependent mixed time-delays*, IEEE Trans. Autom. Control, **55(7)** (2010), 1656–1662.

[19] H. N. Wu and K. Y. Cai, *Mode-independent robust stabilization for uncertain Markovian jump nonlinear systems via fuzzy control*, IEEE Trans. Man, Cybern., **36(3)** (2006), 509–519.

[20] L. G. Wu, P. Shi and H. J. Gao, *State estimation and sliding mode control of Markovian jump singular systems*, IEEE Trans. Autom. Control, **55(5)** (2010), 1213–1219.

[21] L. Q. Wu, P. Shi, H. J. Gao and C. H. Wang, *Robust H_∞ filtering for 2-D Markovian jump systems*, Automatica, **44(7)** (2008), 1849–1858.

[22] L. G. Wu, X. J. Su, P. Shi and J. B. Qiu, *A new approach to stability analysis and stabilization of discrete-time T-S time-varying delay systems*, IEEE Trans. Syst. Man, Cybern., **40(1)** (2011), 273–286.
[23] Z. J. Wu, X. J. Xie, P. Shi and Y. Q. Xia, *Backstepping controller design for a class of stochastic nonlinear systems with Markovian switching*, Automatica, **45(4)** (2009), 997–1004.
[24] Z. J. Wu, X. J. Xie and P. Shi, *Adaptive tracking for stochastic nonlinear systems with Markovian switching*, IEEE Trans. Autom. Control, **55(9)** (2010), 2135–2141.
[25] Y. Q. Xia, M. Y. Fu, P. Shi, Z. J. Wu and J. H. Zhang, *Adaptive backstepping controller design for stochastic jump systems*, IEEE Trans. Autom. Control, **54(12)** (2009), 2853–2859.
[26] J. L. Xiong, J. Lam, H. J. Gao and D. W. C. Ho, *On robust stabilization of Markovian jump systems with uncertain switching probabilities*, Automatica, **41(5)** (2005), 897–903.
[27] S. Y. Xu and J. Lam, *Robust H_∞ control for uncertain discrete-time-delay fuzzy systems via output feedback controllers*, IEEE Trans. Fuzzy Syst., **13(1)** (2005), 82–93.
[28] D. Yue, and Q. L. Han, *Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching*, IEEE Trans. Autom. Control, **50(2)** (2005), 217–222.
[29] Y. S. Zhang, S. Y. Xu and J. H. Zhang, *Delay-dependent robust H_∞ control for uncertain fuzzy Markovian jump systems*, Int. J. Control, Autom., Syst., **7(4)** (2009), 520–529.
[30] Y. H. Zhang, S. Y. Xu and B. Y. Zhang, *Robust output feedback stabilization for uncertain discrete-time fuzzy Markovian jump systems with time-varying delays*, IEEE Trans. Fuzzy Syst., **17(2)** (2009), 411–420.
[31] Y. Zhao, H. J. Gao, J. Lam and B. Z. Du, *Stability and stabilization of delayed T-S fuzzy systems: a delay partitioning approach*, IEEE Trans. Fuzzy Syst., **17(4)** (2009), 750–762.

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